Eisenstein Series and Dehn Surgery

M. Kapovich

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1. Introduction.

1.1. In spite of its algebraic contents this paper was motivated by a purely geometric question. Let $N$ be a hyperbolic 3-manifold of finite volume with several cusp ends $E_1, \ldots, E_p$. Consider a "Dehn filling" (or "generalized Dehn surgery") of one of these cusps (say $E_1$) which is just a compactification of $E_1$ by adding a circle. It was proved by W.Thurston [Thu] that for all but a finite number of parameters of the Dehn filling the resulting manifolds possess complete hyperbolic structures of finite volume.

**Question 1.** What happens with the geometry of other cusps $E_2, \ldots, E_p$ after the Dehn filling? More restrictively, do the conformal moduli, or rather the Teichmuller parameters, of these cusp tori remain the same?

A positive answer for Question 1 was given as a remark in the thesis of C.Hodgson [Ho]; however, his proof contains a fatal gap. The answer is more complicated and depends on the topology of $N$. In principle, the problem can be solved in terms of ideal triangulations by formulas ([Thu, Chapter 4], [NZ]); particular manifolds can be considered in this way. W.Neumann and A.Reid [NR] constructed examples of arithmetic manifolds with 2 cusps for which the answer is positive. In the general case the answer depends on the topology of the given manifold.

Basically there are two sources of these examples [NR]. First let $N_1$ be a hyperbolic 3-orbifold of finite volume which has two cusps $E_1 \cong E_2 \cong \mathbb{T}^2 \times [0, \infty)$ and suppose that there exists a rigid
hyperbolic 2-dimensional suborbifold \( O \subset N_1 \) dividing \( E_1 \) from \( E_2 \). Another example starts with a hyperbolic 3-orbifold \( N_2 \) of finite volume which has two cusps \( E_1 \equiv O_1 \times [0, \infty) \), \( E_2 \equiv O_2 \times [0, \infty) \) where \( O_1 \) is a nonrigid Euclidean orbifold while \( O_2 \) is rigid. In the both cases any Dehn surgery on one cusp has no influence on the geometry of another one. The tricky question is to find manifold covers over \( N_1 \), \( N_2 \) which still have only two cusps (see [NR]).

In this paper we use the machinery of the Eisenstein series to attack the question on the "infinitesimal level". The idea of this approach is explained below.

1.2. Let \( N = \mathbb{H}^3/\Gamma \), where \( \Gamma \subset \text{PSL}_2(\mathbb{C}) \). Let \( A_1, A_2, \ldots, A_p \) denote the maximal parabolic subgroups of \( \Gamma \) corresponding to the cusps \( E_1, E_2, \ldots, E_p \). The hyperbolic Dehn filling of the cusp \( E_1 \) corresponds to the deformation of the holonomy representation \( \rho_0: \pi_1(N) \to \Gamma \subset \text{PSL}_2(\mathbb{C}) \) in the variety

\[
R(\Gamma) = \text{Hom}(\Gamma)/\text{ad}(\text{PSL}_2(\mathbb{C}))
\]

which preserves traces of elements of the groups \( A_2, \ldots, A_p \). Denote by \( u_j \) the real Lie algebra of the group \( U_j \) which is the maximal unipotent subgroup of \( \text{PSL}_2(\mathbb{C}) \) such that \( A_j \subset U_j \). The tangent space to \( R(A_j) \) at the point \([id]\) is isomorphic to \( H^1(A_j, \rho \text{ad}_2(\mathbb{C})_{\text{Ad}}) \cong H^1(u_j, \rho \text{ad}_2(\mathbb{C})_{\text{Ad}}) \cong \mathbb{C}^2 \).

The split torus \( \mathbb{T}_j \) which normalizes the group \( U_j \) acts on \( H^1(u_j, \rho \text{ad}_2(\mathbb{C})_{\text{Ad}}) \cong \mathbb{C}^2 \) via two characters \( \xi_1, \xi_2 \) of the weights \((-4)\) and \(0\), respectively. The eigenspaces of these characters are \( W_1, W_2 \). The space \( W_2 \) is tangent to the deformations of \( A_j \) which preserve traces of elements. On the infinitesimal level we have

**Question 2.** Let \( \omega \in H^1(\Gamma, \rho \text{ad}_2(\mathbb{C})_{\text{Ad}}) \) be such that the projection \( \omega_{1,1} \) of \( \omega \) on \( W_{1,1} \) doesn't vanish while \( \omega_{j,1} = 0 \) for
every $j \neq 1$. Is $\omega_{j,2}$ equal to zero for each $j \neq 1$?

Following [NR], if $\omega_{j,2} = 0$ then we call the cusp $E_j$ infinitesimally isolated from $E_1$.

**Remark 1.** The deformations of $[\rho_0]$ corresponding to Dehn fillings of the cusp $E_1$ (while other cusps remain to be complete) lie in the variety $R(\Gamma \mid E_2, \ldots, E_p) = \{ \rho : \Gamma \to SL_2(\mathbb{C}) : \text{Tr}(\rho(\alpha_j)) = \text{Tr}(\rho_0(\alpha_j)) , j = 2, \ldots, p \} / SL_2(\mathbb{C})$.

By transversality this variety is a smooth submanifold of $R(\Gamma)$ and the class $\omega$ above belongs to its tangent space. Thus a negative answer to Question 2 implies a negative answer to Question 1.

The Question 2 can be treated by the following analytic tools, extracted from [Ha 1].

Denote by $E$ the space $\rho_{\mathfrak{sl}_2}(\mathbb{C})$ and let $\text{Ad}: \text{Isom}(X = \mathbb{H}^3) \to GL(E)$ be the adjoint representation. Define the complex of automorphic forms $A^q(\Gamma, X, \text{Ad}) = \{ \omega \in \Omega^q(X, E) : \text{Ad}(\gamma) \circ R_\gamma \omega = \omega \text{ for every } \gamma \in \Gamma \}$, where $R_\gamma$ is the right action and $d: A^q(\Gamma, X, \text{Ad}) \to A^{q+1}(\Gamma, X, \text{Ad})$ is the exterior differential. Put $T_\gamma = \text{Ad}(\gamma) \circ R_\gamma$. Suppose that the stabilizer of the point $z \in \partial \mathbb{H}^3$ in the group $\Gamma$ is the parabolic subgroup $\Gamma_\infty = A_1$. Denote by $X(1)$ the horosphere in $\mathbb{H}^3$ such that $X(1) = \{ (x_1, x_2, 0) : (x_1, x_2) \in \mathbb{R}^2 \}$. Now let $[\varphi] \in H^1(A^*(X(1), \Gamma_\infty, \text{Ad}) \mid \Gamma_\infty \Gamma_\infty) \cong H^1(\Gamma_\infty, \text{Ad} \mid \Gamma_\infty \Gamma_\infty)$.

Our first aim is to extend $\varphi$ to a differential form $\tilde{\varphi}$ on the whole hyperbolic space $X$.

(a) Consider the vertical unit vectorfield $\tau(z)$ on $X(1) \ni z$. Let $\tilde{\varphi}(\tau(z)) = 0$. Suppose that $\text{Ad}(u) \circ R_u \varphi = \varphi$ for every $u \in U_1$.

(b) Pick $s \in \mathbb{C}$. For $t \in \mathbb{C}^*$ we put
\[ \Lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \text{SL}(2, \mathbb{C}). \text{ Then the formula} \]

\[ \text{Ad}(\Lambda(t)) \circ \varphi_S(z \Lambda(t)) = \tilde{\varphi}(z) t^{-1-s} \]

, where \( \Lambda(t) \) acts on \( X \ni z \in X(1) \) in the usual way, defines the extension \( \varphi_S \) of the form \( \tilde{\varphi} \) to the hyperbolic space. However, \( \varphi_S \) is not \( \Gamma \)-automorphic. Then the Eisenstein series

\[ (c) \quad E(\varphi_S; x) = E(\varphi, s; x) = \sum_{\gamma \in \Gamma \setminus \Gamma} (T_\gamma \varphi_S)(x) \]

is the automorphic extension of \( \varphi \). This form-valued function admits the meromorphic continuation \( E(\varphi, s; x) \) to the whole complex plane \( \mathbb{C} \ni s \). However, the form \( E(\varphi, s; x) \) is closed only for two values of \( s \).

The torus \( \mathfrak{T} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}^*_+ \right\} \) acts on \( H^1(\Gamma\infty, \text{Ad}|_{\Gamma\infty}) \) via the adjoint representation with the weights \( 0 \) and \( (-4) \). The eigenspace \( H^1_{(-4)}(\Gamma\infty, \text{Ad}|_{\Gamma\infty}) \) is tangent to the "loxodromic" deformations of the group \( \Gamma\infty \) in \( \text{PSL}(2, \mathbb{C}) \), i.e. the images of \( \Gamma\infty \) under these deformations are semisimple subgroups of \( \text{PSL}(2, \mathbb{C}) \).

It follows from [Ha 2] that for \( [\varphi] \in H^1_{(-4)}(\Gamma\infty, \text{Ad}|_{\Gamma\infty}) \) we have:

(i) the form-valued function \( s \mapsto E(\varphi, s; \lambda) \) is holomorphic for \( s = 3 \);

(ii) \( dE(\varphi, 3; x) = 0 \);

(iii) the cohomology class \( [E(\varphi, 3; x)] \) is not trivial and its restriction to \( U_j/A_j \) is equal to

\[ \left[ \int_{U_j/A_j} T_u E(\varphi, 3; x) du \right]. \]
Now we are interested in the restrictions of \( E(\varphi, 3; x) \) to the cusps \( A_j \) of the group \( \Gamma \). They are given by the operator

\[
C : \varphi \mapsto (\ldots, \int_{U_j/A_j} T_u E(\varphi, 3; x) du, \ldots) \in \bigoplus_{j=1}^p H^1(A_j, \text{Ad}
|_{A_j})
\]

We can consider \( C \) as the endomorphism of

\[
\bigoplus_{j=1}^p H^1(A_j, \text{Ad}
|_{A_j})
\]

if we put \( \varphi = (\varphi, 0, \ldots, 0) \in \bigoplus_{j=1}^p H^1(A_j, \text{Ad}
|_{A_j}). \)

Moreover, we have:

(a) \( j \)-th component \( C(\varphi)_j \) belongs to the eigenspace \( H^1_{(0)}(A_j, \text{Ad}) \) for \( j \neq 1 \);

(b) \( C(\varphi)_1 = \varphi + \psi \), where \( \psi \in H^1_{(0)}(A_1, \text{Ad}) \).

Thus, the cocycle \( E(\varphi, 3; x) \) gives the desired tangent vector to the Dehn fillings of the cusp \( A_1 \). To answer on the Question 2 we have to decide whether the elements

\[
(*) \quad \left[ \int_{U_j/A_j} T_u E(\varphi, 3; x) du \right]
\]

of the groups \( H^1(A_j, \text{Ad}
|_{A_j}) \) are nontrivial.

Unfortunately, the computation of this integral in the general case is very nontrivial problem. If we restrict ourselves to torsion free congruence subgroups \( \Gamma \) of Bianchi groups then the result can be obtained in terms of special values of some \( L \)-functions. In spite of the transcendental nature of the \( L \)-functions, it is easier to prove that they do not vanish than to calculate the integral (\( \ast \)) directly. Namely, in Theorem 1 we show that the natural restriction maps
\[ \text{res}_j : H^1(\Gamma, \text{Ad}) \to H^1(\mathcal{A}_j, \text{Ad}) \]

are \textbf{onto}. This means that for every cusp \( E_j \) we can find another cusp \( E_{k(j)} \neq E_j \) such that the variation of the modulus of \( E_j \) is nontrivial after the Dehn filling of the cusp \( E_{k(j)} \), i.e. \( E_j \) isn't infinitesimally isolated form \( E_{k(j)} \). Theorem 2 claims that Dehn fillings of every cusp \( E_j \) give nontrivial variation of another cusp \( E_{k(j)} \neq E_j \) (probably for all other cusps variation is nontrivial too). Application of the transfer extends the statements of Theorems 1 and 2 to all finite-index subgroups in \( \Gamma \).

1.3. There are several other interesting problems related to the main question of the present paper; they are concerned with the Kleinian subgroups of \( \text{PSL}(2, \mathbb{C}) \). For example:

Let \( N \) be a complete hyperbolic 3-manifold of finite volume with one cusp and suppose that \( \partial N \) is a totally geodesic surface \( S \) (which can be disconnected). Consider the manifolds \( N_{p/q} \) which results from \( N \) by Dehn filling of the cusp. Then, according to Thurston, for all but finite \( p/q \) the manifolds \( N_{p/q} \) admit complete hyperbolic structures with totally geodesic boundary. Denote the new hyperbolic boundary surfaces by \( S_{p/q} \). The question is:

\textbf{Question 3.} Are the surfaces \( S_{p/q} \) isometric to \( S \)?

This question was considered by M. Fujii in [F]. He constructed an example of \( N \) for which the variation of Teichmuller parameters of \( S \) is nontrivial even on the infinitesimal level. Another example of nontrivial variation can be found in section 5 of present paper. Examples of trivial variations of \( S \) are constructed in [NR]. Question 3 can also be treated via Eisenstein series. In this context, it should be noted that the Eisenstein cohomology classes for Kleinian subgroups of \( \text{PSL}_2(\mathbb{C}) \) were
considered by I.Kra [Kr] from the analytical point of view.

**Conjecture.** Let $\Gamma \subset \text{PSL}_2(\mathbb{C})$ be a lattice and $A \subset \Gamma$ be an arbitrary maximal abelian subgroup. There exists a subgroup of finite index $\Gamma_0 \subset \Gamma$ such that

$$\text{res}_A: H^1(\Gamma_0, \text{Ad}) \to H^1(A \cap \Gamma_0, \text{Ad})$$

is onto.

Theorem 1 establishes this conjecture for arithmetic groups $\Gamma$ when $A$ is unipotent.

1.4. Except for section 5 we shall closely follow the lines of the articles of G.Harder [Ha 1-4]. In this sense the current paper is just a commentary to [Ha 1-4]. Our problem was to make explicit the topological meaning of the articles of G.Harder.

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2. Notations and formulation of main theorem.

Consider the totally imaginary number field $F= \mathbb{Q}(\sqrt{-m})$, where $m$ is a positive square-free integer. Let $\mathcal{O}= \mathcal{O}_m$ be the ring of integers of $F$, $\mathfrak{A} < \mathcal{O}$ be an integer ideal. Then the group

$$\Gamma(\mathfrak{A}) = \{ \gamma \in \text{GL}_2(\mathcal{O}) : \gamma \equiv 1 \pmod{\mathfrak{A}} \}$$

is a congruence subgroup of the level $\mathfrak{A}$. Let $G_0/F = \text{GL}_2/F ; B_0/F$ be the upper-triangular Borel
subgroup of $GL_2/F$. The groups $G/\mathcal{O}$, $B/\mathcal{O}$ and so on result from $G_0$, $B_0$ etc. by the restriction of scalars. For any subgroup $H \subset G$ put $H^{(1)} = \{ h \in H : \det(h) = 1 \}$. For ring $L$ we denote by $L^\times$ its group of units (invertible elements). For any subgroup $H_0 \subset G_0$ and ring $L \supset \mathcal{O}$ we put $H(L) = H_0(L \otimes \mathcal{O}F)$. Any parabolic subgroup $B_j(\mathbb{A})$ of $\Gamma(\mathbb{A})$ has the type $B_j(\mathbb{A}) = B(\mathcal{O}) \cap b_j \Gamma(\mathbb{A}) b_j^{-1}$ where $b_j$ belongs to $B(\mathcal{O})$. Let $Ad = \rho_0 : G_0 \to GL(M_0)$ be the adjoint representation, where $M_0 = \rho gl_2(F)$. Denote by $\Delta = \Gamma(\mathbb{A}) \cap (Z(G_0) \cong \mathcal{O}^\times)$ the center of the group $\Gamma(\mathbb{A})$.

**THEOREM 1.** Suppose that $\Gamma(\mathbb{A})/\Delta$ is torsion-free. Then the natural restriction maps

$$res_{j,0}(\mathbb{A}) : H^1(\Gamma(\mathbb{A}), M_0) \longrightarrow H^1(B_j(\mathbb{A}), M_0)$$

are epimorphisms.

3. Preliminary results for proof of theorem 1.

3.1. We shall use constructions, notations and statements from the papers of G. Harder [Ha 2-4]. In this section we introduce the following notations.

For nonarchimedean valuations $\psi$ of $F$ put $K_\psi(\mathcal{O}) = GL_2(\mathcal{O}_\psi)$ and $K_\psi = K_\psi(\mathbb{A}) = \{ \gamma \in GL_2(\mathcal{O}_\psi) \mid \gamma \equiv 1 \pmod{\mathbb{A}_\psi} \}$; for the infinite place we put $G_\infty = G(\mathbb{R}) \cong GL_2(\mathbb{C})/\mathbb{R}$; $K_\infty = U(2) \cdot \mathbb{C}^*$, where $\mathbb{C}^*$ is the center of $G_\infty$; then $K_f = K_f(\mathbb{A}) = \prod_{\psi \text{ is finite}} K_\psi$, $K = K_\infty \cdot K_f$.

**Remark 2.** The group $K_f(\mathbb{A})$ is normal in $K_f(\mathcal{O})$.

Consider the rings of adeles $\mathbb{A}_F$, $\mathbb{A} = \mathbb{A}_\mathcal{O} = \mathbb{A}_{\mathcal{O},\infty} \times \mathbb{A}_f$ and the groups of ideles $I_F = \mathbb{I}_{F,\infty} \times \mathbb{I}_f$, $I_\mathcal{O} = \mathbb{I}_{\mathcal{O},\infty} \times \mathbb{I}_{\mathcal{O},f}$ for the fields $F$, $\mathcal{O}$.

Put $G = G \times \mathcal{O}$ and extend the representation $\rho_0$ to the
representation \( \rho : G \to GL(M' = M_0 \otimes_\mathbb{Q} \mathbb{Q}) \) which splits as

\[
\rho : G = GL_2 / \mathbb{Q} \to GL(M) \otimes GL(M') = 1_{\mathbb{T}} \otimes Ad_{\mathbb{T}} \otimes Ad_{\mathbb{T}} \otimes 1_{\mathbb{T}} \to \mathbb{Q}
\]

\( = \rho \otimes \rho' \). Here \( 1_{\mathbb{Q}} , Ad_{\mathbb{Q}} \) are trivial and adjoint representations; \( \sigma \) is an embedding of \( F \) in \( \mathbb{Q} \). It suffices to prove that the natural restriction map

\[
res_j(\mathbb{R}) : H^1(\Gamma(\mathbb{R})/\Delta , M) \to H^1(\mathbb{B}_j(\mathbb{R})/\Delta , M)
\]

is surjective. The modulus \( M \) decomposes as \( 1_{\mathbb{T}} \otimes Ad = M_0 \otimes M_0 \), where \( M_0 \) is isomorphic to the \((d_0 + 1)\)-dimensional (over \( \mathbb{Q} \)) vector space of homogeneous polynomials of the degree \( d_0 \) with the action:

\[
\rho_\sigma \left( \begin{array}{ll}
a & b \\
c & d \\
\end{array} \right) : P(X_\sigma , Y_\sigma) \to P(aX_\sigma + cY_\sigma , bX_\sigma + dY_\sigma) \cdot det \left( \begin{array}{ll}
a & b \\
c & d \\
\end{array} \right)^{\nu(\sigma)}
\]

where \( d_\sigma = \begin{cases} 0, \text{ if } \sigma = \mathbb{T} \\
2, \text{ if } \sigma = \mathbb{T} \end{cases} \) and \( \nu_\sigma = \begin{cases} 0, \text{ if } \sigma = \mathbb{T} \\
-1, \text{ if } \sigma = \mathbb{T} \end{cases} \) (cf. [Ha 4]).

3.2. Geometry of associated locally-symmetric space.

Attach to \( \Gamma(\mathbb{R}) \) the locally-symmetric space

\[
S_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K \cong GL_2(F) \backslash GL_2(\mathbb{A}_F) / K
\]

which is not connected, its components are "counted" by the map

\[
det : S_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K \to \mathbb{I}_{F,F} / F^* \cdot det(K_F),
\]

where \( det(x_f^{(j)}) \) are representatives of cosets \( \mathbb{I}_{F,F} / F^* \cdot det(K_F) \) and \( x_f^{(j)} = \begin{pmatrix} \det(x_f^{(j)}) & 0 \\
0 & 1 \\
\end{pmatrix} \). The connected components of the symmetric space \( X_K = G(\mathbb{A}) / K \) are \( \mathbb{H}^3_{(j)} = G \cdot x_f^{(j)}K_F / K \cong \mathbb{H}^3 \).

Suppose that \( \mathbb{H}^3_{(j)} \) is invariant under \( q \in G(\mathbb{Q}) \). Then
\( q \mathcal{L}_f^{(j)} K_f = a_f^{(j)} K_f \), hence \( q \in a_f^{(j)} K_f (a_f^{(j)})^{-1} \cap G(\mathcal{Q}). \) Thus, the individual components of \( S_K \) have the form:

\[
(a_f^{(j)} K_f (a_f^{(j)})^{-1} \cap G(\mathcal{Q})) \setminus \mathcal{H}_{(j)}^3 = \Gamma_{(j)}(\mathcal{A}) \setminus \mathcal{H}_{(j)}^3.
\]

In particular, \( \Gamma(\mathcal{A}) \equiv \Gamma_{(1)}(\mathcal{A}) = \{ \gamma \in G(\mathcal{Q}) \mid \gamma \equiv 1 \pmod{\mathcal{A}} \} \); the subgroup \( \Delta \subset \Gamma_{(j)}(\mathcal{A}) \) acts trivially on \( \mathcal{H}_{(j)}^3 \) and the 1-st component of \( S_K \) is homeomorphic to \( (\Gamma(\mathcal{A})/\Delta) \setminus \mathcal{H}^3 \).

By means of the representation \( \rho \) we define the sheaf \( \tilde{\mathcal{M}} \) of local sections of the fiber bundle over \( S_K \) (with the fiber \( M \)). Then

\[
H^*(\Gamma(\mathcal{A})/\Delta \setminus \mathcal{H}^3, \tilde{\mathcal{M}}) \cong H^*(\Gamma(\mathcal{A})/\Delta, M).
\]

3.3. **Geometry of ideal boundary of \( S_K \).**

Denote by \( \overline{\partial S_K} \) the compactification of \( S_K \) such that \( \partial \overline{S_K} \) is the collection of tori.

**LEMMA 1.** (i) The space \( \partial S_K \) is homotopy equivalent to \( B(\mathcal{Q}) \setminus G(\mathcal{A})/K. \) (ii) The components of \( B(\mathcal{Q}) \setminus G(\mathcal{A})/K \) are fibers of the map \( \det \cdot \rho_f : \)

\[
B(\mathcal{Q}) \setminus G(\mathcal{A})/K \xrightarrow{\rho_f} B(\mathcal{Q}) \setminus G(A_\mathcal{Q})/K_f \xrightarrow{\det} \mathbb{I}_{F,f}/F^* \cdot \det(K_f)
\]

**PROOF.** See [Ha 3, Proposition 3.1]

**CONVENTION 1.** To simplify notations we shall identify \( S_K \) and \( B(\mathcal{Q}) \setminus G(\mathcal{A})/K \); below we shall drop the bar sign for \( \partial S_K \).

Consider the commutative diagram:

\[
\begin{array}{ccc}
B(\mathcal{Q}) \setminus G(\mathcal{A})/K \xrightarrow{r} & B(\mathcal{Q}) \setminus G(A_\mathcal{Q})/K_f & \\
\downarrow q \circ r & & \downarrow q \\
B(A_\mathcal{Q}) \setminus G(A_\mathcal{Q})/K_f & & \end{array}
\]

where \( q, r \) are the natural projections. Put \( \partial_1 S_K = (q \circ r)^{-1}[1] \), then \( \partial_1 S_K \) is equal to \( B(\mathcal{Q}) \setminus B(\mathcal{A})/K^B \cdot K_f^B \) where \( K^B = K_f \cap B_\mathcal{Q} \).
\[ K_f^B = K_f \cap B(\mathbb{A}_f) \] [Ha 3, proof of theorem 1].

Consider the (right) action of \( \text{GL}_2(\mathcal{O}_f) \) on \( \partial S_K \):

\[ [g] \in B(\mathcal{O}) \backslash G(\mathbb{A}) / K_\infty K_f \longmapsto [g \cdot s_f] \], where \( s_f \in \text{GL}_2(\mathcal{O}_f) \);

this action is defined correctly since \( K_f \) is a normal subgroup of \( \text{GL}_2(\mathcal{O}_f) \). Certainly, this action of \( \text{GL}_2(\mathcal{O}_f) \) by right translations is equivariant with respect to \( q \circ r \):

\[
\begin{array}{ccc}
B(\mathcal{O}) \backslash G(\mathbb{A}) / K_\infty K_f & \xrightarrow{S_f} & B(\mathcal{O}) \backslash G(\mathbb{A}) / K_\infty K_f \\
\downarrow q \circ r & & \downarrow q \circ r \\
B(\mathbb{A}_f) \backslash G(\mathbb{A}_f) / K_f & \xrightarrow{S_f} & B(\mathbb{A}_f) \backslash G(\mathbb{A}_f) / K_f
\end{array}
\]

Hence \( \text{GL}_2(\mathcal{O}_f) \) acts by right translations on fibers of the map \( q \circ r \), in particular:

\[
\partial S_K \xrightarrow{S_f} B(\mathcal{O}) \backslash G(\mathbb{A}_f) s_f K_f / K_\infty K_f \cong \frac{B(\mathcal{O}) \cdot B(\mathbb{A}_f) s_f}{K_\infty K_f}.
\]

**REMARK 3.** The decomposition of \( \partial S_K \) on connected components induces the decomposition of \( \partial S_K \). The action of \( \text{SL}_2(\mathcal{O}_f) = G_1(\mathcal{O}_f) \) preserves the determinant; hence the last decomposition is invariant under \( \text{SL}_2(\mathcal{O}_f) \).

**LEMMA 2.** \( G(\mathbb{A}_f) = B(\mathbb{A}_f) \cdot G(\mathcal{O}_f) \).

**PROOF.** \( G(\mathbb{A}_f) = B(\mathbb{A}_f) \cdot \text{SL}_2(\mathbb{A}_f, f) \) = \( B(\mathbb{A}_f) \cdot \text{SL}_2(F) \cdot G(\mathcal{O}_f) \) (by the strong approximation for \( \text{SL}_2 \)). However, \( \text{SL}_2(F) \subset B(\mathcal{O}) \cdot \text{SL}_2(\mathcal{O}) \subset B(\mathbb{A}_f) \cdot G_1(\mathcal{O}_f) \) which implies the assertion of the lemma.

**COROLLARY 1.** (i) The group \( G(\mathcal{O}_f) \) acts transitively on fibers of \( q \circ r \). (ii) The stabilizer of \( \partial S_K \) in the group \( G(\mathcal{O}_f) \) is equal to \( B(\mathcal{O}_f) \cdot K_f \).

**PROOF.** Let \( B(\mathbb{A}_f) g_f K_f = B(\mathbb{A}_f) K_f \), where \( g_f \in G(\mathcal{O}_f) \). Hence \( g_f = b_f k_f \in B(\mathbb{A}_f) \cdot K_f \cap G(\mathcal{O}_f) \). However, \( K_f \subset G(\mathcal{O}_f) \); therefore \( b_f \in G(\mathcal{O}_f) \) and \( g_f \in B(\mathcal{O}_f) \cdot K_f \).
3.4. Geometry of $\partial_1 S_K$.

3.4.1. Consider the fiber bundle

$p: B(\mathcal{O}) \setminus B(\mathcal{A})/ K_\infty^{B,B_f} \to T(\mathcal{O}) \setminus T(\mathcal{A})/ K_\infty^{T,T_f}$, where $K_\infty^T = K_\infty \cap T$, $K_f^T = K_f \cap T(\mathcal{A}_f)$, $T$ is the group of diagonal matrices. The fiber of $p$ is the 2-torus $T^2 = U(\mathcal{O}) \setminus U(\mathcal{A})/ K_f^U = \Gamma_U \setminus U_\infty$, where $\Gamma_U = U(\mathcal{O}) \cap K_f^U$.

$U(\mathcal{O}) = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in F \right\}$, $\Gamma_U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathcal{O}, \lambda \equiv 0 \pmod{2} \right\}$ is the lattice on the complex plane $U_\infty \subseteq \mathbb{C}$. In particular, $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} = \pi_1(\text{component of } \partial_1 S_K)$, hence every component of

$\mathcal{I} = T(\mathcal{O}) \setminus T(\mathcal{A})/ K_\infty^{T,T_f}$ is contractible.

3.4.2. Continue the sheaf $\tilde{\mathcal{M}}$ to the ideal boundary $\partial_1 S_K$.

Define the sheaf $\mathcal{H}^1(T^2, \tilde{\mathcal{M}})$ of local sections of bundle over $\mathcal{I}$ with the fiber $\mathcal{H}^1(T^2, \tilde{\mathcal{M}})$, associated with the action of $T(\mathcal{O})$ on $\mathcal{H}^1(T^2, \tilde{\mathcal{M}})$. Then $\mathcal{H}^1(\partial_1 S_K, \tilde{\mathcal{M}}) \cong H^0(\mathcal{I}, \mathcal{H}^1(T^2, \tilde{\mathcal{M}}))$ (cf. [Ha 4, 2.3]).

3.4.3. $\mathcal{H}^1(T^2, \tilde{\mathcal{M}})$ as $T$-modulus.

Notice that $\mathcal{H}^1(T^2, \tilde{\mathcal{M}}) \cong \mathcal{H}^1(\mathfrak{u}, M) \cong \mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{u}$ is the Lie algebra of the group $U(\mathcal{O})$ [Ha 4]. Then decompose $\mathcal{H}^1(\mathfrak{u}, M)$ as the direct sum of the eigenspaces for the action of the torus $T(\mathcal{O})$:

$\mathcal{H}^1(\mathfrak{u}, M) = \mathcal{H}^1(\mathfrak{u}, M)(\chi_1) \oplus \mathcal{H}^1(\mathfrak{u}, M)(\chi_2) = \mathfrak{g} \chi_1 \oplus \mathfrak{g} \chi_2$,

and the direct calculation [Ha 4] of the characters $\chi_j|_{T(\mathcal{O})}$ gives:

$\chi_1: \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto t_2^{d(\tau)+\nu(\tau)+1} \cdot t_1^{\nu(\tau)-1} \cdot t_1^{-d(\tau)} = t_2^{\nu(\tau)-1} \cdot t_1^{\nu(\tau)} = t_2^{\nu(\tau)-1}$

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\[
\chi_2 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \begin{pmatrix} t_1 \ t_2 \\ - \end{pmatrix} \nu(\tau) \cdot \begin{pmatrix} 1 \\ - \end{pmatrix} \nu(\tau) - 1 \cdot \begin{pmatrix} - t_1 \\ t_2 \end{pmatrix} \nu(\tau) + \nu(\tau) + 1 = \]
\[
\begin{pmatrix} t_1 / t_1 \\ t_2 / t_2 \end{pmatrix} ; \chi_j = \chi_j(t_1) \cdot \chi_j(t_2). \text{ If } T^{(1)} = \left\{ \begin{pmatrix} t \\ 0 \\ t^{-1} \end{pmatrix} : t \in F \right\}
\]
then \[
\chi_j \bigg|_{T^{(1)}} = t^{-4}, \quad \chi_2 \bigg|_{T^{(1)}} = t^{-2}t^2, \quad \chi_j \bigg|_{\mathbb{Z}(G)} = 1 \text{ (cf. the item 1.2)}. \]

**Remark 4.** The characters \( \chi_1 = \lambda_{01}^{-1}, \chi_2 = \lambda_{10}^{-1} \) are balanced and \( \nu(\chi_i) = -4 \) (in sense of G. Harder [Ha 4]).

3.4.4. Fix any isomorphisms \( \Theta_{\chi_j} \approx \Omega \). Notice that \( U(A) = \text{Ker}(p) \) is a normal subgroup in \( B(A) \); hence \( B(A_f) \) acts on \( T(A) \) via right translations and \( U(A) \) lies in the kernel of this action. Therefore, it is possible to identify the action of \( B(A) \) on \( T(A) \) with the action of \( T(A) \) on itself, which drops on \( \mathcal{I} = T(\mathbb{Q}) \setminus T(A)/K^T \).

Now, following [Ha 4] we describe in convenient terms the cohomology \( H^0(T(\mathbb{Q}) \setminus T(A)/K^T_{\mathbb{Q}} \setminus \chi_j) \). Consider the set of Hecke characters \( \Sigma_j = \{ \phi : T(\mathbb{Q}) \setminus T(A)/K^T \to \Omega : \phi = \phi_\infty \cdot \phi_f, \phi_\infty = (\chi_j)^{-1} \} \), in other words, \( \phi \in \Sigma_j \) has the type \( \chi_j \) at infinity. If \( \phi \in \Sigma_j \) then \( \phi_f \big|_{T(\mathbb{Q})} = \chi_j \). The space
\[
H^0(T(\mathbb{Q}) \setminus T(A)/K^T_{\mathbb{Q}} \setminus \chi_j)
\]
is isomorphic to the space of locally constant sections \( e : T(A)/K^T \to \Omega \) such that: \( e(qt) = \chi_j(q) \cdot e(t) \) for every \( q \in T(\mathbb{Q}) \).

Put \( I_f = T(\mathbb{Q}) \setminus T(A_f)/K^T_f \).

**Assertion 1.** Every section \( e(t) \) as above can be described as a linear combination of \( e_\phi(t) = \phi_f(t) \), \( \phi = (\chi_j^{-1}, \phi_f) \in \Sigma_j \).

**Proof.** The space of sections is isomorphic to the space of \( \Omega \)-valued functions \( \Omega[I_f]^\# \) on the finite abelian group \( I_f \). The
space $\mathfrak{g}[I_f]^\#$ is spanned by the set of $\mathfrak{g}$-valued characters of the
group $I_f$. This implies the assertion.

3.4.5. The items 3.4.3 and 3.4.4 imply that

$$H_B^1(\tilde{N}) = H^1(\partial_1 S_K, \tilde{N}) \otimes \mathfrak{g}_\mathfrak{e}_\phi \otimes \mathfrak{g}_\mathfrak{e}_\phi \phi \in \Sigma^1 \phi \in \Sigma^2$$

(3.4.5)

REMARK 5. The group $B^{(1)}(A_f)$ acts transitively on $I_f$.

Now we can define the action of the group $B(A_f)$ on $H^1(\partial S_K, \tilde{N})$.

Namely, the action of $B(A_f)$ on $I$ induces the following action on

$$H^0(I, H^1(\overline{V}^2, \tilde{N})) : R(b_f)e\phi(t) = e\phi(t b_f) = \phi_f(b_f) e\phi(t).$$

So the decomposition (3.4.5) is invariant under the action of $B(A_f)$
and the restrictions of the action to 1-dimensional subspaces $\mathfrak{g}_\mathfrak{e}_\phi$
are defined by the characters $\phi_f : B(A_f) \to \mathfrak{e}^*$, where $\phi_f|U(A_f) = 1$.

3.4.6. The final formula for $H^1(\partial S_K, \tilde{N})$ as $B(A_f)$-module.

Define $\text{Ind}_{B(A_f), K_f} H_B^1(\tilde{N}) = \left\{ \psi : G(A_f) \to H_B^1(\tilde{N}) : \psi(b_f g_f k_f) = R(b_f) \psi(g_f), \text{ for every } b_f \in B(A_f), g_f \in G(A_f), k_f \in K_f \right\}$.

G(O_f)

LEMMA 3. $\text{Ind}_{B(O_f), K_f} H_B^1(\tilde{N})$ is isomorphic to $\text{Ind}_{B(A_f), K_f} H_B^1(\tilde{N})$.

PROOF. Consider the linear restriction map

$$\theta : \text{Ind}_{B(A_f), K_f} H_B^1(\tilde{N}) \to \text{Ind}_{B(O_f), K_f} H_B^1(\tilde{N}), \theta(\psi) = \psi|_{G(O_f)}.$$

(a) Show that $\theta$ is surjective. Recall that $G(A_f) = B(A_f) G(O_f)$. Then for $\psi \in \text{Ind}_{B(A_f), K_f} H_B^1(\tilde{N})$ we put $\tilde{\psi}(g_f) = b_f \cdot o_f = R(b_f) \psi(o_f)$, where $o_f \in G(O_f)$. Check that this definition is correct. Let $g_f = b_f o_f = a_f n_f$, where $a_f \in B_f$, $n_f \in G(O_f)$. Then

$$c_f =$$

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\[ a_f^{-1} b_f = n_f g_f^{-1} \in G(O_f) \cap B(A_f) = B(O_f); \quad n_f = c_f o_f. \] Now \( R(a_f) \psi(n_f) = R(a_f) R(c_f) \psi(o_f) = R(a_f c_f) \psi(o_f) = R(b_f) \psi(o_f). \) Hence \( \tilde{\psi} \) doesn't depend on the decomposition and our definition is correct.

(b) Show that \( \text{Ker}(\theta) = 0 \). Let \( \psi \in \text{Ind}_{B(A_f)}^{G(A_f)} \text{K}_f \text{H}_B^1(\tilde{\mathcal{H}}), \psi|_{G(O_f)} = 0. \) Then \( \psi(g_f = b_f o_f) = R(b_f) \psi(o_f) = 0 \) implies that \( \psi(o_f) = 0. \) ■

Thus we obtained the decomposition:

\[ H^1(\partial S_K, \tilde{\mathcal{H}}) \cong \prod_{\phi \in \Sigma_1 \cup \Sigma_2} \{ \psi: G(A_f) \to \mathbb{Q} \phi \cong \mathbb{Q}: \psi(b_f g_f k_f) = \phi(b_f) \psi(g_f), \text{for every } b_f \in B(A_f), k_f \in K_f \}. \]

A decomposition of such kind is given in [Ha 4, Theorem 1] after transition to the inverse limit by \( K_f \).

**CONVENTION 2.** By the symbol \( e_\phi \) we shall denote the element of \( H^1(\partial S_K, \tilde{\mathcal{H}}) \) which is zero outside \( B(A_f) K_f \) and whose restriction to \( B(A_f) K_f \) is equal to \( e_\phi \).

4. Description of boundary homomorphism after Harder

and proof of main theorem

4.1. Let \( \phi \in \Sigma_1; \omega_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G(\mathfrak{g}), \text{ad}(\omega_0) \) is a nontrivial element of Weil group. Define the character \( |\alpha|: B(\mathbb{A}) \to \mathbb{C}^* \) as

\[ |\alpha|: \begin{pmatrix} t & u \\ 0 & t_2 \end{pmatrix} \to |t_1/t_2| = \prod_{\nu} \| (t_1/t_2)_\nu \|_\nu \] (where \( \nu \) runs through all valuations of \( F \)). Obviously we have

\[ (4.1) \quad |\alpha| \big|_{B(\mathbb{A}) \cap K} = 1 \]

since the matrix elements of \( K_{\psi} \cap T(\mathbb{A})_{\psi} \) are units of \( O_{\psi} \).
Put \( w_0 \cdot \phi = |a| \cdot \phi^w_0 \), where \( \phi^w_0(t) = \phi(w_0tw_0^{-1}) \); therefore:

\[
\phi^w_0 \bigg|_{T^{(1)}(A)} = \phi^{-1} \bigg|_{T^{(1)}(A)}.
\]

Notice that: (a) \( w_0 \cdot \phi \) is a Hecke character. Really, \( w_0 \cdot \phi \big|_{T(\mathbb{Q})} = 1 \) follows from \( |a| \big|_{T(\mathbb{Q})} = 1 \) (see [Lg, p. 85-86]). Next, (4.1) implies that \( |a| \big|_{K_T} = 1 \). (b) \( w_0 \cdot \phi \) has the type \( \chi_2 \) at infinity because:

1. \( \phi_\infty = \phi \bigg|_{T_\infty} = \chi_1^{-1} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto (t_1/t_2)^2; \)

2. \( \text{Ad}(w_0) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}; \)

3. \( |a|_\infty : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \frac{-t_1}{t_2}. \)

So \( |a|_\infty (\chi_1^{-1}) w_0 \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = (t_1/t_2)^2 t_1/t_2 \chi_2^{-1} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \)

and \( w_0 : \phi \in \Sigma_1 \mapsto w_0 \cdot \phi \in \Sigma_2 \). The same arguments show that:

\( w_0 : \phi \in \Sigma_2 \mapsto w_0 \cdot \phi \in \Sigma_1 \).

4.2. The results of (4.1) and (3.4.6) give us the isomorphism:

\[
H^1(\partial S_K, \tilde{M}) \cong \bigoplus_{(\phi, w_0 \cdot \phi)} \left( \text{Ind}_{B(\mathbb{A}_f), K_f}^{G(\mathbb{A}_f)} \otimes e_\phi \otimes \text{Ind}_{B(\mathbb{A}_f), K_f}^{G(\mathbb{A}_f)} \otimes e_{w_0 \cdot \phi} \right)_{\phi \in \Sigma_1}
\]

\[
= \bigoplus_{\phi \in \Sigma_1} V_{\phi} \otimes V_{w_0 \cdot \phi}.
\]

Now decompose \( V_{\phi_f}, V_{w_0 \cdot \phi_f} \) in local factors. We have \( \phi_f(b_{\mathbf{f}}) = \)

\[
\phi_{\mathbf{f}}(b_{\mathbf{f}}), b_{\mathbf{f}} \in B(\mathbf{f}) \) . Put \( B_{\mathbf{y}} = B(\mathbf{f}), G_{\mathbf{y}} = GL_2(\mathbf{f}), K_{\mathbf{y}} \)

is the projection of \( K_f \) in \( G_{\mathbf{y}} \), i.e.
\[ K_\psi = \left\{ \gamma \in \text{GL}_2(F_\psi) : \gamma \equiv 1 \pmod{\mathbb{A}_\psi} \right\}. \] Furthermore: \( V_{\phi_\psi} = \) 
\[ \{ \psi : G_\psi \to \mathbb{C} : \psi(b_\psi g_\psi k_\psi) = \phi_\psi(b_\psi) \psi(g_\psi), \text{ for every } b_\psi \in B_\psi, \quad k_\psi \in K_\psi \} \]

**Assertion 2.** The space \( V_{\phi_\psi} \) is isomorphic to \( V_{\phi_\psi} \) if \( \psi \) is finite \( \phi_\psi \).

**Proof.** For \( \psi \in V_{\phi_\psi} \) we put \( \xi_\psi(\psi) = \psi|_{G_\psi} \).

\[ \xi(\psi) = \int_{\psi \text{ is finite}} \psi \in V_{\phi_\psi} \quad \text{if } \psi \text{ is finite} \]

but finite places; hence the product \( \xi(\psi) \) is convergent.

The map \( \xi \) is surjective and has zero kernel. \( \Box \)

**4.3. Local intertwining operators** \( T_\psi : V_{\phi_\psi} \to V_{\phi_\psi} \).

**4.3.1.** Suppose that the quasicharacter \( \phi^{(1)} = \mu_1 / \mu_2 \) is ramified over \( \psi \), i.e. \( \phi \big|_{T^{(1)}(O_{\psi})} \) is not trivial. Then put:

\[ (4.3.1) \quad T_\psi(\psi_\psi)(g_\psi) = \int_{U(F_\psi)} \psi_\psi(\psi_\psi u g_\psi) du_\psi. \]

where \( \psi_\psi \in V_{\phi_\psi} \); \( U(F_\psi) \) is isomorphic to \( F_\psi \) and \( du_\psi \) is the natural measure on \( U(F_\psi) \) induced by this isomorphism i.e.:

(1) \( du_\psi \) is invariant under translations;

(2) \( \int du_\psi = 1; \quad (3) \text{mes}(\|z\|_\psi \cdot E_\psi) = \|z\|_\psi \cdot \text{mes}(E_\psi) \) for every \( z \in F_\psi \)

and measurable \( E_\psi \subset F_\psi \).

To show that the integral in (4.3.1) is finite we reproduce some calculations from [Ha 2]. Let \( g_\psi \in G(O_{\psi}). \)

\[ T_\psi(\psi_\psi)(g_\psi) = \int_{U(F_\psi)} \psi_\psi(\psi_\psi u g_\psi) du_\psi = \int_{U(O_{\psi})} \psi_\psi(\psi_\psi u g_\psi) du_\psi + \]

\[ \sum_{N=1}^{\infty} \|\pi_\psi^{-N}\|_\psi \cdot \int_{O_{\psi}} \psi_\psi(\psi_\psi u_\psi g_\psi) du_\psi \]

where \( \text{ord}_\psi(\pi_\psi) = 1. \)
The first summand is finite since \( O_\psi \) is compact and \( \psi_\psi \) is continuous. Then consider the infinite sum. Notice that:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & x_\psi \cdot \pi_\psi^{-N} \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 \\
-1 & -x_\psi \cdot \pi_\psi^{-N}
\end{pmatrix}
= 
\begin{pmatrix}
-x_\psi^{-1} \cdot \pi_\psi^N & 1 \\
0 & -x_\psi \cdot \pi_\psi^{-N}
\end{pmatrix}
\begin{pmatrix}
1 \\
x_\psi^{-1} \cdot \pi_\psi^N
\end{pmatrix}
\]

hence \( x_\psi^{-1} \cdot \pi_\psi^N \equiv 0 \pmod{\mathbb{Z}_\psi} \) for all \( N \geq N_0 \) (since \( \mathbb{Z}_\psi \) is a power of \( \pi_\psi \) in \( O_\psi \)). Therefore, for \( N \geq N_0 \) the matrix \( \begin{pmatrix}
1 & 0 \\
x_\psi^{-1} \cdot \pi_\psi^N & 1
\end{pmatrix} = k_\psi \)
belongs to \( K_\psi \). Hence

\[
\sum_{N=N_0}^{\infty} \|\pi_\psi^{-N}\|_\psi \cdot \int_{O_\psi}^x \psi_\psi(w_0, \begin{pmatrix}
1 & x_\psi \cdot \pi_\psi^{-N} \\
0 & 1
\end{pmatrix} g_\psi) \, dx_\psi =
\sum_{N=N_0}^{\infty} \phi_\psi^{(1)}(-\pi_\psi^N) \|\pi_\psi^{-N}\|_\psi \psi_\psi(g_\psi) \cdot \left( \int_{O_\psi}^x \phi_\psi^{(1)}(x_\psi) \, dx_\psi = 0 \right) = 0 \text{ ([Ig]).}
\]

4.3.2. Suppose that the quasicharacter \( \phi_\psi^{(1)} = \mu_1 / \mu_2 \in \Sigma_j \) is unramified over \( \psi \), i.e. \( \phi \big|_{T_\psi(t)} \) is trivial. Decompose \( \phi_\psi \) as

\[
\phi_\psi : \begin{pmatrix}
t_1 & 0 \\
0 & t_2
\end{pmatrix} \mapsto \mu_1\psi(t_1) \cdot \mu_2\psi(t_2).
\]

For \( G_\psi \ni g_\psi = b_\psi k_\psi \in B(F_\psi) \cdot GL_2(O_\psi) \) we put:

\[
\psi_\psi^{(1)}(g_\psi) = \phi_\psi^{(1)}(b_\psi k_\psi) = \phi_\psi(b_\psi) \mu_1\psi(\det(k_\psi))
\]

\( \psi_\psi^{(1)} \) is a "spherical" function. Show that this definition is correct. Suppose that \( b_\psi k_\psi = b_\psi k_\psi \). Then (1) \( \mu_1\psi(\varepsilon) = \mu_2\psi(\varepsilon) \)
for every \( \varepsilon \in \mathcal{O}^X \): (2) \( k_\psi = k'_\psi \tilde{k}_\psi \), where \( k'_\psi = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \), \( \tilde{b}_\psi = b_\psi k'_\psi \).

Hence \( \mu_{1\psi}(\det(k'_\psi)) = \mu_{1\psi}(k_1) \cdot \mu_{2\psi}(k_2) \); \( \phi_{1\psi}(b_\psi) \mu_{1\psi}(\det(k'_\psi)) = \phi_{1\psi}(b_\psi) \mu_{1\psi}(k_1) \cdot \mu_{2\psi}(k_2) \mu_{1\psi}(\det(k'_\psi)) = \phi_{1\psi}(\tilde{b}_\psi) \mu_{1\psi}(\det(k'_\psi)) \). Therefore the value of \( \psi_{0,\psi}^{(1)} \) does not depend on the decomposition of \( g_\psi \). Notice that \( |\alpha|_\psi \) is unramified, hence \( w_0 \cdot \phi_{1\psi}^{(1)} \) and \( \phi_{1\psi}^{(1)} \) are unramified simultaneously. Now put:

\[
T_\psi : \psi_{0,\psi}^{(1)} \in V_{\phi_\psi} \mapsto \psi_{0,\psi}^{(2)} \in V_{w_0 \cdot \phi_\psi}
\]

This operator extends linearly to an isomorphism between the \( B_\psi \)-modules \( V_{\phi_\psi} \) and \( V_{w_0 \cdot \phi_\psi} \) [Ha 3, 4]. Put

\[
T_\psi(\psi_\psi)(g_\psi) = (L_\psi(0)/L_\psi(-1)) \cdot \int \psi_\psi(w_0 u_\psi g_\psi) du_\psi,
\]

where \( L_\psi(s) = L_\psi(s, \phi_{1\psi}^{(1)}) = \left(1 - \phi_{1\psi}^{(1)}(\pi_\psi)/N((\pi_\psi))^S\right)^{-1} \) and for \( s \in \mathbb{Z}_+ \) we have \( N((\pi_\psi))^S = \text{(number of elements of } \mathcal{O}/(\pi_\psi)^S) = \|\pi_\psi\|_\psi^{-S} \), valuation \( \psi \) is determined by the prime ideal \( (\pi_\psi) \). The function \( L_\psi(s, \phi_{1\psi}^{(1)}) \) admits a meromorphic continuation to the whole complex plane. Calculate the value \( T_\psi(\psi = \psi_{0,\psi}^{(1)})(g_\psi) \) for \( g_\psi \in G^{(1)}(\mathcal{O}_\psi) = \text{SL}_2(\mathcal{O}_\psi) \).

Consider the function

\[
\psi_\psi = \psi_{0,\psi}^{(1)}(b_\psi k_\psi) = (|\alpha|_\psi^{s/2} \cdot \phi_{1\psi}(b_\psi)) \cdot |\det(k_\psi)|_\psi^{s/2} \mu_{1\psi}(\det(k_\psi))
\]

which is a spherical function with respect to the character

\[
^s \phi_{1\psi} = |\alpha|_\psi^{s/2} \cdot \phi_{1\psi}(b_\psi).
\]

According to (4.3.1) we have:
\[
\int_{U(F_{\mathcal{O}_{\mathcal{O}_y}})} \psi_{0,y,s}(w_0 u \cdot g_y \cdot du_y = \left( \int_{U(P_{\mathcal{O}_y})} \psi_y(w_0 u \cdot g_y) \cdot du_y = 1 \right) + \\
\sum_{N=1}^{\infty} f^{(1)}_y (\pi^N_{\mathcal{O}_y}) \| \pi^{-N}_{\mathcal{O}_y} \| \cdot \psi_y(g_y) \cdot \left( \int_{O_{\mathcal{O}_y}} f^{(1)}_y (x_y) dx_y = 1 - \| \pi_{\mathcal{O}_y} \| \right) = \\
\left(1 - f^{(1)}_y (\pi_{\mathcal{O}_y}) \| \pi^s_{\mathcal{O}_y} \| \right) \left(1 - f^{(1)}_y (\pi_{\mathcal{O}_y}) \| \pi^{s-1}_{\mathcal{O}_y} \| \right)^{-1} = \\
L_y(s-1, \phi^{(1)}_y) / L_y(s, \phi^{(1)}_y).
\]

Remark 6. Apriori this infinite sum is divergent for \( s = 0 \) however it is convergent for sufficiently large \( \text{Re}(s) \) ; we can use a regularization by a meromorphic continuation to the whole complex plane to obtain the desired value \( L_y(s-1, \phi^{(1)}_y) / L_y(s, \phi^{(1)}_y) \big|_{s=0} \). In fact, as we shall see later, this series is convergent.

So \( T_{\mathcal{O}_y}(\psi = \psi^{(1)}_{0,y})(g_y \in SL_2(\mathcal{O}_y)) = 1 \) . On other hand, \( \psi^{(2)}_{0,y}(g_y) = 1 \) for \( g_y \in G^{(1)}(\mathcal{O}_y) = SL_2(\mathcal{O}_y) \). Direct calculations show that \( T_{\mathcal{O}_y}(\psi_y)(b_y g_y) = w_0 \cdot \phi_y(b_y) T_{\mathcal{O}_y}(\psi_y)(g_y) \) and \( T_{\mathcal{O}_y} R(g_y) = R(g_y) \cdot T_{\mathcal{O}_y} \). This implies that \( T_{\mathcal{O}_y}(\psi^{(1)}_{0,y}) = \psi^{(2)}_{0,y} \) and we really obtained the desired extension.

4.4. The formula for a global intertwining operator is

\[
T_{\phi} = \left\{ \begin{array}{ll}
\phi & \text{is finite} \\
T_{\phi} : V_{\phi} \rightarrow V_{w_0 \cdot \phi} & \text{is unramified}
\end{array} \right.
\]

Let \( L_F(S, \phi^{(1)}_y) = \\
\left[ \begin{array}{ll}
L_y(S, \phi^{(1)}_y) & \text{is finite} \\
\phi^{(1)}_y & \text{is unramified}
\end{array} \right] \\
\]

where the local factors

\[
L_y(s, \phi^{(1)}_y) = \left(1 - \phi^{(1)}_y (\pi_{\mathcal{O}_y}) / N((\pi_{\mathcal{O}_y}))^{-s} \right)^{-1}
\]

were introduced in 4.3.2.
Then the function $L_F(s, \phi^{(1)})$ admits a meromorphic continuation to the whole complex plane. Introduce the "constant"

$$c_{\phi} = L(-1, \phi^{(1)}_\psi)/L(0, \phi^{(1)}_\psi) \in \mathbb{C} \cup \{\infty\}.$$ 

**THEOREM H (Harder [Ha 4]).** Let

$$res : H^1(S_K, \tilde{\mathcal{H}}) \rightarrow H^1(\partial S_K, \tilde{\mathcal{H}})$$

be the restriction homomorphism. Then:

1. $\text{Im}(res) = \bigoplus_{\phi \in \Sigma_1} \text{Im}(res_{\phi}) \subset V_{\phi_\sigma} \otimes V_{\phi_\sigma} \cdot \phi_\sigma$;
2. $\text{Im}(res_{\phi}) = \left\{ (\psi, \alpha_{\phi} \cdot c_{\phi} \cdot T_{\phi} (\psi)) \mid \psi \in V_{\phi_{\sigma}} \right\}$,

where $\alpha_{\phi} \in \mathbb{C}^*$ is a constant.

**PROOF.** The field $F$ is a totally imaginary quadratic extension of $\mathbb{Q}$ and $\phi_\sigma = \lambda_{\sigma_1}$ (see the items 3.4.2, 3.4.3). Then the character $\phi$ is "balanced" (in the sense of Harder). Therefore we can apply the theorem 2 from [Ha 4].

QED.

For the completeness we repeat here briefly the arguments of [Ha 4, proof of theorem 2; Ha 2, Ha 3]. Put $\omega(g, \phi, \psi) = \omega\left((b_\omega, k_\omega, g_\sigma), \phi, \psi\right) = \phi_\omega(b_\omega) \psi(g_\omega) \left(\text{ad}_{p_\omega}^* (k_\omega^{-1}) \circ \rho_\omega(k_\omega^{-1}) e_{\phi_\sigma}\right)$, where $p_\omega$ is the orthogonal complement to the Lie algebra of $K_\omega$ in $\mathfrak{g}_2^\omega(\mathbb{C})$, $e_{\phi_\sigma} \in \text{Hom}(p_\omega, M)$ has the weight $\phi_\omega^{-1}$ with respect to the adjoint representation of $T(\mathbb{R})$. Let

$$E(g, \phi, \psi) = \sum_{a \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \omega(a \cdot g)$$

Consider the "constant term"

$$E^0(g, \phi, \psi) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(ug, \phi, \psi) du$$

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where $du$ is the Tamagawa measure on $U(A)$. Then the restriction of $E(g, \phi, \psi)$ to $U(Q) \setminus U(A)$ is a closed 1-form which is cohomologous to the restriction of $E^0(g, \phi, \psi)$. The computation of the integral (*) is relatively simple in our case since we have of the Bruhat decomposition $G(Q) = B(Q) + B(Q)w_0U(Q)$. Then

$$
\int_{U(Q) \setminus U(A)} E(ug, \psi) du = \sum_{a \in B(Q) \setminus G(Q)} \omega(a \cdot ug) du = \omega(g) + \sum_{u \in U(Q)} \omega(w_0u \cdot g)
$$

$$
= \omega(g) + \int_{U(A)} \omega(w_0u \cdot g) du \text{ since } \text{Vol}(U(Q) \setminus U(A)) = 1.
$$

The form $\omega$ decomposes as $\omega_\psi \otimes \psi^*$ is finite and the integral

$$
\int_{U(A)} \omega(w_0u \cdot g) du \text{ is equal to } d_F \cdot \left( \int_{U(F_\psi)} \psi_\psi(w_0u \cdot g) du \right) \times \int_{U(F_\infty)} \omega_\infty (w_0u \cdot g) du \infty,
$$

where the nonzero number $d_F$ is coming from the Tamagawa measure. Direct calculations [Ha 4] show that the last multiple (for $g=1$) is equal to $\beta_\phi \epsilon w_0 \cdot \phi_\infty$ where $\beta_\phi \in \mathbb{C}^*$ is a scalar depending only on the character $\phi_\infty$. Then $\alpha_F = d_F \cdot \beta_\phi$.

According to 4.3.1, 4.3.2 for any finite $\psi$ we have:

$$
\int_{U(F_\psi)} \psi_\psi(w_0u \cdot g) du = \begin{cases} T_\psi(\psi_\psi)(g), & \text{if } \phi_\psi \text{ is ramified} \\
(L_\psi(-1, \phi_\psi(1))/L_\psi(0, \phi_\psi(1))) \cdot T_\psi(\psi_\psi)(g_\psi) & \text{else}
\end{cases}
$$

This calculation proves the theorem.
4.5. Lemma 4. $c_{\phi} \neq \{0, \infty\}$.

Proof. First we notice that $c_{\phi} \neq \infty$, $L(0, \phi^{(1)}) \neq \{0, \infty\}$ which follows from [Ha 4, proof of theorem 2, p.82]. So we only have to show that $L(-1, \phi^{(1)}) \neq 0$. Denote $\phi^{(1)}$ by $\eta$. For $L(s, \eta)$ we put [Lg]:

$$\Lambda(s, \eta) = ((2\pi)^{-2}d_{\eta})^{S/2}\Gamma(s_{\infty}/2)L(s, \eta)$$

where $d_{\eta} \in \mathbb{Q}^{*}$, the complex number $s_{\infty}$ will be defined later and the function $\Lambda(s, \eta)$ satisfies the functional equation:

$$|\Lambda(s, \eta)| = |\Lambda(1-s, \eta)|.$$ 

Hence $|\Lambda(-1, \eta)| \in \mathbb{C}^{*} \iff |\Lambda(2, \eta)| \in \mathbb{C}^{*}$.

Assertion 3. $L(2, \eta) \in \mathbb{C}^{*}$.

Proof. The function $\Lambda(s, \eta)$ is holomorphic and $L'(s)/L(s)$ is bounded for Re$(s) > 1+1/2$ [Lg]. Hence $L(2, \eta) \neq 0, \infty$. $\blacksquare$

Therefore we have $|L(-1, \eta)| = |\Gamma(2_{\infty}/2)\Gamma((-1)/2)|^{-1}\cdot \text{const}$, where const $\neq \{0, \infty\}$. Consider $\Gamma(s_{\infty}/2)$, $s \in \{2, -1\}$. The map $s \mapsto s_{\infty}$ is constructed via the character $\eta_{\infty}(z) = z^{4}$, $z \in \mathbb{C}$ in the following way.

We decompose $\eta_{\infty}(z) = z^{-4}$ as $z^{-4} = (z/|z|)^{m_{\infty}}|z|^{2c} = z^{m_{\infty}}|z|^{2c-m_{\infty}} = z^{m_{\infty}+c}|z|^{-c_{\infty}}$ (see [Lg]). Then $m_{\infty} = -4$, $s_{\infty}/2 = s + |m|/2$ (since $|F: \mathbb{Q}| = 2$), $s_{\infty} = s + 2$; $2_{\infty}/2 = 4$, $(-1)/2 = 1$. However $\Gamma(4) \cdot \Gamma(1)^{-1} \in \mathbb{C}^{*}$. Therefore $|L(-1, \eta)| \neq 0$. Lemma is proved. $\blacksquare$

4.6. Recall that $H^{1}(\partial S_{K}, \tilde{\mathcal{M}}) \approx \text{Ind}_{B(\mathcal{O}_{f})K_{f}}^{G(\mathcal{O}_{f})} H^{1}(\partial S_{K}, \tilde{\mathcal{M}})$. We have to single out the different components of $\partial S_{K}$ in this induced modulus. The group $G(\mathcal{O}_{f})$ acts transitively on the decomposition $\partial S_{K} = \partial_{1}S_{K} \cup \partial_{2}S_{K} \cup \ldots$. This decomposition corresponds to $G(\mathcal{O}_{f}) = B(\mathcal{O}_{f})K_{f} + B(\mathcal{O}_{f})g_{f}^{(2)}K_{f} + \ldots$ (where $g_{f}^{(1)} = 1$). The restriction of $H^{1}(\partial S_{K}, \tilde{\mathcal{M}})$ on $\partial_{j}S_{K}$ is equivalent to the restriction of the
function \( \psi : G(\mathbb{A}_f) \to H_B^1(\tilde{\mathbb{N}}) \) on \( B_f g_f^{(j)} K_f \). Then the image of \( H^1(\partial S_K, \tilde{\mathbb{N}}) \) in \( H^1(\partial S_K, \tilde{\mathbb{N}}) \) corresponds to
\[
\{ \psi : G(\mathbb{O}_f) \to H_B^1(\tilde{\mathbb{N}}) : \text{Supp}(\psi) = B_f K_f, \ \psi(b_f k_f) = R(b_f) \psi(1) \}.
\]
Respectively, \( H^1(\partial S_K, \tilde{\mathbb{N}}) \subset H^1(\partial S_K, \tilde{\mathbb{N}}) \) corresponds to
\[
\{ \psi : G(\mathbb{O}_f) \to H_B^1(\tilde{\mathbb{N}}) : \text{Supp}(\psi) = B_f K_f g_f^{(j)}, \ \psi(b_f g_f^{(j)} k_f) = R(b_f) \psi(g_f^{(j)}) \}.
\]
In the same way, on the level of \( \text{Ind}_{B(\mathbb{A}_f), K_f}^G(\mathbb{A}_f) \) we can single out the space \( H^1_j(\tilde{\mathbb{N}}) \) as the space of functions with support at \( B(\mathbb{A}_f), g_f^{(j)} K_f \).

4.7. **Assertion 4.** If for every \( j \) the map
\[
\text{res} : H^1(S_K, \tilde{\mathbb{N}}) \to H^1_j(\partial S_K, \tilde{\mathbb{N}})
\]
is onto then \( \text{res} : H^1(S_K, \tilde{\mathbb{N}}) \to H^1(\mathbb{T}^2_{ij}, \tilde{\mathbb{N}}) \) is onto for every boundary torus \( \mathbb{T}^2_{ij} \subset \partial S_K \).

**Proof.** \( H^1_j(\partial S_K, \tilde{\mathbb{N}}) = \bigoplus_i H^1(\mathbb{T}^2_{ij}, \tilde{\mathbb{N}}) \).

The space \( H_B^1(\tilde{\mathbb{N}}) \) and hence \( H_j^1(\tilde{\mathbb{N}}) \) is the direct sum of two subspaces where \( B(\mathbb{A}) \) acts by two (quasi)characters which are different at the infinite place. Denote these subspaces by \( V^1_j, V^2_j \) (where 1, 2 correspond to \( \chi_1, \chi_2 \) - values at infinity), \( V^i = \bigoplus_j V^i_j \).

4.8. **Principal lemma.**

**Lemma 5.** \( V^2_j \subset \text{Im} \left( \text{res}_j : H^1(S_K, \tilde{\mathbb{N}}) \to H^1_j(\partial S_K, \tilde{\mathbb{N}}) \right) \) for every \( j \neq 1 \).

**Proof.** Let \( \phi \in \Sigma_1, \phi_f = \mu_1 f \mu_2 f \), where \( \mu_k f \) are defined on \( \mathbb{I}_f ; \phi^{(1)}_f = \mu_1 f(t_f)/\mu_2 f(t_f), t_f \in \mathbb{I}_f \).

4.8.1. Let \( \psi \in \tilde{H}_1^1(\tilde{\mathbb{N}}) \setminus \{0\} \) be an eigenfunction of \( B(\mathbb{A}_f) \) corresponding to the quasicharacter \( \phi \). Then
\[
T_{\phi_f} : \psi \mapsto 0 \cdot w_0 \cdot \phi_f \subset V_0 \cdot \phi_f.
\]
Our aim is to show that \( T_{\phi_f} (\psi) \mid B_f g_f^{(j)} K_f \neq 0 \). If it is done

then \( \text{Span}(T_{\phi_f} (\psi) \mid B_f g_f^{(j)} K_f) = \text{Span}(e_{\omega_0 \cdot \phi_f} \mid B_f g_f^{(j)} K_f) = V_j^2 \).

4.8.2. Thus, take \( \psi \in V_{\phi_f}, \text{Supp}(\psi) = B_f K_f \). The space \( M_\phi \) of such functions is 1-dimensional (over \( \mathbb{Q} \)) then we can choose \( \psi \) in the following way:

(a) If there exists a place \( \eta \) where \( \phi^{(1)}_f \) is ramified then we take \( \psi_\eta = e_{\phi_\eta} \) for all such places; for other places \( \xi \) (where \( \phi^{(1)}_f \) is unramified) we take \( \psi_\xi = \psi^{(1)}_{0, z} \) (see the Convention from 3.4.6).

(b) If \( \phi^{(1)}_f \) is unramified at every place, then we pick a prime \( \pi \in \mathcal{O}_F \) and put: \( \psi_\eta = e_{\phi_\eta} \) (for \( \eta = (\pi) \)) and \( \psi_\xi = \psi^{(1)}_{0, z} \) (for every \( \xi \neq \eta \)).

In any case, the function \( \psi = \bigotimes_{\eta \text{ finite}} \psi_\eta \) belongs to \( M_\phi \).

4.8.3. If \( \psi_\xi = \psi^{(1)}_{0, z} \) then \( T_{\phi_\xi} (\psi_\xi) (g_\xi) = 0 \) for every \( g_\xi \). So, our problem is reduced to the calculation of the term

\[
\text{const} \cdot T_{\psi_\eta} (g_\eta^{(j)}) \int_{U(F_\eta)} \psi_\eta (w_0 \nu \eta g_\eta^{(j)}) d\psi_\eta \quad \text{in the case} \quad \psi_\eta = e_{\phi_\eta}.
\]

Remark 7. If necessary we use here the regularization (4.3.2) of the integral above.

Fix the Bruhat decomposition: \( G(F_\eta) = B(F_\eta) + U(F_\eta) \cdot w_0 B(F_\eta) \). The element \( g_\eta^{(j)} \) belongs to the second summand since \( j \neq 1 \). Then:

\[
g_\eta = g_\eta^{(j)} = \begin{pmatrix} 1 & u' & (0 \quad 1) & \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \\ 0 & 1 & (-1 \quad 0) \end{pmatrix}
\]

and we have:
\[
\int_{U(F_\psi)} \psi_\psi(w_0 u_\psi g^{(1)}) du_\psi = \\
\int_{U(F_\psi)} \psi_\psi\left(w_0 \begin{pmatrix} 1 & u_\psi + u' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}\right) du_\psi = \\
\int_{U(F_\psi)} \psi_\psi\left(\begin{pmatrix} 1 \\ u_\psi + u' \end{pmatrix} \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}\right) du_\psi = \\
\int_{U(F_\psi)} \psi_\psi\left(\begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix}\right) du_\psi, \text{ where } z = (u_\psi + u') b_1 b_2^{-1}.
\]

Recall that \(\text{Supp}(\psi_\psi) = B_\psi K_\psi\), then \(\psi_\psi\left(\begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix}\right)\)

vanishes unless \(\begin{pmatrix} a_1 & a_3 \\
0 & a_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix} = 1 \pmod{\pi_\psi^N} \) for some

\(\begin{pmatrix} a_1 & a_3 \\
0 & a_2 \end{pmatrix} \in B(F_\psi)\). However this is equivalent to

\(a_2 \equiv a_1 \equiv 1 \pmod{\pi_\psi^N}, a_3 \equiv (u_\psi + u') b_1 b_2^{-1} \equiv 0 \pmod{\pi_\psi^N}\) and

\((u_\psi + u') b_1 b_2^{-1} \in \pi_\psi^N O_\psi\). Hence, \(u_\psi\) runs through the set \(b_2 b_1^{-1} \pi_\psi^N O_\psi + u' = \Xi\). Notice that \(\text{mes}(\Xi) = \|b_1 b_2^{-1} \pi_\psi^N \|_\psi \cdot (1 - \|\pi_\psi\|_\psi) \neq 0\)

and

\(\psi_\psi\left(\begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1-b_3 z & -b_3^2 z \\ z & b_3 z + 1 \end{pmatrix}\right) = \phi_\psi\left(\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\right)\).

Therefore, \(T_\psi \psi(g^{(1)}) = \text{const} \cdot \phi_\psi\left(\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\right) \text{mes}(\Xi) \neq 0\). So, Lemma is proved.
4.9. **COROLLARY 2.** \( V_j^2 \subset \text{Im}(\text{res}_j) \) for every \( j = 1, 2, \ldots \).

**PROOF.** The group \( G(O_f) \) acts on \( S_K, \partial S_K \) (and on \( H^1(S_K, \tilde{\mathcal{N}}), H^1(\partial S_K, \tilde{\mathcal{N}}) \)) and this action commutes with the restriction maps. However, this action is transitive on \( \{ \partial_j S_K, j = 1, 2, \ldots \} \).

Therefore, the corollary follows from Lemma 5. \( \blacksquare \)

4.10. **COROLLARY 3.** \( V_1^1 \subset \text{Im}(\text{res}_1) \).

**PROOF.** Let \( \psi \in H^1(\tilde{\mathcal{N}}) \cap V_{\phi_f} \), then \( \text{Supp}(\psi) = B(A_f)K_f \) and

\[
(\psi, c_{\phi_f}^T \psi \Big|_{B(A_f)K_f}) = (\psi, \text{const.} \int_{\psi \text{ is finite}} \psi_{\psi} w_0 u_{\psi} b_{\psi} k_{\psi} du_{\psi})
\]

\[
= (\psi, \text{const.} \cdot c_{\phi_f} \psi \Big|_{B_fK_f}) \in \text{Im}(\text{res}_{\phi_f}) \Big|_{B_fK_f}.
\]

**Remark.** The scalar \( \text{const.} \) vanishes iff \( \phi_f^{(1)} \) is ramified over some place as it follows from (4.3.1).

However, according to Corollary 1, every vector \((0, \varphi)\) belongs to \( \text{Im}(\text{res}_{\phi_f}) \Big|_{B_fK_f} \); therefore

\[
\left\{ (\psi, 0) : \psi \in H^1(\tilde{\mathcal{N}}) \cap V_{\phi_f} \right\} \subset \text{Im}(\text{res}_{\phi_f}) \Big|_{B_fK_f}
\]

for every Hecke character \( \phi \in \Sigma_1 \). This proves the corollary. \( \blacksquare \)

4.11. Now we apply the action of the group \( G(O_f) \) in the same manner as in the section 4.9 to obtain:

**COROLLARY 4.** \( V_j^1 \subset \text{Im}(\text{res}_j) \) for every \( j = 1, 2, \ldots \).

This corollary finishes the proof of Theorem 1. \( \blacksquare \)

4.12. **Proof of theorem 2.**

Below we use notations introduced in 1.2.
Theorem 2. Let $\mathcal{T}^2_{nj} \subset \partial S_K$. Then for every $\omega_{nj,1} \in H^1(\pi_1(\mathcal{T}^2_{nj}), \mathcal{M}_0 = \rho gl_2(\mathbb{C}))$ there exists a class $\omega \in H^1(\Gamma = \Gamma(\mathbb{H}), \mathcal{M}_0)$ such that:

(i) projection of $\omega$ to $W_{nj,1}$ is equal to $\omega_{nj,1}$;
(ii) $\omega_{k1,1} = 0$ for every torus $\mathcal{T}^2_{k1} \neq \mathcal{T}^2_{nj}$;
(iii) for every $k \neq j$ there exists $l = l(k)$ such that the projection $\omega_{k1,2}$ doesn't vanish.

Proof. Using the isomorphism $\mathcal{M}_0 \cong \mathcal{M}$ and preserving notations for sub (upper)scripts we identify classes the $\omega_{k1,p}$ with elements of $H^1(\Gamma, \mathcal{M})$; the eigenvectors of the character $\xi_1$ correspond to elements of $V^1$ under the isomorphism $H^1(\partial S_K, \tilde{\mathcal{M}}) \cong V^1 \oplus V^2$. Applying the action of $SL_2(O_{F,1})$ we can suppose that $j = 1$.

$$|\mathcal{T}_f|$$

Then $V^1_1 = \oplus_{s=1}^{[\mathcal{T}_f]} W_{s1,1}$ (under our convention on notations above).

Put $T = \oplus_{f \phi} T_{\phi}^f$. For $\omega_{n1,1} \equiv \psi \in V^1_1$ let $\omega \in H^1(\Gamma, \rho gl_2(\mathbb{C}))$ be a class corresponding to $(\psi, T\psi) \in \text{Im}(\text{res})$. Notice that the operator $\phi \in V^1_1 \mapsto (T\phi)_k \in V^2_k$ is injective (4.8), therefore the class $\omega$ has the desired properties.

Thus, after infinitesimal deformation of the group $\Gamma = \Gamma(\mathbb{H})$ corresponding to a Dehn surgery on $\mathcal{T}^2_{nj}$, we have nontrivial infinitesimal variation of Euclidean structure on some other cusp tori of a connected component of $S_K$ containing $\mathcal{T}^2_{nj}$.

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5. Example.

5.1. Example. Let $M$ be the complete hyperbolic manifold homeomorphic to the complement of the "Figure 8" knot. Then (according to C. Maclachlan [M]) $M$ contains an immersed closed totally geodesic surface $\Sigma$. Let

$$R(\pi_1 M) = \Hom(\pi_1 M, \text{SL}_2(\mathbb{C}))/\text{SL}_2(\mathbb{C})$$

**ASSERTION 5.1.** Under deformations of holonomy representation of $\pi_1 M$ the group $\pi_1 \Sigma$ varies nontrivially. So, under small deformations of $\text{id} = \rho_o : \pi_1 M \to \text{SL}_2(\mathbb{C})$ the image of $\rho_o(\pi_1 \Sigma)$ becomes quasifuchsian.

**PROOF.** As it was shown by R. Riley [Ri] (see also [KK]) in the space $R(\pi_1 M)$ there is an irreducible complex curve $\sigma$, which contains $[\rho_o]$ and some solvable representation $[\rho_s]$. Hence there exists an element $\gamma \in \pi_1 \Sigma$ such that $\rho_s(\gamma) = 1$ while $\rho_o(\gamma) \neq 1$. This means that $\rho_o(\gamma)$ varies nontrivially under the deformation of $\rho_o$ along $\sigma$. \hfill \blacksquare

Let $\tilde{M} \to M$ be a finite -sheeted covering such that $\Sigma$ lifts in the embedded closed totally geodesic surface $\tilde{\Sigma}$ in $\tilde{M}$. Then the assertion above means that after a Dehn filling of $\tilde{M}$ with sufficiently large parameter the surface $\tilde{\Sigma}$ deforms to non-totally geodesic incompressible surface. In general case the surface $\tilde{\Sigma}$ doesn't split $\tilde{M}$. Let $N$ be the hyperbolic manifold with totally geodesic boundary which is the compactification of $\tilde{M} \setminus \tilde{\Sigma}$. Then the Mostow's rigidity theorem implies that the answer to Question 3 (see Introduction) is negative for the manifold $N$. 

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USSR, 680002, Khabarovsk, Frunze 72,
Institute for Applied Mathematics

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

MSRI, Centennial Drive, Berkeley,
CA 94720, USA

Email: kapovich@msri.org
kapovich@umd.sofya.edu

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