On the dynamics of pseudo-Anosov homeomorphisms on representation varieties of surface groups *

Michael Kapovich

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Abstract. We study the action of pseudo-Anosov homeomorphisms $f : R \to R$ on the character varieties of $SL(2, \mathbb{C})$-representations of the fundamental groups $\pi_1(R)$ of closed orientable hyperbolic surfaces R. We prove that the representation $\pi_1(R) \hookrightarrow SL(2, \mathbb{C})$ corresponding to the holonomy representation of the hyperbolic structure on the mapping torus of f is a hyperbolic fixed point for the action of f on the character variety $X(\pi_1(R))$.

1 Introduction

Suppose that R is a closed oriented hyperbolic surface, $f : R \to R$ is an orientation-preserving pseudo-Anosov homeomorphism inducing an automorphism ϕ of the fundamental group $\pi_1(R)$. According to Thurston's hyperbolization theorem the mapping torus M_f of f admits a hyperbolic structure (see [O]). The fundamental group $\pi_1(M_f)$ is the semidirect product $G = \pi_1(R) \rtimes \mathbb{Z}$, where a generator t of $\mathbb{Z}$ acts on $\pi_1(R)$ by the conjugation $x \mapsto t^{-1}xt$, this action is the same as the automorphism ϕ. We use the hyperbolic structure on M_f to realize the group G as a discrete group of isometries of the hyperbolic 3-space $\mathbb{H}^3$. Let $X(\pi_1(R)) = \mathrm{Hom}(\pi_1(R), SL(2, \mathbb{C}))//SL(2, \mathbb{C})$ be the character variety. Note that the isomorphism $\phi : \pi_1(R) \to \pi_1(R)$ induces a holomorphic automorphism

$$\Phi : X(\pi_1(R)) \to X(\pi_1(R)), \quad \Phi : [\rho] \mapsto [\rho \circ \phi]$$

where $[\rho]$ denotes the $SL(2, \mathbb{C})$-equivalence class of a representation $\rho : \pi_1(R) \to SL(2, \mathbb{C})$. The equivalence class of the identity embedding $\iota : \pi_1(R) \hookrightarrow G \subset SL(2, \mathbb{C})$ is a fixed point of Φ.

The main goal of this paper is to prove the following theorem that was conjectured by Curt McMullen in [Mc]:

Theorem 1.1 *$[\iota]$ is a hyperbolic fixed point of Φ, i.e. the derivative*

$$d\Phi : T_{[\iota]}X(\pi_1(R)) \longrightarrow T_{[\iota]}X(\pi_1(R))$$

has no eigenvalues λ such that $|\lambda| = 1$.

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Remark 1.2 *McMullen proved in [Mc] that there are at least two eigenvalues of* $d\Phi$ *which do not belong to the unit circle.*

Unlike McMullen's approach, our proof is mostly cohomological, we study the action of ϕ on the cohomology group $\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$ of $\pi_1(R)$, where $sl(2,\mathbb{C})$ is the Lie algebra of $SL(2,\mathbb{C})$ and $\pi_1(R)$ acts on $sl(2,\mathbb{C})$ via the adjoint representation $ad \circ \iota$. We prove the following theorem which implies Theorem 1.1:

Theorem 1.3 *The action* $\phi^* : \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C})) \to \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$ *has no eigenvalues* λ *such that* $|\lambda| = 1$.

Note however that not every fixed point of Φ is hyperbolic. For instance, we can choose ϕ that acts trivially on $\mathrm{H}_1(R,\mathbb{Z})$, thus the trivial representation of $\pi_1(R)$ to $SL(2,\mathbb{C})$ is not a hyperbolic fixed point of the corresponding mapping Φ. More generally, for each $m \geq 1$ there are hyperbolic 3-manifolds M fibered over circles so that the character variety $X(\pi_1(M))$ contains a smooth complex m-dimensional submanifold Y^m. Let $\pi_1(R)$ denote a normal surface subgroup in $\pi_1(M)$ and $\Phi : X(\pi_1(R)) \to X(\pi_1(R))$ the automorphism corresponding to the fibration of M over $\mathbb{S}^1$ with the fiber R. Thus, for each $[\rho] \in Y^m$ the point $[\rho|_{\pi_1(R)}]$ is a fixed point of Φ and the derivative of Φ at $[\rho|_{\pi_1(R)}]$ has at least m-dimensional fixed subspace.

As a warm-up to the proof of Theorem 1.3 we prove that ϕ^* has no roots of unity as eigenvalues. Indeed, if λ is an eigenvalue of ϕ^* such that $\lambda^n = 1$, then $(\phi^n)^*$ has a nonzero invariant vector in $\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$. Note that for every $n \geq 1$ the group $G_n := \pi_1(R) \rtimes \langle t^n \rangle$ acts on the hyperbolic 3-space as a uniform lattice. Now the Serre-Hochschild exact sequence (see [Br, Corollary 6.4]) implies that $\mathrm{H}^1(G_n, sl(2,\mathbb{C})) \neq 0$ since it is isomorphic to the space of $(\phi^n)^*$-invariants on $\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$, this contradicts the Calabi-Weil infinitesimal rigidity theorem [Ra].

Below is an outline of our proof in the general case: Suppose that we have a cocycle $\sigma \in Z^1(\pi_1(R), sl(2,\mathbb{C}))$ such that $\phi^{\#}(\sigma) = \lambda\sigma$, $\lambda \in \mathbb{C}^*$. Then there is a smooth vector-field ξ on $\mathbb{H}^3$ and its lift $\tilde{\xi}$ to the group $SL(2,\mathbb{C})$ such that:

(a) $ad(\gamma)\xi - \xi = \sigma_\gamma$ for all $\gamma \in \pi_1(R)$.

(b) $ad(t)\tilde{\xi} = \lambda^{-1}\xi$.

If $|\lambda| = 1$ it follows that ξ is a *quasiconformal vector-field* on $\mathbb{H}^3$ in the sense of Ahlfors. Then we construct a *tangential extension* ξ_∞ of ξ to the ideal boundary $\mathbb{S}^2$ of $\mathbb{H}^3$ so that ξ_∞ is a quasiconformal vector-field on $\mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\}$ which still satisfies the property (a) with respect to the action of $\pi_1(R)$ on $\mathbb{S}^2$. Then $\mu = \bar{\partial}\xi_\infty$ is a $\pi_1(R)$-invariant Beltrami differential, hence the Sullivan's rigidity theorem [Su] implies that $\mu = 0$ almost everywhere.[1] Therefore ξ_∞ is actually a Moebius vector-field and the property (a) implies that the cocycle σ is a coboundary. This proves that the only solution τ of the equation

$$\phi^*(\tau) = \lambda\tau, \quad |\lambda| = 1, \quad \tau \in \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$$

is the trivial cohomology class which concludes the proof.

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[1] Note that we have to use Sullivan's theorem since it is the only rigidity theorem which deals with discrete groups of infinite covolume such as $\pi_1(R) \subset SL(2,\mathbb{C})$.

The reader will notice that our arguments in the proof of Theorem 1.3 provide an alternative proof of the Calabi-Weil infinitesimal rigidity theorem a la Mostow. Namely, suppose that $\Gamma \subset \mathbf{Isom}(\mathbb{H}^n)$ is a uniform lattice ($n \geq 3$). Then the Calabi-Weil infinitesimal rigidity theorem states that $\mathrm{H}^1(\Gamma, ad) = 0$. The usual way to prove this is to take a harmonic representative α of a class $[\alpha] \in \mathrm{H}^1(\Gamma, ad)$ and then verify that $\alpha = 0$ via Bochner's technique. Instead we take a quasiconformal vector-field ξ on $\mathbb{H}^n$ representing $[\alpha]$, extend ξ (tangentially) to the sphere at infinity and then check (using for instance the Sullivan's rigidity theorem) that this extension is actually a Moebius vector-field . This proves that α is a coboundary.

The main technical difficulty in our proof is to establish existence of a "continuous extension" of quasiconformal vector-fields from the open ball in $\mathbb{R}^n$ to its boundary. We prove that under some extra condition[2] there is a *tangential* continuous extension. This extension will suffice for our purposes. The nontrivial analytical ingredient of our construction of the *tangential* extension is the *Semenov's stability theorem* (see Theorem 3.5), which is a part of the general stability theory for spatial quasiconformal mappings. This theorem proves that k-quasiconformal vector-fields in the unit open ball in $\mathbb{R}^n, n \geq 3$, are bounded. We discuss this and some basic facts about quasiconformal vector-fields in Section 3. In Section 4 we establish existence of tangential extensions of quasiconformal vector-fields . Theorem 1.3 is proven in Section 5. In Section 2 we show how to deduce Theorem 1.1 from Theorem 1.3.

The proof of Theorem 1.3 given here is not entirely satisfactory: it does't tell how eigenvalues of ϕ^* are related to combinatorial invariants of the pseudo-Anosov homeomorphism $f : R \to R$ (like its Perron-Frobenius matrix, see [FLP]). Another interesting question is to describe stable and unstable manifolds of the fixed point $[\iota]$ of the mapping Φ. Note that Φ preserves the natural symplectic structure on $X(\pi_1(R))$ (see [Go]). Thus ϕ^* has $3g-3$ eigenvalues whose absolute value is less than 1, and the same number of eigenvalues outside the unit disc (where g is the genus of R). In particular, the (complex) dimension of stable and unstable manifolds E^s, E^u of Φ at $[\iota]$ is $3g-3$. McMullen in [Mc] proved the following theorem about E^s and E^u:

Theorem 1.4 *Take a pair of singly degenerate Kleinian groups* $F^+, F^- \subset SL(2, \mathbb{C})$*, whose ending laminations* L^+, L^- *are those of the discrete doubly degenerate group* $\pi_1(R) \subset G \subset SL(2, \mathbb{C})$*. Then the Teichmuller spaces* $T(F^+), T(F^-)$ *are open subsets in* E^s *and* E^u*.*

However it is unclear how E^s, E^u behave away from the domains $T(F^+), T(F^-)$.

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[2] The vector-field must be *automorphic* under the action of a discrete group of Moebius transformations whose limit set is the boundary of the ball, see Definition 4.4.

Contents

2 Local dynamics on the representation variety

Most of the material of this section is fairly well-known, we present it here for the sake of completeness. Let $\pi_1(R), \phi$ be as in the Introduction. Let

$$ad(g)\xi = g^{-1}\xi g$$

denote the adjoint action of the group $SL(2,\mathbb{C})$ on its Lie algebra $sl(2,\mathbb{C})$. We consider the action of the automorphism ϕ on the variety $\mathrm{Hom}(\pi_1(R), SL(2,\mathbb{C}))$ given by:

$$\phi : \rho \mapsto \rho \circ \phi, \rho \in \mathrm{Hom}(\pi_1(R), SL(2,\mathbb{C})) \quad .$$

This action of ϕ projects to a holomorphic automorphism Φ of the *character variety* $X(\pi_1(R)) = Hom(\pi_1(R), SL(2,\mathbb{C}))//SL(2,\mathbb{C})$.

Remark 2.1 *The character variety is the quotient of* $\mathrm{Hom}(\pi_1(R), SL(2,\mathbb{C}))$ *in the sense of geometric invariants theory and in general is different from the "naive" set-theoretic quotient* $Hom(\pi_1(R), SL(2,\mathbb{C}))/SL(2,\mathbb{C})$*. However if we restrict ourselves to the open subvariety* $\mathrm{Hom}(\pi_1(R), SL(2,\mathbb{C}))^-$ *consisting of Zariski dense representations then the projection* $X(\pi_1(R))^-$ *of* $\mathrm{Hom}(\pi_1(R), SL(2,\mathbb{C}))^-$ *to* $X(\pi_1(R))$ *is naturally isomorphic to the set-theoretic quotient* $Hom(\pi_1(R), SL(2,\mathbb{C}))^-/SL(2,\mathbb{C})$*, see [JM, Theorem 1.1]. Moreover* $X(\pi_1(R))^-$ *is a smooth complex manifold of the dimension* $6g-6$*, see [W, Go].*

Suppose that $[\rho_0] \in X(\pi_1(R))$ is a fixed point for Φ and the image of ρ_0 is Zariski dense in $SL(2,\mathbb{C})$. Then there is an element $t \in SL(2,\mathbb{C})$ such that

$$t\rho_0(\phi(\gamma))t^{-1} = \rho_0(\gamma), \text{ for all } \gamma \in \pi_1(R) \ .$$

We consider the induced action $T_0(\Phi)$ of Φ on the tangent space $T_{[\rho_0]}$ to $X(\pi_1(R))$ at $[\rho_0]$. Our goal is to identify this action with the natural action ϕ^* of ϕ on the 1-st cohomology group

$$\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C})) = Z^1(\pi_1(R), sl(2,\mathbb{C}))/B^1(\pi_1(R), sl(2,\mathbb{C})) \ .$$

Recall that the action ϕ^* is defined by

$$\phi^{\#}(c)_\gamma = ad(t^{-1})c_{\phi(\gamma)}, \quad \phi^{\#} : Z^1(\pi_1(R), sl(2,\mathbb{C})) \longrightarrow Z^1(\pi_1(R), sl(2,\mathbb{C}))$$

$$\phi^*[c] = [\phi^{\#}(c)], \quad [c] \in \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$$

where $c : \gamma \mapsto c_\gamma \in sl(2,\mathbb{C})$ are cocycles in $Z^1(\pi_1(R), sl(2,\mathbb{C}))$.

Remark 2.2 *Here and in what follows we use the notation c_γ for the value of the cocycle c on the element $\gamma \in \pi_1(R)$. Note that c_γ belongs to $sl(2,\mathbb{C})$ and is a vector-field on $SL(2,\mathbb{C})$ and $\mathbb{H}^3$. Thus given $g \in SL(2,\mathbb{C})$ or $x \in \mathbb{H}^3$ the evaluations $c_\gamma(g)$ and $c_\gamma(x)$ are tangent vectors in $T_g(SL(2,\mathbb{C}))$ and $T_x\mathbb{H}^3$ respectively.*

It was first noticed by Andre Weil in [W] (see also [Go]) that the (Zariski) tangent space $T_{[\rho_0]}$ is naturally isomorphic to $\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$.

Lemma 2.3 *Under the natural isomorphism between $T_{[\rho_0]}$ and $\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$ we have: $T_0(\Phi) = \phi^*$.*

Proof: Recall (see [Go]) that curves of representations ρ_ϵ through the point ρ_0 can be described as

$$\rho_\epsilon(g) = \exp(\epsilon c_\gamma + O(\epsilon^2))\rho_0(\gamma), \quad \epsilon \to 0$$

where c belongs to $Z^1(\pi_1(R), sl(2,\mathbb{C}))$. The isomorphism

$$T_{\rho_0}\mathrm{Hom}(\pi_1(R), SL(2,\mathbb{C})) \cong Z^1(\pi_1(R), sl(2,\mathbb{C}))$$

is given by:

$$\left.\frac{d}{d\epsilon}\rho_\epsilon\right|_{\epsilon=0} \mapsto c$$

Thus

$$\rho_\epsilon(\phi(\gamma)) \approx \exp(\epsilon\ c_{\phi(\gamma)})\rho_0(\phi(\gamma)) = t^{-1}\exp(\epsilon\ ad(t^{-1})c_{\phi(\gamma)})\rho_0(\gamma)t$$

The last has the same projection to $X(\pi_1(R))$ as $\exp(\epsilon ad(t^{-1})c_{\phi(\gamma)})\rho_0(\gamma)$, which has the tangent vector $\phi^{\#}(c)$. Thus

$$T_0(\Phi)(c) - \phi^{\#}(c) \in B^1(\pi_1(R), sl(2,\mathbb{C}))$$

which concludes the proof. □

Corollary 2.4 *Theorem 1.3 is equivalent to Theorem 1.1.*

3 Quasiconformal vector-fields and the S-operator

For the proof of Theorem 1.3 we will need definitions and properties of *quasiconformal vector-fields* that were introduced by Lars Ahlfors in [Ah1] under the name of *quasiconformal deformations.* Our discussion here will follow [Ah1] and [Ah2].

Let U denote the upper half-space $\mathbb{R}^n_+ = \{(x_1, ..., x_n) : x_n > 0\}$, we will identify it with the hyperbolic n-space $\mathbb{H}^n$ with the hyperbolic metric $|dx|/x_n$. Let $\mathbf{Mob}_n$ denote the group of Moebius transformations of $\overline{\mathbb{R}^n}$ and $\mathbf{Mob}_n(U)$ denote the stabilizer of U in $\mathbf{Mob}_n$. The group $\mathbf{Mob}_n(U)$ acts as the group of isometries $\mathbf{Isom}(\mathbb{H}^n)$ of $\mathbb{H}^n$. We shall use the notations mob_n and $isom(\mathbb{H}^n)$ for the Lie algebras of the groups $\mathbf{Mob}_n$ and $\mathbf{Isom}(\mathbb{H}^n)$. We will realize mob_n and $isom(\mathbb{H}^n)$ as subalgebras in the space of all vector-fields on $\mathbb{R}^n$ and U.

If X is a smooth manifold and $A : X \to X$ is a diffeomorphism of X then we shall denote by $A_* : \xi \mapsto A_*(\xi)$ the action of A on vector-fields on X. In particular, the action of Moebius transformations A on vector-fields ξ is given by the formula:

$$A_*(\xi(x)) = (DA_x)^{-1}\xi(Ax)$$

where DA_x is the Jacobian matrix of A at x. Then $A_*(\xi) = ad(A)(\xi)$ for all $\xi \in mob_n$. Similarly, if we identify mob_n with the Lie algebra of left-invariant vector-fields on $\mathbf{Mob}_n$ then $A_*(\xi) = ad(A)(\xi)$. Elements ξ of mob_n have the form of "2-nd degree polynomials":

$$\xi(\overrightarrow{x}) = \overrightarrow{u} + (Q + aI)\overrightarrow{x} + |\overrightarrow{x}|^2\overrightarrow{b} - 2\langle\overrightarrow{x}, \overrightarrow{b}\rangle\overrightarrow{x} \quad .$$

Here $\langle \bullet, \bullet \rangle$ is the scalar product in $\mathbb{R}^n$, $\overrightarrow{u}$, $\overrightarrow{b}$ are vectors in $\mathbb{R}^n$, $Q = -Q^T$, $a \in \mathbb{R}$ and I is the identity matrix. If $\xi \in isom(\mathbb{H}^n)$, then:

$$\langle\overrightarrow{u}, \overrightarrow{e}_n\rangle = \langle\overrightarrow{b}, \overrightarrow{e}_n\rangle = 0, \quad Q\overrightarrow{e}_n = \overrightarrow{0} \quad .$$

We define the norm $\|A\|$ for $n \times n$-matrices as $\|A\| = \sqrt{Tr(AA^T)}$. Define the supremum-norm $\|h\|_\infty$ for $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ by $\|h\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$ using the Euclidean metric on $\mathbb{R}^n$ (the issue here is that we have to consider hyperbolic metrics on domains in $\mathbb{R}^n$ as well). We shall denote by $B_x(r)$ the metric ball with the radius r and center x in $\mathbb{R}^n$, let $B(1) = B_0(1)$ and $S(1) = \partial B(1)$. We shall identify $S(1)$ with the sphere $\mathbb{S}^{n-1}$.

Consider a domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ and a continuous vector-field $f : \Omega \to \mathbb{R}^n$ on Ω. Suppose that f has locally integrable distributional partial derivatives. Define the matrix $S = S_n[f](x)$ by the formula:

$$S = \frac{1}{2}(Df + Df^T) - \frac{1}{n}Tr(Df) \quad .$$

The operator $S[f]$ is called the *Ahlfors* S*-operator*, it was introduced by Lars Ahlfors in [Ah1]. If $n = 2$ then the S-operator becomes the $\bar{\partial}$-operator:

$$S[f] = \begin{bmatrix} Re(\partial f/\partial\bar{z}) & Im(\partial f/\partial\bar{z}) \\ Im(\partial f/\partial\bar{z}) & -Re(\partial f/\partial\bar{z}) \end{bmatrix}$$

under the standard matrix realization of $\mathbb{C}$.

Definition 3.1 *A vector-field f above is called a k-***quasiconformal vector-field** *if $\|S[f]\| \in L_\infty$ and $\|S[f]\|_\infty \leq k\sqrt{n}$.*

Quasiconformal vector-fields are infinitesimal analogues of quasiconformal mappings in $\mathbb{R}^n$. More precisely, suppose that h_t, $t \in [0,1]$, is a smooth family of quasiconformal homeomorphisms such that $h_0 = id$. Then $\frac{d}{dt}h_t|_{t=0} = f$ is a quasiconformal vector-field . If $M_t = M(h_t)$ is the *Beltrami differential* of h_t then $\frac{d}{dt}M_t|_{t=0} = S[f]$, see [Ah1, Ah2] for more details. Similarly to quasiconformal mappings quasiconformal vector-fields are differentiable a.e. in their domains.

Assume that $n = 2$, $f(z)$ transforms as a vector-field under Moebius mappings α. Thus $\mu = S[f] = \bar{\partial}(f)$ transforms as a Beltrami differential under α:

$$\alpha_* : \mu(z) \mapsto \mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z)$$

More generally, if $n \geq 2$ then

$$A_* S[f] := S[A_*(f)] = (DA)^{-1}(S[f] \circ A) DA \tag{1}$$

(see [Ah1]). This implies:

$$\|S[A_*(f)]\| = \|S[f] \circ A\| \tag{2}$$

Therefore Moebius transformations send k–quasiconformal vector-fields to k–quasiconformal vector-fields. This allows to define quasiconformal vector-fields on the extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$.

Note that the matrix $S[f]$ is symmetric and traceless. Suppose that $S(z) = S[f](z) \neq 0$, let φ_z be the corresponding quadratic form. Then $S(z)$ determines a *nontrivial* splitting of $\mathbb{R}^n$ (which we identify with the tangent space at z):

$$\mathbb{R}^n = P_{S,z} \oplus N_{S,z} \tag{3}$$

where the restriction of φ_z to P_z is a positive-definite quadratic form and the restriction of φ_z to N_z is a non-positive quadratic form. Thus the formula (1) implies

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^n$ be a domain where $S \neq 0$ almost everywhere. Suppose that $A : \Omega \to \Omega$ is a Moebius transformation and $A_* S = S$. Then a.e. defined splitting $P_{S,z} \oplus N_{S,z}$ of the tangent bundle $T\Omega$ is invariant under A.*

The following lemma is the usual Weyl's lemma if $n = 2$, for $n \geq 3$ it was proven by Ahlfors in [Ah2]:

Lemma 3.3 *If f is a quasiconformal vector-field on a connected domain $\Omega \subset \overline{\mathbb{R}^n}$ and $S[f] = 0$ a.e. in Ω, then f is a conformal vector-field . If $n \geq 3$ or $\Omega = \overline{\mathbb{R}^n}$ this implies that f is a Moebius vector-field , i.e. $f \in mob_n|_\Omega$.*

J. Sarvas proved in [Sar] that k-quasiconformal vector-fields have the following "convergence property" similar to one for quasiconformal mappings:

Lemma 3.4 *(J. Sarvas, [Sar].) Suppose that f_j is a sequence of k-quasiconformal vector-fields in a domain $\Omega \subset \mathbb{R}^n$ such that $\|f_j\| \leq C$ for all j. Then $\{f_j\}$ contains a subsequence which is convergent uniformly on compacts in Ω to a k-quasiconformal vector-field on Ω.*

The following theorem was proven by V. Semenov in [Se]:

Theorem 3.5 *(Semenov's Stability Theorem, [Se, Lemma 1].) Suppose that $n \geq 3$. Then there exists a universal constant c_n depending only on the dimension n of $\mathbb{R}^n$ so that the following holds:*

Let f be a k-quasiconformal vector-field on the **open** *unit ball $B(1) \subset \mathbb{R}^n$. Then there exists $\lambda \in mob_n$ such that $\|f - \lambda\|_\infty \leq c_n k$.*

In particular, every quasiconformal vector-field in $B(1)$ is bounded (this obviously fails if $n=2$).

In what follows we will need to establish a relation between the operators S_n and S_{n-1} in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. Let $n \geq 2$ and $\xi(x) = (\xi_1, ..., \xi_n)$ be a differentiable vector-field in a domain $\Omega \subset \mathbb{R}^n$, take the hyperplane $L = \{x_n = 0\}$ which intersects Ω along a domain $\Omega' \neq \emptyset$. Define a vector-field $\hat{\xi}$ on the domain Ω' by

$$\hat{\xi}(x') = (\xi_1, ..., \xi_{n-1}), \ x' = (x_1, ..., x_{n-1}) \quad .$$

Lemma 3.6 $\|S_n[\xi]\| \geq \|S_{n-1}[\hat{\xi}]\|$.

Proof: Let $diag(D\xi)$ denote the diagonal matrix whose diagonal entries are the diagonal entries of $D\xi$. Then

$$\|S_n[\xi]\|^2 = \|\frac{1}{2}(D\xi + D\xi^T) - diag(D\xi)\|^2 + \|diag(D\xi) - \frac{1}{n}Tr(diag(D\xi))\|^2 \ .$$

Let $q_n := \|diag(D\xi) - \frac{1}{n}Tr(diag(D\xi))\|^2$. It is obvious that

$$\|\frac{1}{2}(D\xi + D\xi^T) - diag(D\xi)\|^2 \geq \|\frac{1}{2}(D\hat{\xi} + D\hat{\xi}^T) - diag(D\hat{\xi})\|^2 \ .$$

Thus what we need to prove is: $q_n \geq q_{n-1}$. We introduce the notations: $z_i = \partial\xi_i/\partial x_i$, $\tau_n = z_1 + ... + z_n$, then $\tau_{n-1} = \tau_n - z_n$. Clearly

$$q_n = \sum_{j=1}^{n} z_j^2 - \frac{1}{n}\tau_n^2$$

$$q_{n-1} = \sum_{j=1}^{n-1} z_j^2 - \frac{1}{n-1}\tau_{n-1}^2$$

and

$$q_n - q_{n-1} = z_n^2 - \frac{1}{n}\tau_n^2 + \frac{1}{n-1}\tau_{n-1}^2 = \frac{1}{n-1}(nz_n^2 + \frac{1}{n}\tau_n^2 - 2\tau_n z_n) \quad .$$

The determinant of the matrix of the quadratic form $Q(z_n, \tau_n) = nz_n^2 + \frac{1}{n}\tau_n^2 - 2\tau_n z_n$ is zero and diagonal entries are positive, thus $q_n - q_{n-1} = Q(z_n, \tau_n) \geq 0$. $\square$

4 Tangential extension of quasiconformal vector-fields

It will be crucial for us to construct continuous extensions of quasiconformal vector-fields from the open unit ball $B(1)$ to its boundary $S(1)$. The extension that we will construct *is not* a vector-field on $\mathbb{R}^n$ but rather a tangent vector-field on the sphere $S(1) \cong \mathbb{S}^{n-1}$. The only extension results for quasiconformal vector-fields that I know are theorems of L. Ahlfors [Ah3] and J. Sarvas [Sar] below:

Theorem 4.1 *(L. Ahlfors, [Ah3].) Suppose that f is a quasiconformal vector-field on the open unit ball $B(1) \subset \mathbb{R}^n$, such that:*

(a) $|f(x)|$ is uniformly bounded in $B(1)$,

(b) $\lim_{|x|\to 1}\langle f(x), x\rangle = 0$, i.e. f is asymptotically tangential.

Then f admits a continuous extension to the unit sphere $S(1)$ and, by reflection in $S(1)$, an extension to a quasiconformal vector-field on $\overline{\mathbb{R}^n}$.

Theorem 4.2 *(J. Sarvas, [Sar].) Isolated singularities of quasiconformal vector-fields in $\overline{\mathbb{R}^n}$ $(n \geq 3)$ are removable.*

Remark 4.3 *To understand the difficulty note that for any holomorphic function $f(z)$ defined in the upper half-plane $U := \{Im(z) > 0\} \subset \mathbb{C}$ the vector-field $f(z)\partial/\partial z$ is 0-quasiconformal in U, but usually it can not be continuously extended to the boundary.*

As far as I can see, Ahlfors' theorem is not strong enough to suffice for our purposes. Namely, quasiconformal vector-fields that we consider are not apriori *asymptotically tangential*. On the other hand, if ξ is a quasiconformal vector-field on $B(1)$, then its projection $\mathcal{P}_p(\xi)$ (defined below) is not in general a quasiconformal vector-field on $B(1)$.

We will prove existence of a continuous *tangential* extension under the following technical assumption:

Definition 4.4 *We say that a continuous vector-field f in $B(1)$ is* **automorphic** *if the following holds:*

There exists a discrete group F of Moebius transformations of $B(1)$ whose limit set equals $S(1)$ and a 1-cocycle $\sigma : F \to isom(\mathbb{H}^n)$ so that

$$A_* f - f = \sigma_A$$

for all $A \in F$.

To describe the extension we will need several preliminary constructions. Pick a point $p \in S(1)$ and consider the family $\mathcal{U}_p$ of horospheres U_p in $\mathbb{H}^n = B(1)$ centered at p. Let $\nu_{x,p}$ denote the unit (with respect to the Euclidean metric) normal vector to the sphere U_p at the point $x \in U_p$. We assume that $\nu_{x,p}$ is directed outside U_p. Let ξ be a vector-field on $B(1)$. We define another vector-field $\mathcal{P}_p(\xi)$ in $B(1)$ by the formula

$$\mathcal{P}_p(\xi)(x) = f(x) - \langle f(x), \nu_{x,p}\rangle \nu_{x,p}$$

where $\langle \bullet, \bullet\rangle$ is the standard scalar product in $\mathbb{R}^n$. Clearly $\mathcal{P}_p(f)(x)$ is tangent to the sphere U_p at the point x and is obtained via the orthogonal projections of $f(x)$ to $T_x(U_p)$.

Proposition 4.5 *Suppose that f is a smooth automorphic k-quasiconformal vector-field on the open unit ball $B(1)$ in $\mathbb{R}^n$, $n \geq 3$. Then for any $p \in S(1)$ the field $\mathcal{P}_p(f)$ admits a continuous extension to $[B(1) \cup S(1)] - \{p\}$ whose restriction to the unit sphere $S(1) \cong \mathbb{S}^{n-1}$ is a vector-field $f_{p,\infty}$ which is tangent to $S(1)$ and is k-quasiconformal on $\mathbb{S}^{n-1} - \{p\} \cong \mathbb{R}^n$.*

Proof: Let F denote the discrete group and σ the cocycle so that f is automorphic with respect to F and σ.

Step 1. We first prove that the field f has well-defined limits (with respect to the conical topology) on a dense countable subset in $S(1)$, which consists of fixed points of loxodromic elements of the group F.

Lemma 4.6 *Let $L \subset S(1)$ be the collection of fixed points of loxodromic elements of the group F. Then the vector-field f admits a continuous (with respect to the conical topology) extension to L.*

Proof: It is enough to consider the upper half-space model $U = \mathbb{R}^n_+$ of the hyperbolic n-space $\mathbb{H}^n$, assume that the origin O is a fixed point of a loxodromic element $A : \overrightarrow{x} \mapsto \lambda W \overrightarrow{x}$ of the group F. We assume that $0 < \lambda < 1$, the orthogonal matrix W belongs to $O(n-1)$ (the stabilizer of ∂U in $SO(n)$).

Pick a sequence of points $z_k \in U$ that is convergent to the origin O in the conical topology, i.e there is $r > 0$ so that z_k belong to the hyperbolic r-neighborhood of the geodesic $\{(0,0,...,0,x_n), x_n > 0\} \subset \mathbb{H}^n$. Let K denote the metric ball in $\mathbb{H}^n$ with the center at $\overrightarrow{e_n}$ and hyperbolic radius $r + \exp(\lambda)$. Then each z_k belongs to the $\langle A \rangle$-orbit of K:

$$A^{-k}(z_k) = y_k \in K, \quad \text{for some} \quad k \in \mathbb{Z} \ .$$

Let $\sigma = \sigma_A \in isom(\mathbb{H}^n)$ be the value of the cocycle σ at A, then

$$(A^k_* f)(y) = \sum_{j=0}^{k-1} (A^j_* \sigma)(y) + f(y)$$

for each $y \in K$. This means:

$$f(A^k y) = \lambda^k W^k f(y) + \lambda^k W^k \sum_{j=0}^{k-1} (A^j_* \sigma)(y)$$

Since $0 < \lambda < 1$ it is clear that $\lim_{k\to\infty} \lambda^k U^k f(y) = 0$ uniformly for $y \in K$. Note that

$$|(A^j_* \sigma)(y)| = \lambda^{-j} |\sigma(A^j y)| \ .$$

To estimate $|\sigma(A^j y)|$ we let

$$\sigma(x) = \overrightarrow{v} + (T + sI)\overrightarrow{x} + (|x|^2 \overrightarrow{b} - 2\langle \overrightarrow{x}, \overrightarrow{b} \rangle \overrightarrow{x}) = \sigma_{(-1)}(x) + \sigma_{(0)}(x) + \sigma_{(1)}(x) \ .$$

Then

$$\sigma_{(0)}(A^j \overrightarrow{y}) = (T + sI) A^j \overrightarrow{y} = A^j (T_j + sI) \overrightarrow{y} = \lambda^j W^j (T_j + sI) \overrightarrow{y}$$

where T_j is a sequence of skew-symmetric matrices which have uniformly bounded norm ($T_j = A^{-j} T A^j$). Thus

$$|(T + sI) A^j \overrightarrow{y}| \leq Const \cdot \lambda^j$$

and

$$|\lambda^k W^k \sum_{j=0}^{k-1} A^j_* \sigma_{(0)}(y)| = |\lambda^k W^k \sum_{j=0}^{k-1} \lambda^{-j} (T + sI) A^j \overrightarrow{y}| \leq Const \cdot k \cdot \lambda^k$$

which tends to zero as $k \to \infty$. The estimate for $|\sigma_{(1)}(A^j y)|$ is:

$$|\sigma_{(1)}(A^j y)| \leq 3\lambda^{2j} |\overrightarrow{b}| \cdot |\overrightarrow{y}|^2 .$$

Thus

$$|\lambda^k W^k \sum_{j=0}^{k-1} \lambda^{-j} \sigma_{(1)}[A^j \overrightarrow{y}]| \leq 3|\overrightarrow{b}| \cdot |\overrightarrow{y}|^2 \lambda^k \frac{\lambda^{2k}-1}{\lambda^2-1}$$

which again tends to zero as $k \to \infty$ uniformly for $y \in K$. We conclude that

$$\lim_{k\to\infty} f(A^k y) = \lim_{k\to\infty} \sum_{j=0}^{n-1} \lambda^k W^k A_*^j \sigma_{(-1)}(y) =$$

$$= \lim_{k\to\infty} \sum_{j=0}^{k-1} \lambda^{k-j} W^{k-j} \overrightarrow{v} = -\overrightarrow{v} + \sum_{j=0}^{\infty} \lambda^j W^j \overrightarrow{v} = [1-A]^{-1} \overrightarrow{v} - \overrightarrow{v} .$$

Therefore

$$\lim_{k\to\infty} f(A^k y) = [1-A]^{-1} \overrightarrow{v} - \overrightarrow{v} = ([1-A]^{-1} - 1)\sigma(0)$$

uniformly for $y \in K$. Hence, for the sequence z_k convergent to O in the conical topology we get:

$$\lim_{k\to\infty} f(z_k) = \lim_{k\to\infty} f(A^k y_k) = ([1-A]^{-1} - 1)\sigma(0)$$

where $A^{-k} z_k = y_k \in K$. $\Box$

Step 2. Now we change the coordinates by an isometry of $\mathbb{H}^n = U$ so that the point p becomes the point $\infty \in \overline{\mathbb{R}^n}$. Then $f : U \to \mathbb{R}^n$ is a k-quasiconformal deformation whose sup-norm is bounded in $U \cap B(R)$, $0 < R < \infty$.

We will prove that $\mathcal{P}_p(f)$ has a continuous extension $f_{p,\infty}$ to $cl(U) \cap B(R/2)$, so that $f_{p,\infty}|_{B(R/2)\cap\{x_n=0\}}$ is a k-quasiconformal vector-field in $\mathbb{R}^{n-1}$.

For each $0 < t < 1$ we consider the restriction f_t of $\mathcal{P}_p(f)$ to the unit $(n-1)$-disc $B_{C_t}(1) \subset \{x_n = t\}$ with the center at $C_t = (0, ..., 0, t)$. We regard f_t as a vector-field on $B^{n-1}(R/2) \subset \mathbb{R}^{n-1}$ by "forgetting" the last coordinate x_n. Lemma 3.6 implies that f_t is a k-quasiconformal vector-field .

Note that sup-norms of the k-quasiconformal vector-fields f_t are uniformly bounded (Theorem 3.5). Hence, according to Lemma 3.4, up to a subsequence t_j, f_t are convergent to a k-quasiconformal vector-field $f_{p,\infty}$ on $B(R/2) \cap \mathbb{R}^{n-1}$. The limit $f_{p,\infty}$ is supposed to be the boundary value of the extension of $\mathcal{P}_p(f)$ to the hyperplane $\{x_n = 0\}$. The problem is that for different sequences $t_j \to 0$ we may get distinct limits. However we know that different limits must coincide on a dense set of points (Step 1). Thus the vector-field $\mathcal{P}_p f$ admits a continuous extension to $B(R/2) \cap \{x_n = 0\}$, which is k-quasiconformal in $\mathbb{R}^{n-1}$. $\Box$

Note however that $f_{p,\infty}$ is not defined at the point $p = \infty$. Thus we have to consider extensions corresponding to another family of horospheres. Let $q \in \mathbb{S}^{n-1} - \{p\}$. From Theorem 4.5 we know that both vector-fields $\mathcal{P}_p(f)$, $\mathcal{P}_q(f)$ admit continuous extensions $f_{p,\infty}$, $f_{q,\infty}$ to the sphere $\mathbb{S}^{n-1}$.

Proposition 4.7 *$f_{p,\infty} = f_{q,\infty}$ on $\mathbb{S}^{n-1} - \{p, q\}$.*

Proof: We go back to the unit ball model of the hyperbolic space. Pick a pair of horospheres $U_p \in \mathcal{U}_p$, $U_q \in \mathcal{U}_q$, let ρ denote the minimum of their Euclidean radii. Take a point $x \in U_p \cap U_q$. Then obviously

$$\lim_{\rho \to 1} |\nu_{x,p} - \nu_{x,q}| = 0$$

where $|\cdot|$ denotes the Euclidean norm.

$$\lim_{\rho \to 1} |\mathcal{P}_p(f)(x) - \mathcal{P}_q(f)(x)| \leq \lim_{\rho \to 1} Const |\nu_{x,p} - \nu_{x,q}| = 0 \ . \ \Box$$

Thus for each automorphic vector-field f there is a k-quasiconformal vector-field f_∞ tangent to $\mathbb{S}^{n-1}$ which coincides with each $f_{p,\infty}$ on $\mathbb{S}^{n-1} - \{p\}$. Therefore we have proved

Theorem 4.8 *Suppose that f is a smooth automorphic k-quasiconformal vector-field on the open unit ball $B(1)$ in $\mathbb{R}^n$, $n \geq 3$. Then f admits a continuous* **tangential extension** *f_∞ to $S(1)$. The vector-field f_∞ is again a k-quasiconformal vector-field on the sphere $S(1) = S^{n-1}$.*

Proposition 4.9 *Let f be a quasiconformal vector-field on $B = B(1)$ which is automorphic with respect to a group $F \subset \mathbf{Mob}_n$ and a cocycle σ: $A_* f - f = \sigma_A$ for all $A \in F$. Assume that F is torsion-free. Then*

$$A_* f_\infty - f_\infty = \sigma_A|_{\mathbb{S}^{n-1}} \ \ \textit{for all } A \in F \ .$$

Proof: Let $A \in F - \{1\}$, then A has a fixed point $p \in \mathbb{S}^{n-1}$. Thus the family of horospheres $\mathcal{U}_p$ is invariant under A. Therefore

$$A_* \mathcal{P}_p(f) = \mathcal{P}_p(A_* f) \ .$$

Since $\mathcal{P}_p(f)$ has continuous extension f_∞ to $\mathbb{S}^{n-1}$ we conclude that

$$A_* f_\infty - f_\infty = (\sigma_A)_\infty = \sigma_A|_{\mathbb{S}^{n-1}} \ . \ \Box$$

Remark 4.10 *It is easy to check that the above proposition is also valid for groups with torsion.*

5 Proof of the main theorem

In what follows we will identify of the hyperbolic 3-space with the quotient space $SU(2)\backslash SL(2,\mathbb{C})$, let $P : SL(2,\mathbb{C}) \to \mathbb{H}^3$ denote the projection. Let Vec be the space of C^∞-vector-fields on $SL(2,\mathbb{C})$ which are invariant under the left $SU(2)$-action. Clearly we have the well-defined projection $P_* : Vec \to \mathrm{H}^0(T\mathbb{H}^3)$ to the space of vector-fields on the hyperbolic 3-space. This projection commutes with the (right) action of the group $SL(2,\mathbb{C})$ on vector-fields. Note that Vec is invariant under the multiplication by complex numbers (since the left action of $SU(2)$ on $T(SL(2,\mathbb{C}))$ is $\mathbb{C}$-linear). We shall identify the Lie algebra $sl(2,\mathbb{C})$ of $SL(2,\mathbb{C})$ with the subalgebra of left-invariant vector-fields on $SL(2,\mathbb{C})$.

The following simple lemma is critical for proving quasiconformality of the vector-field ξ that we will construct in Theorem 5.2 (Assertion (c)). Note that Lemma clearly fails without the assumption $|\lambda| = 1$.

Lemma 5.1 *Let $\tilde{\xi} \in Vec$, $U \subset \mathbb{H}^3$ is a metric ball, $\tilde{U} = P^{-1}(U)$. Then there exists a constant $C = C(\xi, U)$ such that:*

$$\|S[P_*(\lambda \cdot \tilde{\xi}|_{\tilde{U}})]\| \leq C$$

for all $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

Proof: The family of vector-fields $\lambda \cdot \tilde{\xi}|_{\tilde{U}}$, $|\lambda| = 1$, is compact (with respect to the C^1-topology of uniform convergence). Thus the assertion follows from continuity of the S-operator. □

We consider the action of the group $G = \pi_1(R) \rtimes \langle t \rangle$ on the hyperbolic 3-space as in the Introduction. This action extends to the (right) action of G on $SL(2, \mathbb{C})$. Our main goal is to prove the following

Theorem 5.2 *Suppose that $\sigma \in Z^1(\pi_1(R), sl(2, \mathbb{C}))$ is such that $\phi^\#(\sigma) = \lambda\sigma$, $\lambda \in \mathbb{C}^*$. Then there is a smooth vector-field ξ on $\mathbb{H}^3$ and its lift $\tilde{\xi} \in Vec$ to the group $SL(2, \mathbb{C})$ such that:*

(a) $ad(\gamma)\xi - \xi = \sigma(\gamma)$ for all $\gamma \in \pi_1(R)$.
(b) $ad(t)\tilde{\xi} = \lambda^{-1}\tilde{\xi}$.
(c) The vector-field ξ is quasiconformal.

Proof: To construct the vector-field ξ we will need *a partition of unity $\tilde{\eta}$* on the space $SL(2, \mathbb{C})$ corresponding to the subgroup $\pi_1(R)$ and invariant under the action of t:

Lemma 5.3 *There exists a positive bounded C^∞-function $\tilde{\eta}$ on $SL(2, \mathbb{C})$ which satisfies the following properties:*

(1) *$\tilde{\eta}$ is invariant under the right action of t and the left action of $SU(2)$.*
(2) *$\sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) = 1$ for all $g \in SL(2, \mathbb{C})$.*

Proof: Our proof follows [Kr, Chapter V, Lemma 3.1]. Take a finite covering $\{D_j\}$ of the manifold $SU(2)\backslash SL(2, \mathbb{C})/G$ by open metric balls D_j which satisfy the condition:

The radii of the balls D_j are smaller than the injectivity radius of $\mathbb{H}^3/G$.

Let m denote the multiplicity of the covering $\{D_j\}$ (i.e. the maximal number of balls which have nonempty intersection). The group G acts on the lift of this covering to $\mathbb{H}^3$ with trivial stabilizers. Let $\{\bar{\eta}_j\}$ denote the partition of unity on $\mathbb{H}^3/G$ corresponding to the covering $\{D_j\}$. For each D_j choose a connected component V_j of its lift to $\mathbb{H}^3$ and let η_j be the lift of $\bar{\eta}_j$ to V_j. Extend the function η_j to $\mathbb{H}^3$ by:

$$\eta_j(x) = \begin{cases} \eta_j(t^n x) & \text{if } x \in t^{-n}(V_j) \\ 0 & \text{otherwise} \end{cases}$$

Finally let

$$\eta(x) = \sum_j \eta_j(x) \quad .$$

It is clear that η is invariant under the action of t, $\sum_{\gamma \in \pi_1(R)} \eta(\gamma(x)) = 1$ for all x and $0 \leq \eta \leq 1$. Then we lift the function η to $\tilde{\eta} = \eta \circ P : SL(2, \mathbb{C}) \to \mathbb{R}$. □

Suppose that $\sigma \in Z^1(\pi_1(R), sl(2, \mathbb{C}))$ is such that $\phi^\#(\sigma) = \lambda\sigma$, $\lambda \in \mathbb{C}^*$ (at this stage we do not need the assumption $|\lambda| = 1$).

Lemma 5.4 *Under the assumptions above there exists a vector-field* $\tilde{\xi} \in Vec$ *which satisfies the following properties:*

(i) $t_* \tilde{\xi} = \lambda^{-1} \tilde{\xi}$;

(ii) $\alpha_* \tilde{\xi} - \tilde{\xi} = \sigma_\alpha$, *for all* $\alpha \in \pi_1(R)$.

Proof: Our proof again follows [Kr, Chapter V, Theorem 3.2] (see [KM] for more general constructions). Let

$$\tilde{\xi}(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot \sigma_\gamma(g)$$

(Here $\sigma_\gamma(g)$ is a tangent vector to $SL(2, \mathbb{C})$ at g, see Remark 2.2.) Note that for every $g \in SL(2, \mathbb{C})$ not more than m terms of this series are different from zero (where m is the multiplicity of the covering $\{D_j\}$). Hence the infinite series which we use to define $\tilde{\xi}$ is convergent to a smooth vector-field on $SL(2, \mathbb{C})$. To show that $\tilde{\xi}$ is invariant under the (left) $SU(2)$-action we note that the elements σ_γ of the Lie algebra are left-invariant under $SL(2, \mathbb{C})$ and the function $\tilde{\eta}$ on $SL(2, \mathbb{C})$ is invariant under the left action of $SU(2)$.

Now we verify (i). We have:

$$\tilde{\xi}(g) = - \sum_{\phi(\gamma) \in \pi_1(R)} \tilde{\eta}(g\phi(\gamma)) \cdot \sigma_{\phi(\gamma)}(g)$$

$$t_*^{-1} \tilde{\xi}(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot t_*^{-1} \sigma_{\phi(\gamma)}(g)$$

since $\tilde{\eta}$ is t-invariant.

$$- \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot t_*^{-1} \sigma_{\phi(\gamma)}(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot (\phi^{\#} \sigma)_\gamma(g)$$

The latter equals

$$-\lambda \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\gamma) \cdot \sigma_\gamma(g) = \lambda \tilde{\xi}(g)$$

which proves (i). The second assertion of Lemma again follows from the direct computation:

$$(\alpha_* \tilde{\xi})(g) = - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\alpha\gamma)(\alpha_* \sigma_\gamma)(g) =$$

and hence, by the cocycle condition $\sigma_{\gamma \circ \alpha} - \sigma_\alpha = \alpha_* \sigma_\gamma$,

$$= \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\alpha\gamma) \sigma_\alpha(g) - \sum_{\gamma \in \pi_1(R)} \tilde{\eta}(g\alpha\gamma) \sigma_{\gamma \circ \alpha}(g) = \sigma_\alpha(g) + \tilde{\xi}(g) \quad . \square$$

Thus $\xi := P_*(\tilde{\xi})$ is a smooth vector-field on $\mathbb{H}^3$, where $\tilde{\xi}$ is the vector-field constructed in the previous lemma.

Lemma 5.5 *The field* ξ *satisfies the following properties:*

(i) $\alpha_* \xi - \xi = \sigma_\alpha$ *for all* $\alpha \in \pi_1(R)$.

(ii) $\|S[\xi(x)]\| \leq Const$ *for all* $x \in \mathbb{H}^3$.

Proof: The first assertion follows from the statement (ii) of Lemma 5.4. Let's prove the second assertion. Recall that the quotient $\mathbb{H}^3/G$ is compact, pick a compact $K \subset \mathbb{H}^3$ which is a fundamental domain for the action of G. Let $C_1 = \max \|S[\xi](x)\|, x \in K$. The equality $\gamma_*\xi - \xi = \sigma_\gamma \in sl(2,\mathbb{C})$ implies:

$$S[\gamma_*\xi] = S[\xi]$$

for all $\gamma \in \pi_1(R)$. Hence it follows from the equality (2) in Section 3 that

$$\|S[\xi]\| = \|S[\xi] \circ \gamma\|$$

and $\|S[\xi]\|$ is bounded by C_1 along the $\pi_1(R)$-orbit of K. Similarly,

$$\|S[\xi](t^n x)\| = \|S[P_*(\lambda^{-n}\tilde{\xi})](x)\|, \ n \in \mathbb{Z},$$

thus Lemma 5.1 implies that $\|S[\xi](t^n x)\| \leq C_2 = C(\xi, K)$ for all $n \in \mathbb{Z}$. Therefore $\|S[\xi(x)]\| \leq Const = \max(C_1, C_2)$ for all $x \in \mathbb{H}^3$. □

This concludes the proof of Theorem 5.2. □

Now we can start proving Theorem 1.3. We assume that ϕ^* has an eigenvalue λ such that $|\lambda| = 1$, let $\tau \in \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C})) - \{0\}$ be an eigenvector: $\phi^*(\tau) = \lambda\tau$. As we proved in the Introduction, $\lambda \neq 1$.

Lemma 5.6 *Under the conditions above there exists a cocycle* $\sigma \in Z^1(\pi_1(R), sl(2,\mathbb{C}))$ *such that* $\tau = [\sigma]$ *and* $\phi^\#(\sigma) = \lambda\sigma$.

Proof: Choose bases in $\mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$ and $B^1(\pi_1(R), sl(2,\mathbb{C}))$; the automorphism $\phi^* : \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C})) \to \mathrm{H}^1(\pi_1(R), sl(2,\mathbb{C}))$ is represented by a matrix D. Hence, $\phi^\# : Z^1(\pi_1(R), sl(2,\mathbb{C})) \to Z^1(\pi_1(R), sl(2,\mathbb{C}))$ is represented by a matrix:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

Note that A is the matrix of the action of $\phi^\#$ on the complex 3-dimensional space $B^1(\pi_1(R), sl(2,\mathbb{C}))$. The element $t \in SL(2,\mathbb{C})$ is loxodromic, the eigenvalues of its adjoint representation are: $1, \mu^2, \mu^{-2}$, where $|\mu| \neq 1$, hence none of these numbers equals λ. The spectrum of the action of ϕ on $B^1(\pi_1(R), sl(2,\mathbb{C}))$ is the same as the spectrum of the adjoint representation of t^{-1} (since the subgroup $\pi_1(R) \subset SL(2,\mathbb{C})$ is Zariski dense). Now existence of the eigenvector σ follows from the linear algebra. □

We apply Theorem 5.2 to the cocycle σ (given by Lemma 5.6) and construct a quasiconformal vector-field ξ on $\mathbb{H}^3$ such that

$$ad(\gamma)\xi - \xi = \sigma_\gamma \ , \quad \text{for all} \ \ \gamma \in \pi_1(R) \ .$$

Hence by Theorem 4.8 the vector-field ξ admits a tangential extension $\zeta = \xi_\infty$ to $\mathbb{S}^2 = \partial B$, which is a quasiconformal tangent vector-field on $\mathbb{S}^2$ such that

$$\gamma_*\zeta - \zeta = \sigma_\gamma \ , \ \text{for all} \ \gamma \in \pi_1(R) \tag{4}$$

(see Proposition 4.9). Recall that $S[\chi] = 0$ for any conformal vector-field χ. Thus, by applying the operator S to the both sides of the equation (4), we get:

$$\gamma_* S[\zeta] - S[\zeta] = 0 \ , \quad \text{for all} \ \ \gamma \in \pi_1(R) \ .$$

The matrix-valued function $S[\zeta]$ is measurable and belongs to $L_\infty(\overline{\mathbb{C}})$. Let L_S be the subset of $\overline{\mathbb{C}}$ where $S[\zeta] \neq 0$. This subset is measurable, $\pi_1(R)$-invariant and the field $S[\zeta]$ determines a nontrivial $\pi_1(R)$-invariant measurable splitting of the tangent bundle TL_S (see Lemma 3.2). Now we can apply

Theorem 5.7 *(Sullivan's rigidity theorem, [Su], see also [O].) Suppose that $F \subset PSL(2, \mathbb{C})$ is a discrete finitely-generated group whose limit set is $\overline{\mathbb{C}}$ and $L \subset \overline{\mathbb{C}}$ is an F-invariant measurable subset of nonzero measure. Then there are no F-invariant measurable line-fields defined on L.*

We let $F = \pi_1(R)$, then Sullivan's theorem implies that the set L_S has zero measure. This means that $S[\zeta] = 0$ almost everywhere and, according to Lemma 3.3, ζ is a Moebius vector-field : $\zeta \in mob_2 = sl(2, \mathbb{C})$. We conclude that the cocycle σ is a coboundary:

$$\delta_\zeta(\gamma) = ad(\gamma)\zeta - \zeta = \sigma_\gamma$$

which contradicts our assumption that the class $[\sigma] = \tau \in \mathrm{H}^1(\pi_1(R), sl(2, \mathbb{C}))$ is nontrivial. This proves Theorem 1.3. □

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Michael Kapovich: Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA ; kapovich@math.utah.edu