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5 **Discreteness is undecidable**

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14 We prove that the discreteness problem for two-generated nonelementary subgroups of  
 15  $SL(2, \mathbb{C})$  is undecidable in the Blum–Shub–Smale (BSS) computability model.

16 *Keywords:* Discrete subgroups; decidability.

17 *Mathematics Subject Classification:* 22E40, 20F10

18 **1. Introduction**

19 This paper is motivated by the following basic question:

20 **Question 1.1.** Let  $G$  be a connected Lie group and let  $\mathcal{A} = (A_1, \dots, A_k)$  be a  
 21 finite ordered subset of  $G$ . Is the discreteness problem for the subgroup  $\Gamma_{\mathcal{A}} :=$   
 22  $\langle A_1, \dots, A_k \rangle < G$  decidable?

23 This question, in the case of  $G = \mathrm{PSL}(2, \mathbb{C})$ , was raised, most recently, in the  
 24 paper [8] by Gilman and Keen, who noted that “it is a challenging problem that  
 25 has been investigated for more than a century and is still open.” The decidability  
 26 problem was solved in the case  $G = \mathrm{PSL}(2, \mathbb{R})$  by Riley [20] and, more efficiently,  
 27 in the case of two-generated subgroups, by Gilman and Maskit [9] and Gilman [6],  
 28 (*cf.* [7] for a comparison of the two approaches).

29 To make the general decidability question more precise, one has to specify the  
 30 model of computability. There are several computability models over the real num-  
 31 bers; we refer the reader to [1, 21] for summaries of these and in-depth treatment  
 32 of the Blum–Shub–Smale (BSS) and the *bit-computability* approaches, respectively.  
 33 In this paper, we address decidability of the discreteness problem in the *real-RAM*  
 34 or BSS computability model as it is the closest in spirit to the papers by Gilman,  
 35 Maskit and Keen mentioned above as well as Riley’s work [20]. We will address

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1 decidability of the discreteness problem in the bit-computability model in another  
2 paper [13].

3 **Remark 1.2.** We refer the reader to the paper by Gilman in [7], where several  
4 (semi)algorithms for the discreteness problem in  $\mathrm{PSL}(2, \mathbb{R})$  and  $\mathrm{PSL}(2, \mathbb{C})$  in differ-  
5 ent computability models, including the BSS model, are compared.

6 Briefly, computations in the BSS model over the real numbers are performed  
7 by a BSS *machine*, which is an analogue of a Turing machine except that a BSS  
8 machine can store finite lists of real numbers and do elementary algebraic and  
9 order operation with real numbers: such a machine can add, subtract, multiply and  
10 divide, as well as verify inequalities and equalities  $a < b, a = b$  for real numbers.  
11 (BSS machines are also defined for computations in other rings, but, in this paper  
12 we will use only real numbers.) We refer to [1] for the details.

13 A subset  $E \subset \mathbb{R}^n$  is BSS-*semicomputable* (or the membership problem for  $E$   
14 is BSS-*semidecidable*) if  $E$  is the *halting set* of a BSS machine: there exists a BSS  
15 machine which, given an input vector  $x \in \mathbb{R}^n$ , stops if and only if  $x \in E$ . A  
16 membership problem for  $E$  is BSS-decidable if and only if both  $E$  and  $E^c = \mathbb{R}^n - E$   
17 are BSS-semicomputable. We refer the reader to the book [1] for the details.

18 **Remark 1.3.** In our paper, the input for a BSS machine is a tuple  $\mathcal{A}$  of  $2 \times 2$   
19 complex matrices.

20 The main result about BSS machines needed for our paper is the following  
21 theorem due to Blum, Shub and Smale, see [1, Theorem 1, Chap. 2]:

22 **Theorem 1.4.** *The halting set for a BSS machine is a (computable) countable*  
23 *union of real semialgebraic subsets of  $\mathbb{R}^n$ .*

24 **Remark 1.5.** We note that the proof of this theorem in [1] actually shows more:  
25 Allow a *generalized* BSS machine to do Boolean operations with inequalities, as  
26 well as to compute not only rational functions, but also *real algebraic functions*,  
27 i.e., functions whose graphs are given by finite sets of polynomial equations and  
28 inequalities, e.g.,  $\sqrt{x}$ . Then, the halting set of such a machine is still a countable  
29 union of real semialgebraic subsets.

30 Before stating our main results, we note that the nondiscreteness problem for  
31 one-generator subgroups of  $G = S^1 \subset \mathbb{C}^*$  is *not semidecidable*, since a subgroup  
32  $\langle A \rangle < S^1$  is nondiscrete if and only if  $A$  has infinite order, i.e., is not a root of unity.  
33 The complement in  $S^1$  of the set of roots of unity is clearly not a countable union of  
34 arcs, therefore, it cannot be a halting set of a BSS machine. Thus, the discreteness  
35 problem, strictly speaking, is undecidable already in  $G = \mathrm{PSL}(2, \mathbb{R})$ . To make it  
36 decidable in  $G = \mathrm{PSL}(2, \mathbb{R})$  one has to exclude from  $G^k$  the algebraic subvariety  
37 consisting of tuples of matrices generating (virtually) abelian subgroups. Regarding  
38 subgroups of  $G = \mathrm{PSL}(2, \mathbb{C})$  with two (or more) generators, one has to exclude, for  
39 a similar reason, dihedral subgroups (both finite and infinite). In line with the work

1 of Gilman, Keen, Maskit and Riley, we will, moreover, exclude from consideration  
 2 all tuples  $\mathcal{A}$  which generate *elementary subgroups* of  $G = \mathrm{SL}(2, \mathbb{C})$ . (This exclusion  
 3 also allows for a clean discussion of the *character variety*, which is a quotient of  
 4  $\mathrm{Hom}(F_k, G)$  by the group  $G$  acting *via* conjugation.) Observe also that Riley's  
 5 arguments in [20] (based on the Jørgensen inequality) show that the *nondiscreteness*  
 6 *problem* for nonelementary subgroups of  $\mathrm{PSL}(2, \mathbb{C})$  is BSS-semidecidable.

7 The space  $G^k$  of  $k$ -tuples of matrices  $\mathcal{A}, A_j \in G$ , is naturally identified with the  
 8 *representation variety*, which is the algebraic variety  $\mathrm{Hom}(F_k, G)$ , via the map

$$\phi \mapsto (A_1, \dots, A_k), \quad A_j = \phi(x_j), \quad j = 1, \dots, k,$$

9 where  $F_k = \langle x_1, \dots, x_k \rangle$  is the free group of rank  $k$ . The variety  $\mathrm{Hom}(F_k, G)$  con-  
 10 tains a (closed) real semialgebraic subvariety  $\mathrm{Hom}_e(F_k, G)$  consisting of represen-  
 11 tations  $\phi$  whose images are *elementary subgroups* of  $G$ , i.e., subgroups which either  
 12 fix a point in the hyperbolic three-space or on its ideal boundary sphere or preserve  
 13 a geodesic in the hyperbolic three-space. The complement

$$\mathrm{Hom}_{ne}(F_k, G) = \mathrm{Hom}(F_k, G) - \mathrm{Hom}_e(F_k, G)$$

14 is the space of *nonelementary representations*. This space is the main object of our  
 15 study. We let

$$\mathrm{Hom}_d(F_k, G) \subset \mathrm{Hom}_{ne}(F_k, G)$$

16 denote the subset consisting of nonelementary representations with discrete images.  
 17 Since elementary representations are excluded, the subset  $\mathrm{Hom}_d(F_k, G)$  is known  
 18 to be closed (in the classical topology), see the paper of Jørgensen and Klein [10],  
 19 as well as [11].

20 In this paper, we prove:

21 **Theorem 1.6.** *The subset  $\mathrm{Hom}_d(F_2, \mathrm{SL}(2, \mathbb{C}))$  is not BSS-semicomputable.*

22 Thus, at least in the BSS-computability model, the discreteness problem for  
 23 two-generated subgroups of  $\mathrm{SL}(2, \mathbb{C})$  is undecidable. Our proof is modeled on the  
 24 undecidability result for the Mandelbrot set  $\mathbb{M}$ : The membership problem for  $\mathbb{M}$  is  
 25 BSS-undecidable according to [1, Chap. 2]. The proof of Theorem 1.6 is not difficult,  
 26 but it relies upon three deep results:

- 27 • Description of BSS-computable sets by Blum, Shub and Smale, see [1].
- 28 • Minsky's solution of the ending lamination conjecture for punctured tori [15].<sup>a</sup>
- 29 • Miyachi's theorem [17], proving nonsmoothness (at the "cusps") of the boundary  
 30 of the Maskit slice in the character variety of the punctured torus.

31 The undecidability theorem in this paper should be contrasted with the *semide-*  
 32 *cidability result* for *convex-cocompact faithful representations* into  $\mathrm{PSL}(2, \mathbb{C})$ , cf. [8].

<sup>a</sup>Minsky's work used here was one of the many papers leading, eventually, to the solution of the full Ending Lamination Conjecture by Minsky, Brock and Canary, [16, 3].

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1 We note that a more general semidecidability result for *Morse (Anosov)* represen-  
 2 tations of hyperbolic groups into semisimple Lie groups is proven in the work of the  
 3 author with Leeb and Porti [12, Sec. 7.7].

## 4 **2. Proof of Theorem 1.6**

5 Set  $G = \mathrm{SL}(2, \mathbb{C})$ . We will show that the set in Theorem 1.6 is not a countable union  
 6 of real semialgebraic subsets of  $\mathrm{Hom}_{ne}(F_2, G)$ , where we regard  $G$  as a real alge-  
 7 braic group. First of all, instead of working in  $\mathrm{Hom}(F_2, G)$ , it suffices to work with  
 8 the character variety  $X = X(F_2, G) = \mathrm{Hom}(F_2, G) // G$ . The reason is that there is  
 9 a polynomial map  $\tau : \mathrm{Hom}(F_2, G) \rightarrow X$  whose fibers are the *extended*  $G$ -orbits in  
 10  $\mathrm{Hom}(F_2, G)$ , where  $G$  acts via composition of representations  $F_2 \rightarrow G$  with inner  
 11 automorphisms of  $G$ . Discreteness, of course, is invariant under conjugation. We  
 12 will avoid discussion of the *extended* orbit equivalence and only note that for repre-  
 13 sentations in  $\mathrm{Hom}_{ne}(F_2, G)$  the extended orbit equivalence is the same as the orbit  
 14 equivalence. Therefore, it suffices to work with the character variety. Concretely,  
 15 the map  $\tau$  is given by

$$\tau(A, B) = (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \in \mathbb{C}^3.$$

16 Our next reduction is to the *Maskit slice*  $X_M$  in  $X$ , i.e., the complex-algebraic  
 17 subset given by the following trace conditions:

$$\mathrm{tr}([A, B]) = -2, \quad \mathrm{tr}(A) = 2.$$

18 Since the Maskit slice is algebraic, the problem now reduces to the one in the Maskit  
 19 slice. The Maskit slice of  $X$  is complex one-dimensional, it is biregularly isomorphic  
 20 to the complex line  $\mathbb{C}$  via the map

$$(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \mapsto \mathrm{tr}(B) \in \mathbb{C}.$$

21 We, therefore, identify  $X_M$  with  $\mathbb{C}$  via this map. Recall that geometrically finite rep-  
 22 resentations are dense among all discrete and faithful representations  $\Gamma \rightarrow \mathrm{SL}(2, \mathbb{C})$   
 23 (for any finitely generated group  $\Gamma$ ). This was proven first by Minsky [15] for repre-  
 24 sentations of punctured torus groups, and, hence, in the Maskit slice, which suffices  
 25 for our purposes. The general case is due to the work of many people, most notably,  
 26 Bromberg [4], Brock and Bromberg [2], Namazi and Souto [18] and Ohshika [19].  
 27 We, thus, have:

28 **Proposition 2.1.** *The space  $\mathcal{D} \subset \mathbb{C}$  of equivalence classes of discrete representa-*  
 29 *tions  $[\rho] \in X_M = \mathbb{C}$  has the following structure:*

$$\mathcal{D} = \mathcal{DF} \sqcup C,$$

30 *where  $C$  is a countable subset of nonfaithful geometrically finite representations and*  
 31  *$\mathcal{DF}$  is the set of equivalence classes  $[\rho] \in X_M$  such that  $\rho : F_2 \rightarrow G$  is discrete and*  
 32 *faithful.*

1 Thus, it suffices to show that  $\mathcal{DF}$  is not a countable union of real semialgebraic  
 2 subsets. Due to the work of Minsky [15], the topological boundary of  $\mathcal{DF}$  is a  
 3 topological arc  $\alpha$  properly embedded in  $\mathbb{C}$ . The complement to  $\mathcal{DF}$  in  $\mathbb{C}$  is also  
 4 diffeomorphic to  $\mathbb{R}^2$ .

5 Before proving the next lemma, we recall that an *accidental parabolic element*  
 6 of a representation

$$\rho : F_2 = \pi_1(T^2 - \text{point}) \rightarrow \text{SL}(2, \mathbb{C})$$

7 is an element of  $F_2$  represented by a (necessarily simple) nonperipheral loop  $\gamma$  (not  
 8 representing the conjugacy class of the generator  $A$  of  $F_2$ ) on the punctured torus  
 9  $T^2 - \text{point}$ , such that  $\pi_1(\gamma)$  is a parabolic element of  $\text{SL}(2, \mathbb{C})$ . The equivalence  
 10 class  $[\rho] \in \mathcal{DF}$  of a representation  $\rho$  is called a *cusps* if  $\rho$  has an accidental parabolic  
 11 element. It again follows from Minsky's work ([15, Theorem B]) that *cusps are*  
 12 *dense* in the boundary of  $\mathcal{DF}$  (cf. the earlier work of C. McMullen [14]).

13 **Lemma 2.2.** *The arc  $\alpha$  contains no smooth subarcs (which are not singletons).*

14 **Proof.** Miyachi proved [17] the arc  $\alpha$  is not smooth at each cusp, which are dense  
 15 in  $\alpha$ . Therefore,  $\alpha$  does not contain nondegenerate smooth subarcs.  $\square$

16 We can now conclude the proof of Theorem 1.6.

17 **Proof.** Suppose that  $\mathcal{DF}$  is a countable union

$$\bigcup_{j \in J} E_j$$

18 of real algebraic subsets  $E_j$  of  $\mathbb{C}$ . Each  $E_j$  is either finite or its topological frontier  
 19  $\partial E_j$  in  $\mathbb{C}$  is a finite union of real-algebraic arcs. Since, as noted above, the arc  $\alpha$   
 20 does not contain real-algebraic subarcs, each  $E_j$  intersects  $\alpha$  in a nowhere dense (in  
 21  $\alpha$ ) subset. By the Baire Theorem, the union

$$\bigcup_{j \in J} \partial E_j \cap \alpha$$

22 has empty interior in  $\alpha$ . Therefore, the union of subsets  $E_j$  cannot be equal to  $\mathcal{DF}$ .  
 23 This contradiction concludes the proof of Theorem 1.6.  $\square$

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