# Periods of abelian differentials and dynamics

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**Abstract**

Given a closed oriented surface $S$ of genus $\geq 2$ we describe those cohomology classes $\chi \in H^1(S, \mathbb{C})$ which appear as the period characters of abelian differentials for some choice of complex structure $\tau = \tau(\chi)$ on $S$ consistent with the orientation. In other words, we describe the union

$$\bigcup_{\tau \in T(S)} H^{1,0}(S_\tau, \mathbb{C})$$

where $T(S)$ is the Teichmüller space of $S$. The proof is based upon Ratner's solution of Raghunathan's conjecture.

## 1. Introduction

Let $S$ be a closed[1] connected oriented surface of genus $n \geq 2$. Recall that each complex structure $\tau$ on $S$ (consistent with the orientation) determines the linear subspace $H^{1,0}(S_\tau, \mathbb{C}) \subset H^1(S, \mathbb{C})$ of the complex dimension $n$ (i.e. half of the dimension of the cohomology group). In the down-to-earth terms, the subspace $H^{1,0}(S, \mathbb{C})$ consists of the period characters of abelian differentials $\alpha \in \Omega(S)$:

$$\chi_\alpha = \chi \in H^1(S, \mathbb{C}), \quad \chi(c) = \int_c \alpha, \quad c \in H_1(S, \mathbb{Z}).$$

In this paper we describe the subset

$$\bigcup_{\tau \in T(S)} H^{1,0}(S_\tau, \mathbb{C})$$

where $T(S)$ is the Teichmüller space of $S$. In other words, we give a necessary and sufficient condition for a character $\chi \in H^1(S, \mathbb{C})$ to appear as the period of some abelian differential $\alpha$ on $S_\tau$ for *some* choice of the complex structure $\tau$ on $S$.

**Remark 1.1.** *We note the difference between this question and the Schottky problem which asks for description of the subvariety in the Grassmanian $G(n, 2n)$ that consists of the subspaces $H^{1,0}(S_\tau, \mathbb{C}), \tau \in T(S)$.*

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[1] I.e. compact with empty boundary.

Since the solution is obvious in the case $\chi = 0$ we will consider only the nontrivial characters $\chi$. It turns out that there are precisely two topological obstructions for such $\chi$ to be the character of an abelian differential, the first is classical and is a part of the Riemann bilinear relations (see for instance [Nar92]); the second is less known, although it would not be surprising to find that it appears somewhere in the classical literature on abelian differentials. To describe the first obstruction recall that the Poincaré duality defines a symplectic pairing $\omega : H^1(S,\mathbb{R})^{\otimes 2} \to \mathbb{R}$. This yields a quadratic form $H^1(S,\mathbb{C}) \to \mathbb{R}$ again denoted $\omega$:

$$\omega(\chi) := \omega(Re\chi, Im\chi).$$

If $x_1, y_1, ..., x_n, y_n$ denote the standard (symplectic) basis of $H^1(S,\mathbb{Z})$ then $\omega(\chi)$ equals

$$\sum_{i=j}^{n} Im(\overline{\chi(x_j)}\chi(y_j)).$$

The number $\omega(\chi)$ can be also described as

$$\int_S f^*(dA)$$

where $dA$ is the area form $\frac{i}{2}dz \wedge \overline{dz}$ on $\mathbb{C}$, $f : S \to E$ is a section of the complex line bundle $E$ over $S$ associated with $\chi$. (The form $dA$ is induced on $E$ via the projection $\tilde{S} \times \mathbb{C} \to \mathbb{C}$, where $\tilde{S}$ is the universal cover of $S$.)

Note that in the case when $\chi \neq 0$ is the period character of an abelian differential $\alpha \in \Omega(S)$ we have:

$$\omega(\chi) = \int_S \frac{i}{2}\alpha \wedge \bar{\alpha}$$

is the area of the surface $S$ with respect to the singular Euclidean metric on $S$ induced by $\alpha$. Since this area has to be positive we get

**Obstruction 1.** If $\chi \in H^{1,0}(S_\tau)$ for some $\tau \in T(S)$ then $\omega(\chi) > 0$.

The second obstruction applies only to special characters $\chi$. In what follows we will regard elements of $H^1(S,\mathbb{C})$ as additive characters $\chi$ on $H^1(S,\mathbb{Z})$, this way we have the *image* of $\chi$, which is a 2-generated subgroup of $\mathbb{C}$.

**Obstruction 2.** Suppose that the image $Image(\chi)$ of the character $\chi \in H^1(S,\mathbb{C})$ is a discrete subgroup $A_\chi$ of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$. Thus $\chi$ gives rise to a map

$$\chi : H^1(S,\mathbb{Z}) \to H^1(T^2,\mathbb{Z})$$

where $T^2 = \mathbb{C}/A_\chi$ is the 2-torus. This map is realized by a unique (up to homotopy) map $f : S \to T^2$. Then, for each $\chi \in H^{1,0}(S_\tau)$ the degree of $f$ has to be at least 2.

The reason for this obstruction is that if $\chi$ is the period of some $\alpha \in \Omega(S_\tau)$ then the multivalued solution of the equation $dF = \alpha$ on the Riemann surface $S_\tau$ yields a (nonconstant) holomorphic map $f : S \to T^2$ which induces $\chi : H^1(S,\mathbb{Z}) \to H^1(T^2,\mathbb{Z})$. Since the surface $S$ has genus $\geq 2$, the map $f$ cannot be a homeomorphism, hence its degree is at least 2.

Alternatively, the second obstruction can be described as follows. Assume again that the image $A_\chi$ of the character $\chi$ is a discrete subgroup $\cong \mathbb{Z}^2$. Let $Area(\chi)$ denote $Area(\mathbb{C}/A_\chi)$, the area of the flat torus. Then the requirement $deg(f) \geq 2$ is equivalent to

$$\omega(\chi) \geq 2Area(\chi).$$

Our main result is the following:

**Theorem 1.2.** *If $\chi \in H^1(S, \mathbb{C})$ satisfies the conditions imposed by the 1-st and the 2-nd obstruction then $\chi \in H^{1,0}(S_\tau)$ for some $\tau \in T(S)$.*

Despite the classical appearance, the proof of Theorem 1.2 which we present in this paper relies upon the tool unknown in the 19-th century: the ergodic theory.[2]

In §7 we show that if $\chi$ is a nonzero character which is not the period of any abelian differential, it nevertheless possible to find a complex structure $\tau$ on $S$ so that $\chi$ is the period character of a meromorphic differential with a single simple pole on $S_\tau$. We now identify the additive group $\mathbb{C}$ with the subgroup of $PSL(2, \mathbb{C})$ which consists of translations. Then we can regard $\chi$ as a represenation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$. For such $\rho$ define

$$d(\rho) := \begin{cases} 2g-2, & \text{if Obstructions 1 and 2 are satisfied,} \\ 2g, & \text{otherwise.} \end{cases} \qquad (1.3)$$

We recall that a *branched projective structure* $\sigma$ on a complex curve $S$ is an atlas with values in $\mathbb{S}^2$ where the local charts are nonconstant holomorphic functions (not necessarily locally univalent) and the transition maps are linear-fractional transformations (i.e. elements of $PSL(2, \mathbb{C})$). Thus near each point $z \in S$ (which we identify with $0 \in \mathbb{C}$) the local chart has the form $z \mapsto z^k$. The number $k - 1 = deg(z)$ is called the degree of branching at $z$. We get the *branching divisor* $D$ on $S$ whose degree is called the *degree of branching* $deg(\sigma)$. For each representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ there exists a complex-projective structure $\sigma$ (consistent with the orientation on $S$) which corresponds to *some* complex structure on $S$, so that $\rho$ is the holonomy of $\sigma$. We define $d(\rho)$ to be the least degree of branching for such structures. Note that for the trivial representation $\rho$, $d(\rho) = 2g + 2$ and the branched projective structure is given by the hyperelliptic covering. In this note we compute the function $d(\rho)$ in the very special case of representations with the image in the subgroup of translations. The general case will be treated elsewhere, here we only note that in [GKM00] (see also [Kap95]) it was shown that for each representation $\rho$ with *nonelementary image*[3], $d(\rho) \in \{0, 1\}$ equals the 2-nd Stiefel-Whitney class of $\rho$ (mod 2).

**Corollary 1.4.** *For each nontrivial representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ whose image is contained in the subgroup of translations, the function $d(\rho)$ is given by the formula (1.3).*

[2] In view of the proof of the main theorem in the genus 2 case which requires no ergodic theory at all, it is likely that Theorem 1.2 could be found somewhere in the classical literature on abelian differentials.

[3] I.e. the image does not have an invariant finite nonempty subset in $\mathbb{H}^3 \cup \mathbb{S}^2$.

The lower bounds in this theorem are given by the Riemann-Roch (see §7), while the upper bound follows from Theorems 1.2 and 7.1.

Since the map $P : \alpha \to \chi_\alpha$, which sends the abelian differential to its character, is complex-linear, it suffices to prove Theorem 1.2 for *normalized* characters, i.e. the characters $\chi$ such that $\omega(\chi) = 1$ (hence the 1-st obstruction automatically holds). We let

$$X := \{\chi \in H^1(S, \mathbb{C}) : \omega(\chi) = 1\}$$

and

$$\Sigma := X \cap \bigcup_{\tau \in T(S)} H^{1,0}(S_\tau, \mathbb{C}).$$

Let $\Omega$ denote the vector bundle over $T(S)$ whose fiber over a point $\tau \in T(S)$ consists of abelian differentials $\Omega(S_\tau)$. We let $\Omega'$ denote the submanifold in $\Omega$ consisting of abelian differentials $\alpha$ such that $\omega(\alpha) = 1$. We have the map

$$P : \Omega' \to \Sigma \subset X.$$

To explain the appearance of the ergodic theory in the proof we will need two elementary facts about the subset $\Sigma$ in $X$.

**Fact 1.** (See §2.) The map $P : \Omega' \to X$ is open. In particular, $\Sigma$ is open in $X$.

We let $G = Sp(2n, \mathbb{R})$ denote the group of linear symplectic automorphisms of the symplectic structure $\omega$ on $\mathbb{R}^{4n} = H^1(S, \mathbb{C})$. This is a simple algebraic Lie group which acts naturally on $X$. It is elementary that the action of $G$ on $X$ is transitive. The stabilizer $G_\chi$ of a point $\chi \in X$ is isomorphic to $Sp(2n-2, \mathbb{R})$. Thus $X = Sp(2n)/Sp(2n-2)$. Recall that the integer symplectic group $\Gamma = Sp(2n, \mathbb{Z})$ is a *lattice* in the group $G$.

**Fact 2.** The subset $\Sigma$ is invariant under $\Gamma$.

Recall that the group of orientation-preserving diffeomorphisms $Diff(S)$ acts on $H^1(S, \mathbb{C})$ through the group $\Gamma$. If $\chi \in \Sigma$ is the period character of $\alpha \in \Omega(S_\tau)$ and $\gamma \in \Gamma$ corresponds to a diffeomorphism $h : S \to S$, then $\gamma(\chi)$ is the period character of the abelian differential

$$h^*(\alpha) \in \Omega(S_{h^*(\tau)}),$$

where $h^*(\tau)$ is the pull-back of the complex structure $\tau$ via $h$. Thus $\gamma(\Sigma) = \Sigma$.

Combining the above two facts we see that $\Sigma$ is a (nonempty) open $\Gamma$-invariant subset of $X$. We recall

**Theorem 1.5.** *(C. Moore, see [Zim84].) If $G$ is a semisimple Lie group, $\Gamma$ is a lattice in $G$ and $H$ is a noncompact Lie subgroup in $G$ then $H$ acts ergodically on $\Gamma \backslash G$. Equivalently, $\Gamma$ acts ergodically on $G/H$.*

Thus, since $\Sigma \subset X = Sp(2n)/Sp(2n-2)$ is an open nonempty $\Gamma$-invariant subset, the complement $X - \Sigma$ has zero measure. In particular, $\Sigma$ is dense in $X$. Ergodicity of the action $\Gamma \curvearrowright X$ implies that *generic*[4] points $\chi \in X$ have dense $\Gamma$-orbits. Our objective is to understand the *nongeneric* orbits. This is done by applying Ratner's

[4] In the measure-theoretic sense.

solution of Raghunathan's conjecture. Ratner's theorem implies that there are only few types of nongeneric orbits. We will show that most of them correspond to the characters with discrete image. After we describe other orbits we will show that Obstruction 2 suffices for the existence of an abelian differential with the given period character.

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## 2. Geometric preliminaries

*Geometric interpretation of nonzero abelian differentials $\alpha$.* Each nonzero abelian differential $\alpha \in \Omega(S_\tau)$ determines a singular Euclidean structure on the surface $S$ with isolated singularities at zeroes of $\alpha$, see [Str84]. The local charts for this structure are given by the branches of the indefinite integral

$$F(z) = \int_{z_0}^{z} \alpha$$

where $z_0 \in S$ is a base-point. If $\alpha$ vanishes (at the order $k-1$) at a point $0 \in S$ then the local chart at 0 is a $k$-fold ramified covering $z \mapsto z^k$. The transition maps of the flat atlas on $S - Zero(\alpha)$ are Euclidean translations. Vice-versa, suppose that we are given a flat structure on the (topological) surface $S$ where the local charts have the form $z \mapsto z^k$, $k \geq 1$, and the transition maps away from the branch-points are Euclidean translations. This structure canonically defines a complex structure on $S$ together with an abelian differential $\alpha$ obtained by the pull-back of $dz$ via the local charts. Every such singular Euclidean structure gives rise to a *developing map* $dev : \tilde{S} \to \mathbb{C}$ where $\tilde{S}$ is the universal abelian covering of $S$ and $H := H_1(S, \mathbb{Z})$ acts on $\tilde{S}$ by deck-transformations. The mapping $dev$ is $\chi$-equivariant where $\chi : H_1(S, \mathbb{Z}) \to \mathbb{C}$ is the holonomy of the above structure (it coincides with the character of the associated abelian differential). The space $E(S)$ of the above Euclidean structures has a natural topology: the topology of uniform convergence on compacts of the developing mappings. It is easy to see that with this topology the natural bijection $E(S) \to \Omega - 0_\Omega$ is a homeomorphism.

**Matrix form of the characters.** Given the standard (symplectic) basis in $H_1(S, \mathbb{Z})$, $x_1, y_1, ..., x_n, y_n$, we can identify each character $\chi : H_1(S, \mathbb{Z}) \to \mathbb{C} = \mathbb{R}^2$ with the $2 \times 2n$ matrix

$$M(\chi) := [M_1 M_2 ... M_n],$$

$$M_j = M_j(\chi) := \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}, j = 1, ..., n.$$

Here

$$\chi(x_1, ..., y_n) = (u, v)^t, u = (a_1, b_1, ..., a_n, b_n), v = (c_1, d_1, ..., c_n, d_n),$$

and the vectors $u, v$ are the row-vectors of the matrix $M$. The group $G = Sp(2n)$ acts on the matrices $M$ by multiplying them from the right. The matrix $M(\chi)$ is the *matrix form* of the character $\chi$. Then we define

$$\omega_j(u,v) = det(M_j(\chi)) = \begin{vmatrix} a_j & b_j \\ c_j & d_j \end{vmatrix}, \; j = 1, ..., n;$$

it follows that $\omega(u,v) = \sum_j \omega_j(u,v)$. The group $SL(2) = Sp(2)$ acts on the characters $\chi$ by multiplying their matrices from the left. It is clear that this action commutes with the action of $Sp(2n, \mathbb{Z}) \subset G$ and that it preserves each determinant $\omega_j(\chi)$.

**Lemma 2.1.** $Sp(2)\Sigma = \Sigma$.

*Proof.* Suppose that $\chi \in \Sigma$ is the period character of an abelian differential corresponding to a singular Euclidean structure $\sigma$. Take $A \in Sp(2)$. Composing coordinate charts of $\sigma$ with $A$ deforms $\sigma$ to a new singular Euclidean structure of the same area. The holonomy of this structure is the composition $A \circ \chi$. Hence $A\chi \in \Sigma$. ☐

**Lemma 2.2.** *Suppose that $\chi = (u, v)$ and $u, v \in \mathbb{R}^{2n}$ span a 2-dimensional rational subspace (i.e. a subspace which admits a rational basis). Then the $\mathbb{Z}$-module $\mathcal{M}$ generated by the columns of the matrix $M(\chi)$ has rank 2, i.e. is discrete as a subgroup of $\mathbb{R}^2$.*

*Proof.* The action of $GL(2)$ by the multiplication from the left on the matrix $M(\chi)$ preserves the rank of $\mathcal{M}$. Since $Span(u, v)$ is a rational subspace there exists a matrix $A \in GL(2)$ such that the matrix $AM(\chi)$ has integer entries. The rank of the $\mathbb{Z}$-module generated by its columns is clearly 2. ☐

Define

$$X_+ := \{\chi \in X : \omega_j(\chi) > 0, j = 1, ..., n\}.$$

Our strategy in dealing with the *nongeneric characters* $\chi \in X$ is to find $\gamma \in Sp(2n, \mathbb{Z})$ such that $\omega_j(\gamma\chi) > 0$, $j = 1, ..., n$, i.e. $\gamma\chi \in X_+$. As we will see in Theorem 2.3 the existence of such $\gamma$ would imply that $\chi$ belongs $\Sigma$ (i.e. that $\chi$ is the period character of an abelian differential).

**Theorem 2.3.** $X_+ \subset \Sigma$.

*Proof.* Let $(u, v) \in X_+$, $u = (a_1, b_1 ..., a_n, b_{2n}), v = (c_1, d_1, ..., c_n, d_n)$. We let $z_j := (a_j, c_j), w_j := (b_j, d_j) \in \mathbb{R}^2$, $j = 1, ..., n$. Each pair of vectors $(z_j, w_j)$ determines a fundamental parallelogram $P_j$ in $\mathbb{R}^2$ for the lattice generated by $z_j, w_j$. Using parallel translations place these parallelograms so that $P_j \cap P_{j+1}$ has nonempty interior, $j = 1, ..., n-1$. Then for each pair of parallelograms $P_j, P_{j+1}$ ($j = 1, ..., n-1$) cut both $P_j, P_{j+1}$ open along common segments $\beta_j$ and then glue them along the resulting circles. Call the result $\Phi$. See Figure 1.

Finally, for each parallelogram $P_j$ identify the opposite sides via a parallel translation. The result is a surface $S$ equipped with the projection $\delta : \tilde{S} \to \mathbb{C}$ where $\tilde{S}$ is the universal abelian covering. The surface $\Phi$ is the fundamental domain for the action of $H_1(S, \mathbb{Z})$ on $\tilde{S}$ via deck transformations. The restriction $\delta|\Phi : \Phi \to \mathbb{C}$ is the obvious projection. Note that $\delta$ is a local homeomorphism away from the translates of the end-points of the segments $\beta_j$. Near the end-points of such segments the mapping $\delta$ is a 2-fold ramified covering. The abelian differential $\alpha$ on $S$ is obtained by taking the pull-back of $dz$ from $\mathbb{C}$ to $\tilde{S}$ via $\delta$ and then projecting it to $S$. The edges of the

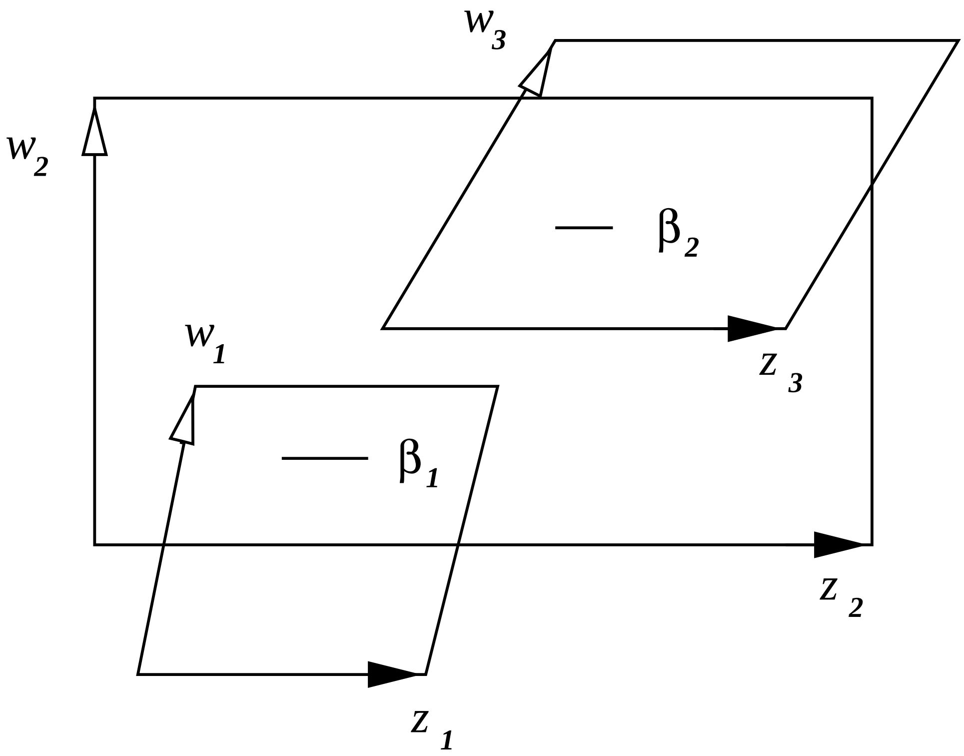


Figure 1:

parallelograms $P_j$ correspond to the standard generators of $H_1(S,\mathbb{Z})$. It is clear that the periods of $\alpha$ over the generators of $H_1(S,\mathbb{Z})$ are given by evaluation of $\chi$ on these generators. □

The above lemma implies that it suffices to show that $\Gamma\chi \cap X_+ \neq \emptyset$ to prove that $\chi \in \Sigma$. Note however that there are characters in $\Sigma$ which do not belong to the orbit $\Gamma X_+$. These are the characters with the discrete image $A_\chi \cong \mathbb{Z}^2$ so that

$$\frac{\omega(\chi)}{Area(\mathbb{C}/A_\chi)} < n.$$

To find abelian differentials corresponding to such characters we need another construction that we describe below.

**Lemma 2.4.** *Suppose that the character $\chi$ has the matrix form*

$$[M_1 M_2 ... M_n], M_1 = \begin{bmatrix} a_1 = \omega(\chi) & 0 \\ 0 & 1 \end{bmatrix}, M_j = \begin{bmatrix} a_j & 0 \\ 0 & 0 \end{bmatrix}, j = 2, ..., n,$$

*where $0 < a_j < a_1$, $j = 2, ..., n$. Then $\chi \in \Sigma$.*

*Proof.* Similarly to the previous lemma we construct complex structure and abelian differential by gluing certain polygons. Let $P_1$ be the fundamental rectangle for the group generated by the vectors $z_1, w_1$ which are the columns of $M_1$. Inside $P_1$ choose pairwise disjoint horizontal segments $\beta_j, \beta_j', j = 2, .., n$, so that the translation via $[a_j 0]$ sends $\beta_j$ to $\beta_j'$. We then cut $P_1$ open along the segments $\beta_j, \beta_j'$ and identify the resulting circles via the translations by $[a_j 0]$, $j = 2, .., n$. Finally, glue the sides of $P_1$

via the horizontal translations, see Figure 2. Analogously to the previous lemma we get a singular Euclidean structure with the holonomy $\chi$. The singular points of this structure correspond to the end-points of the segments $\beta_j$ (the total angle at each of these points is $4\pi$). ☐

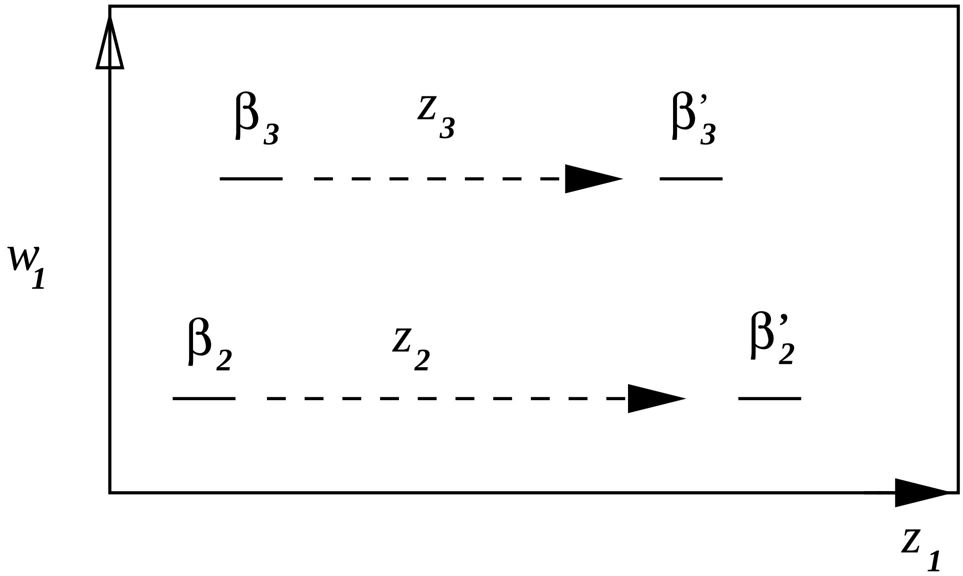


Figure 2:

**Lemma 2.5.** *Suppose that $u, v \in \mathbb{Z}^4$ are vectors such that $\omega(u, v) = 1$. Then this pair of vectors could be completed to an integer symplectic basis in $\mathbb{R}^4$.*

*Proof.* Let $W := Span(u, v)$. Recall that the symplectic projection $Proj_W(z)$ of a vector $z$ to $W$ is given by

$$Proj_W(z) = \omega(z, v)u - \omega(z, u)v.$$

Hence $ker(Proj_W) = W^{\perp}$ is a rational subspace in $\mathbb{R}^4$ and we choose a basis $p, q \in W^{\perp}$ so that the vectors $p, q$ generate the abelian group $\mathbb{Z}^4 \cap W^{\perp}$. The vectors $u, v, p, q$ generate the group $\mathbb{Z}^4$ since the symplectic projection of $\mathbb{Z}^4$ to $W$ and $W^{\perp}$ is contained in $\mathbb{Z}^4 \cap W$ and $\mathbb{Z}^4 \cap W^{\perp}$ respectively. It follows that $\omega(p, q) = 1$ and $x, y, p, q$ form an integer symplectic basis in $\mathbb{R}^4$. ☐

**Lemma 2.6.** *Suppose that $u \in \mathbb{R}^{2n}$ is a nonzero vector. Then there exists $\gamma \in \Gamma$ such that no coordinate of $\gamma(u)$ is zero. If $u, v \in \mathbb{R}^{2n}$ are such that $\omega(u, v) > 0$ then there exists $\gamma \in \Gamma$ such that $\omega_j(\gamma(u), \gamma(v)) \neq 0$ for each $j = 1, ..., n$.*

*Proof.* The projection $Sp(2n) \to \mathbb{R}^{2n} - 0$ given by $g \mapsto g(\overrightarrow{e_1})$ is a real algebraic morphism. The union

$$\bigcup_{j=1}^{n} \{x \in \mathbb{R}^{2n} : x_j = 0\}$$

is a proper (real) algebraic subvariety, hence its inverse image $Y$ in $G = Sp(2n, \mathbb{R})$ is again a proper algebraic subvariety. Since $\Gamma$ is Zariski dense in $G$ we conclude that

$Y$ is not $\Gamma$-invariant. The proof of the second assertion is similar and is left to the reader. $\square$

Recall that $\Omega$ denotes the vector bundle over the Teichmüller space $T(S)$ where the fiber over a point $\tau$ consists of abelian differentials on the Riemann surface $S_\tau$. We have the period map $P : \Omega \to H^1(S, \mathbb{C})$, $\alpha \mapsto \chi_\alpha$. Let $0_\omega$ denote the image of the zero section of $\Omega$.

The following theorem is a variation on the Hejhal-Thurston Holonomy theorem, see [Thu81], [Hej75] and [ECG87], [Gol87]. See also [GKM00, Section 12] for an alternative argument.

**Theorem 2.7.** *(The Holonomy Theorem.) The restriction mapping $P : \Omega - 0_\Omega \to H^1(S, \mathbb{C})$ is open.*

*Proof.* To prove this theorem we need a geometric description of the nonzero abelian differentials $\alpha$. Each $\alpha$ determines a singular Euclidean structure on the surface $S$ with isolated singularities at zeroes of $\alpha$, see [Str84]. The local charts for this structure are given by the branches of the indefinite integral

$$F(z) = \int_{z_0}^{z} \alpha$$

where $z_0 \in S$ is a base-point. If $\alpha$ vanishes (at the order $k$) at a point $0 \in S$ then the local chart at $0$ is a $k$-fold ramified covering $z \mapsto z^{k+1}$. The transition maps of the flat atlas on $S - Zero(\alpha)$ are Euclidean translations. Vice-versa, suppose that we are given a flat structure on the (topological) surface $S$ where the local charts have the form $z \mapsto z^{k+1}$, $k \geq 0$, and the transition maps away from the branch-points are Euclidean translations. This structure canonically defines a complex structure on $S$ together with an abelian differential $\alpha$ obtained by the pull-back of $dz$ via the local charts. Every such singular Euclidean structure gives rise to a *developing map* $dev : \tilde{S} \to \mathbb{C}$ where $\tilde{S}$ is the universal abelian covering of $S$ and $H := H_1(S, \mathbb{Z})$ acts on $\tilde{S}$ by deck-transformations. The mapping $dev$ is $\chi$-equivariant where $\chi : H_1(S, \mathbb{Z}) \to \mathbb{C}$ is the holonomy of the above structure (it coincides with the character of the associated abelian differential). The space $E(S)$ of the above Euclidean structures has a natural topology: the topology of uniform convergence on compacts of the developing mappings. It is easy to see that with this topology the natural bijection $E(S) \to \Omega - 0_\Omega$ is a homeomorphism.

We now prove the holonomy theorem. Let $\sigma \in E(S)$ be a singular Euclidean structure with the period character $\chi$. Let $f : \tilde{S} \to \mathbb{C}$ denote the developing mapping of $\sigma$. Suppose that $\chi_n : H_1(S, \mathbb{Z}) \to \mathbb{C}$ is a sequence of characters converging to $\chi$. Our goal is to find (for large $n$'s) points $\sigma_n \in E(S)$ so that their period characters are $\chi_n$ and $\lim_n \sigma_n = \sigma$.

Choose a triangulation $T$ of $S$ so that each edge is a geodesic arc with respect to the singular Euclidean structure $\sigma$ and each simplex is contained in a coordinate neighborhood of $\sigma$. We will assume that each singular point of $\sigma$ is a vertex of this triangulation. Lift this triangulation to a triangulation $\tilde{T}$ of $\tilde{S}$ of $S$. Pick a finite collection $\Delta_1, ..., \Delta_m$ of 2-simplices in $\tilde{T}$, one for each $H$-orbit. Let $g_i, i = 1, ..., N$, be the elements of the deck-transformation group $H$, so that $g_i(\cup_j \Delta_j) \cap \cup_j \Delta_j \neq \emptyset$. Let $C$ be a compact subset of $\tilde{S}$ whose interior contains both $D := \cup_j \Delta_j$ and its images under $g_i$'s. For each $\chi_n$ construct a continuous $\chi_n$-equivariant mapping $f_n : D \to \mathbb{C}$ so that:

(i) $f_n$ maps each 2-simplex homeomorphically to a Euclidean 2-simplex in $\mathbb{C}$.

(ii) $f_n$'s converge to $f|D$ uniformly on compacts.

Finally, extend each $f_n$ to a $\chi_n$-equivariant mapping $f_n : \tilde{S} \to \mathbb{C}$. It remains to show that each mapping $f_n$ is a local homeomorphism for large $n$ (away from the singular points) and is the $k(x)$-fold ramified cover at each point where $f$ is such a cover. It suffices to check this for points in $D$.

(a) If $x \in int(C)$ belongs to the interior of a 2-simplex in $\cup_i g_i D$, then the claim follows since each $f_n$ is homeomorphism on each simplex.

(b) Suppose $x$ belongs to the interior of a common arc $\eta$ of two 2-simplices $\Delta, \Delta'$ in $\cup_i g_i D$. Since $f$ is a local homeomorphism, $f(\Delta), f(\Delta')$ lie (locally) on different sides of the segment $f(\eta) \subset \mathbb{C}$. Therefore the same holds for $f_n$ if $n$ is sufficiently large. So, $f_n$ does not "fold" along the arc $\eta$ and is a local homeomorphism at $x$.

(c) Lastly, if $x$ is a vertex of a simplex, then the degree of $f$ at $x$ equals $k(x)$, hence for large $n$, the degree of $f_n$ at $x$ is $k(x)$ and it follows from (b) that $f_n$ is a $k(x)$-fold ramified cover at $x$.

Equivariance of $f_n$'s implies that they converge to $f$ uniformly on compacts. ☐

**Line stabilizers in** $Sp(2n)$. In what follows we will need a description of the subgroups $B$ in $Sp(2n)$ with invariant line $L \subset \mathbb{R}^{2n}$. Let $V \subset \mathbb{R}^{2n}$ be a 2-dimensional symplectic subspace containing $L$. To describe the structure of the group $B$ we have to recall several facts about the *Heisenberg groups*. Consider the $2n-2$-dimensional symplectic vector space $(V, \omega|V)$. The Heisenberg group corresponding to this data is the $2n-1$-dimensional Lie group which fits into short exact sequence

$$1 \to \mathbb{R} \to H_{2n-1} \to V \to 1$$

where $V$ is treated as the abelian (additive) Lie group. The normal subgroup $\mathbb{R}$ is central in $H_{2n-1}$. If $g, h \in H_{2n-1}$ project to the vectors $x, y \in V$ then $[g,h] = \omega(x,y) \in \mathbb{R}$. The *Heisenberg multiplication* on this group is the action of the (multiplicative) group $\mathbb{R}_+$ on $H_{2n-1}$ so that $t \in \mathbb{R}_+$ acts on the center $\mathbb{R} \subset H_{2n-1}$ via multiplication by $t^2$ and acts on $V$ via multiplication by $t$. Given this one defines the Lie group $H_{2n-1} \rtimes \mathbb{R}_+$ where $\mathbb{R}_+$ acts on the Heisenberg group via Heisenberg dilation. One can show that the resulting Lie group acts simply-transitively on the complex-hyperbolic space $\mathbb{CH}^n$ of the complex dimension $n$, however we will not need this fact. What we are going to use is the following elementary

**Lemma 2.8.** *The $2n$-dimensional Lie group $CH_{2n} := H_{2n-1} \rtimes \mathbb{R}_+$ contains no lattices.*

*Proof.* Suppose that $\Delta$ is a discrete subgroup of $H_{2n-1} \rtimes \mathbb{R}_+$ with the quotient $M = H_{2n-1} \rtimes \mathbb{R}_+/\Delta$. The unit speed flow on $H_{2n-1} \rtimes \mathbb{R}_+$ along the $\mathbb{R}_+$-factor is volume-expanding and $\Delta$-invariant. Hence it yields a volume-expanding flow on $M$. It follows that $vol(M) = \infty$. ☐

We are now ready to describe the structure of $B$. The group $B$ preserves the span $L + V$ of $L$ and $V$, the projection $L + V \to V$ along the $L$-factor transfers the action of $B$ to the action of the symplectic group $Sp(2n-2)$ on $V$. The kernel of the homomorphism $B \to Sp(2n-2)$ is the group $CH_{2n} = H_{2n-1} \rtimes \mathbb{R}_+$. Here the $\mathbb{R}_+$-factor acts trivially on $V$ and as the maximal torus in $Sp(2) \curvearrowright V^\perp$ preserving $L$. The center $\mathbb{R}$ of the Heisenberg group $H_{2n-1}$ is the kernel of the action $B \curvearrowright L + V$. The whole group $B$ splits as the semidirect product $CH_{2n} \rtimes Sp(2n-2)$ where $Sp(2n-2)$ acts by conjugation on the $V$-factor of $H_{2n-1}$ the same way it acts on the vector space $V$. The subgroup $Sp(2n-2)$ commutes with the subgroup $B_0 := \mathbb{R} \rtimes \mathbb{R}_+$, where $\mathbb{R}$ is the center of $H_{2n-1}$. The proof of these assertions is a straightforward linear algebra computation and is left to the reader.

**Definition 2.9.** The group $H_{2n-1}$ is called the Heisenberg group associated to the flag $(V, L)$ in $(\mathbb{R}^{2n}, \omega)$, where $V$ is a 2-dimensional symplectic subspace and $L$ is a line.

# 3. Ratner's Theorem

Let $G$ be a reductive algebraic Lie group and $U \subset G$ be a connected subgroup generated by unipotent elements[5]. Suppose $\Gamma \subset G$ is a lattice, i.e. a discrete subgroup with the quotient $\Gamma\backslash G$ of finite volume (with respect to the left-invariant measure on $G$). Important examples of lattices in algebraic Lie groups $G$ defined over $\mathbb{Q}$ are given by the *arithmetic groups*, i.e. subgroups commensurable with $G_{\mathbb{Z}}$, the group of integer points in $G$. The group $U$ acts by right multiplications on the manifold $M = \Gamma\backslash G$. On the other hand, the group $\Gamma$ acts by the left multiplication on the manifold $X = G/U$. Given $g \in G$ we let $[g]$ denote its projection to $M$.

**Theorem 3.1.** *(M.Ratner, see [Rat91, Rat95].) Under the above conditions for each $g \in G$ the closure (in the classical topology) of $[g]U$ in $M$ is "algebraic". More precisely, there exists a Lie subgroup $H \subset G$ so that*

- $\overline{[g]U} = [g]H$.
- $H^g \cap \Gamma$ *is a lattice in* $H^g := gHg^{-1}$.

This result is known as *Raghunathan's Conjecture.* Special cases of this conjecture were proven before Ratner by Dani [Dan86] and Margulis [Mar89]. Actually, Ratner's theorem does more than what is stated above: it describes $\Gamma$-invariant ergodic measures on $M$ and uses the ergodic framework to prove Raghunathan's Conjecture. We note that the group $H$ may not be connected, however if $H(0)$ is the connected component of the identity in $H$ then $H(0) \cap \Gamma$ is still a lattice in $H(0)$. Below we reformulate Ratner's theorem in terms of the action of $\Gamma$ on $G/U$. Let $g \in G$ be the element which projects to $x$. Then

$$\overline{\Gamma g U} = \Gamma g H = \Gamma H^g g.$$

Hence

[5] I.e. elements whose adjoint action on the Lie algebra of $G$ is unipotent.

**Corollary 3.2.** *Suppose that $X := G/U$ and $x = gU \in X$. Then the closure of $\Gamma x$ in $X$ equals the $H^g$-orbit of $x$ in $X$, where $H^g$ is a Lie subgroup of $G$ so that $H^g \cap \Gamma$ is a lattice in $H$.*

Note that $gUg^{-1} = G_x$ is the stabilizer of $x$ in $G$. By taking the connected component of the identity we get:

**Corollary 3.3.** *The closure $\overline{\Gamma x}$ in $X$ contains the orbit $\Gamma F_x x$, where $F_x$ is a connected Lie subgroup of $G$ which contains $G_x$ and $\Gamma \cap F_x$ is a lattice in $F_x$.*

Ratner's theorem gives a tool for describing the *exceptional* orbits for the $\Gamma$-action on $X$, still, some work has to be done by analyzing various Lie subgroups $F_x \subset G$ which might appear.

We now specialize to the case $G = Sp(2n, \mathbb{R})$, the automorphism group of the standard symplectic form $\omega$:

$$\omega(a_1, b_1, ..., a_n, b_n) = \sum_{j=1}^{n} a_j b_{j+1} - a_{j+1} b_j,$$

and $X \subset (\mathbb{R}^{2n})^2$ consists of the pairs of vectors $u, v$ such that $\omega(u, v) = 1$.

The stabilizer $U$ of the point $(\overrightarrow{e_1}, \overrightarrow{e_2}) \in X$ is the group $Sp(2n-2, \mathbb{R})$ embedded in $G$ as the subgroup of block-diagonal matrices:

$$\begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ 0 & 1 & 0 \ldots 0 \\ 0 & 0 & Sp(2n-2) \end{bmatrix}.$$

Although the group $U$ is not unipotent itself, it is generated by unipotent elements, hence Ratner's theorem applies. Recall that $\Gamma = Sp(2n, \mathbb{Z})$ is a lattice in $G$, we also note that $\Gamma \cap U$ is a lattice in $U$ as well. Through the rest of the paper we will use the notation $U' = G_\chi$ to denote the stabilizer of the point $\chi \in X$.

**Connected Lie subgroups of $G$ containing $U$.** To apply Ratner's theorem we have to know which Lie subgroups of $G$ contain the Lie subgroup $U'$ (conjugate to $U$). We will list all *maximal* subgroups containing $U$. Recall that a connected Lie subgroup $G_1 \subset G$ is said to be *maximal* if it is not contained in any proper connected Lie subgroup $G_2 \subset G$. We will use a classification of maximal subgroups of classical complex Lie groups done by Dynkin [Dyn52] (the real case was carried out by Karpelevich [Kar55]). In our case the classification of maximal subgroups of $Sp(2n, \mathbb{C})$ easily implies (via the complexification) the needed result for the group of real points $Sp(2n, \mathbb{R})$.

**Theorem 3.4.** *(E. Dynkin, see [GOV94, Ch. 6, Theorems 3.1, 3.2].) Suppose that $H \subset Sp(2n, \mathbb{C})$ is a maximal connected Lie subgroup. Then one of the following holds:*

*(a) $H$ is a maximal parabolic subgroup of $Sp(2n, \mathbb{C})$.*

*(b) $H$ is conjugate to the subgroup $Sp(k, \mathbb{C}) \times Sp(N-k, \mathbb{C})$.*

*(c) $H$ is conjugate to $Sp(s, \mathbb{C}) \otimes SO(t, \mathbb{C})$ where $2n = st$, $s \geq 2, t \geq 3$, $t \neq 4$ or $s = 2, t = 4$.*

Note that in our situation $H$ contains $U \cong Sp(2n-2,\mathbb{C})$, hence we can ignore the case (c). In the case (b) the only possibility is that $F$ is conjugate to the group $Sp(2,\mathbb{C}) \times Sp(2n-2,\mathbb{C})$. In the case (a) the group $H$ has to preserve a complex line in $\mathbb{C}^{2n}$.

We let $\chi = (u, v)$, $u, v \in \mathbb{R}^{2n}$ are so that $\omega(u, v) = 1$. Let $V$ denote $Span(u, v)$. The group $U' = G_\chi \cong Sp(2n-2,\mathbb{R})$ fixes the vectors $u, v$. This group also acts as the full group of linear symplectic automorphisms of the symplectic complement $V^\perp \cong \mathbb{R}^{2n-2}$ of $V$. The maximal subgroups of $G$ which contain $U'$ are:

1. The group $H = Sp(V) \times U'$, where $Sp(V) \cong Sp(2,\mathbb{R})$ is the group of automorphisms of $V$. (The semi-simple case.)

2. The maximal parabolic subgroup $H$ of $G$ which has an invariant line $L \subset \mathbb{R}^{2n}$. (The non semi-simple case.) We note that in this case $L$ is necessarily contained in $V$.

Recall that in each case we have to find connected subgroups $F_\chi \subset H$ which contain $G_\chi = U'$ and such that $F_\chi \cap \Gamma$ is a lattice in $F_\chi$.

## 4. The semi-simple case.

In this case the group $F_\chi \subset Sp(V) \times U'$ containing $U'$ splits as the direct product

$$F_\chi \cong S \times Sp(2n-2,\mathbb{R})$$

where $S \subset Sp(2,\mathbb{R})$. We will need the following

**Theorem 4.1.** *(See [Mar91].) Suppose that $F_1, F_2$ are simple real algebraic Lie groups so that their complexifications do not have isomorphic Lie algebras. Then any lattice $\Delta \subset F_1 \times F_2$ is reducible, i.e. $\Delta \cap F_i$ is a lattice for each $i = 1, 2$.*

We also recall (see [Rag72, Corollary 8.28]):

**Theorem 4.2.** *(M. Raghunathan, J. Wolf) Suppose that $F$ is a connected Lie group whose semisimple part contains no compact factors acting trivially on the radical $R(F)$ of $F$. Then each lattice $\Delta \subset F$ intersects the radical $R(F)$ along a sublattice in $R(F)$. Moreover, the projection of $\Delta$ to $F/R(F)$ is a lattice in this Lie group.*

In our case the group $S$ is either solvable or equals $Sp(2)$, hence combining the two above theorems we conclude that either:

(i) $\Gamma \cap U'$ is a lattice, or

(ii) $n = 2$, $F_\chi \cong Sp(2) \times Sp(2)$ and $\Gamma \cap U'$ is not a lattice[6].

We consider the case (ii) in §6 and here we will concentrate on the more difficult *generic case*, when $\Gamma \cap U'$ is a lattice.

By the Borel density theorem (see e.g. [Zim84]) the intersection $U' \cap \Gamma$ is Zariski dense in $U'$, in particular it contains a diagonalizable matrix $A \in Sp(2n)$ which has the

[6]We note that the group $Sp(2) \times Sp(2)$ contains irreducible lattices, namely the Hilbert modular groups.

eigenvalue 1 of the multiplicity 2. Since $A$ has rational entries, the kernel $ker(A-I)$ is a rational subspace. We recall that the group $U'$ is the pointwise stabilizer of the linear subspace $Span(u,v)$ of $\mathbb{R}^{2n}$ spanned by $u = Re(\chi), v = Im(\chi)$. Hence $Span(u,v)$ is a rational subspace of $\mathbb{R}^{2n}$.

Lemma 2.2 thus implies that the image $A_\chi$ of the character $\chi : H_1(S,\mathbb{Z}) \to \mathbb{C}$ is a discrete subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$. Moreover, without loss of generality we can assume that $A_\chi$ is the standard integer lattice in $\mathbb{C}$ (see Section 2). This might require scaling $\omega(\chi)$ by a positive real number.

We recall that $\omega(u,v) > 0$, where $\chi = (u,v)$,

$$u = (a_1, b_1, ..., a_n, b_n), v = (c_1, d_1, ..., c_n, d_n), a_j, b_j, c_j, d_j \in \mathbb{Z}.$$

**Lemma 4.3.** *There exists $\gamma \in \Gamma$ so that the character $\gamma\chi = \chi' = (u', v')$ satisfies:*

*(i) $\omega_1(u', v') > 0$.*

*(ii) $\omega_j(u', v') = 0$ for each $j \geq 2$ and, moreover,*

$$M_j(\chi') = \begin{bmatrix} a'_j & b'_j \\ c'_j & d'_j \end{bmatrix} = \begin{bmatrix} a'_j & 0 \\ 0 & 0 \end{bmatrix}, a'_j \geq 0.$$

*Proof.* We recall that without loss of generality we can start with $(u,v)$ so that for each $j = 1, ..., n$, $\omega_j(u,v) \neq 0$ or

$$M_j(\chi) = \begin{bmatrix} a_j & 0 \\ 0 & 0 \end{bmatrix}.$$

(Of course, in the beginning of the induction the latter case does not occur.) After multiplying $(u,v)$ by a matrix in $\Gamma \cap Sp(2) \times ... \times Sp(2)$ we can assume that each matrix

$$M_j(\chi) = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} = \begin{bmatrix} a_j & 0 \\ 0 & d_j \end{bmatrix}$$

is diagonal. We now argue inductively. Suppose that $j \in \{2, .., n\}$. We let $d'_j := d_j / lcd(|d_1|, |d_j|)$. Then there are integers $\alpha_j, \beta_j$ such that $\alpha_j d'_j - \beta_j d'_1 = 1$. It follows that

$$\begin{bmatrix} \alpha_j & 0 & \beta_j & 0 \\ a_j & d'_j & -a_1 & -d'_1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \\ a_j & 0 \\ 0 & d_j \end{bmatrix} = \begin{bmatrix} \alpha_j a_1 + \beta_j a_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that the row vectors $p, q$ of the first matrix in the above formula are such that $\omega(p,q) = 1$. Hence, according to Lemma 2.5, there exists a matrix

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ \alpha & 0 & \beta & 0 \\ a_2 & d'_2 & -a_1 & -d'_1 \end{bmatrix}$$

which belongs to $Sp(4,\mathbb{Z})$. We extend the matrix $A$ to a matrix $g \in Sp(2n,\mathbb{Z})$ which preserves all the coordinates except $a_1, b_1$ and $a_j, b_j$. Then the character $\chi' = g\chi$ has $\omega_j(\chi') = 0$. Continuing inductively we find $h \in \Gamma$ so that the character $h\chi$ satisfies:

$$\omega_j(h\chi) = 0, j = 2, 3, ..., n.$$

Note that $\omega_1(h\chi) = \omega(h\chi) = \omega(\chi) > 0$. Recall that $Image(\chi) = \mathbb{Z} \times \mathbb{Z}$. Hence $b'_1 = \chi(y_1) = 1$ (since all other generators $x_1, x_2, y_2, ...$ of $H_1(S, \mathbb{Z})$ are mapped by $h\chi$ to the real numbers. Finally, to get $\gamma\chi$ as required by lemma we multiply $h\chi$ by a diagonal symplectic matrix with diagonal entries in $\{\pm 1\}$ to get $a_j \geq 0$ for $j = 2, ..., n$. □

We again use the notation $\chi$ for the character $\chi'$ obtained in the previous lemma.

**Lemma 4.4.** *There exists $\gamma \in \Gamma$ so that that the character $\gamma\chi$ satisfies:*

*(i)*

$$M_1(\gamma\chi) = \begin{bmatrix} a_1 = \omega(\chi) & 0 \\ 0 & 1 \end{bmatrix}.$$

*(ii) For each $j \geq 2$,*

$$M_j(\gamma\chi') = \begin{bmatrix} a'_j & 0 \\ 0 & 0 \end{bmatrix}, 0 \leq a'_j < a_1.$$

*Proof.* For each $j \geq 2$ there exists $t_j \in \mathbb{Z}$ so that $0 \leq a'_j := a_j - t_j a_1 < a_1$. Then form the symplectic matrix

$$\gamma = \left[\begin{array}{cc|cc|cc|c|cc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & t_2 & 0 & t_3 & \dots & 0 & t_n \\ -t_2 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -t_3 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -t_n & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{array}\right].$$

The reader will note that this matrix belongs to the Heisenberg subgroup of $Sp(2n)$ associated to the flag $(Span(e_1, e_2), Span(e_2))$. Then $\gamma\chi$ has the requires properties:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t_j \\ -t_j & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & 1 \\ a_j & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & 1 \\ a'_j & 0 \\ 0 & 0 \end{bmatrix}. \quad \square$$

We note that for some $j$ we might have $a'_j = 0$. However, since $\omega(\chi) \geq 2 = Area(\mathbb{C}/\mathbb{Z}^2)$ we conclude that there exists at least one $j \geq 2$ so that $a_j > 0$. Rename this index $j$ to make it equal to 2. Rename $\chi' = \gamma\chi$ back to $\chi$ and $a'_j$ back to $a_j$, $j = 2, ..., n$.

**Lemma 4.5.** *There exists $\gamma \in \Gamma$ so that that the character $\gamma\chi$ satisfies: (i)*

$$M_1(\gamma\chi) = \begin{bmatrix} a_1 = \omega(\chi) & 0 \\ 0 & 1 \end{bmatrix}.$$

*(ii) For each $j \geq 2$,*

$$M_j(\gamma\chi) = \begin{bmatrix} a'_j & 0 \\ 0 & 0 \end{bmatrix}, 0 < a'_j < a_1.$$

*Proof.* The required matrix $\gamma$ belongs to the Heisenberg group associated to the flag $(Span(e_3, e_4), Span(e_4))$. For each $j$ such that $a_j \neq 0$ the multiplication by $\gamma$ will not change $a_j$ at all. Suppose that $j \geq 3$, $a_j = 0$. We describe the case $j = 3$ and $n = 3$, the general case is done inductively.

$$\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\gamma M(\chi) = \begin{bmatrix} a_1 & 0 \\ 0 & 1 \\ a_2 & 0 \\ 0 & 0 \\ a_2 & 0 \\ 0 & 0 \end{bmatrix}. \quad \square$$

## 5. The non-semisimple case

In this section we analyze lattices in the non-semisimple Lie subgroups $F$ of $Sp(2n, \mathbb{R})$ that contain $Sp(2n-2, \mathbb{R})$. Recall that each maximal non-semisimple subgroup $B$ of $Sp(2n, \mathbb{R})$ containing $Sp(2n-2, \mathbb{R})$, preserves a line $L \subset V^{\perp}$, where $V = \mathbb{R}^{2n-2}$ is the symplectic subspace invariant under $Sp(2n-2)$. The group $B$ splits as semi-direct product $CH_{2n} \rtimes Sp(2n-2)$, where $CH_{2n} = H_{2n-1} \rtimes \mathbb{R}_+$ and $H_{2n-1}$ is the $2n-1$-dimensional Heisenberg group, see §2.

Now suppose that $F = F_\chi \subset B$ is a Lie subgroup containing $Sp(2n-2)$. Since $Sp(2n-2)$ acts transitively on $V - 0$, the subgroup $F$ has to be one of the following:

(a) $F = B$.

(b) $F = H_{2n-1} \rtimes Sp(2n-2)$.

(c) $F = A \times Sp(2n-2)$ where $A \subset B_0 = \mathbb{R} \rtimes \mathbb{R}_+$.

If $\Delta = F \cap Sp(2n, \mathbb{Z}) \subset F$ is a lattice then its intersection with the subgroup $CH_{2n}$ (case (a)), $H_{2n-1}$ (case (b)) and $A$ (case (c)) is again a lattice (see Theorem 4.2). The first case is impossible by Lemma 2.8. In the third case the intersection $\Delta \cap Sp(2n-2)$ is a lattice as well and we are therefore reduced to the discussion in §5. This leaves us with the case (b), when $Sp(2n, \mathbb{Z}) \cap H_{2n-1}$ is a lattice. Note that there are lattices $\Delta \subset H_{2n-1} \rtimes Sp(2n-2)$ whose intersection with any conjugate of $Sp(2n-2)$ is not a lattice, we leave it to the reader to construct such examples.

Suppose now that $\chi \in X$ is a character (with the real part $u$ and the imaginary part $v$) so that the closure of the orbit $\Gamma\chi$ contains the orbit $F_\chi\chi$ where $F_\chi \cong H_{2n-1} \rtimes Sp(2n-2)$ fixes a line $L$ in $Span(u, v)$. According to the Remark 2.1 it suffices to consider the case $L = Span(u)$. Applying an element $\gamma \in \Gamma$ we can adjust the pair $(u, v)$ so that the vector $u = (a_1, b_1, ..., a_n, b_n)$ has no zero coordinates (see Lemma

2.6). The group $H_{2n-1}$ acts transitively on the set of vectors $v \in \mathbb{R}^{2n}$ satisfying $\omega(u,v) = 1$. Hence we can find $h \in H_{2n-1}$ so that

$$h(v) = \frac{1}{\omega(u,v)}(...., -b_j, a_j, ....).$$

Hence $\omega_j(u, h(v)) = \omega_j(h(u), h(v)) > 0$ for each $j = 1, ..., n$. Since $\overline{\Gamma\chi}$ contains the orbit $F_\chi \chi$, there exists an element $\gamma \in \Gamma$ such that $\omega_j(\gamma(u), \gamma(v)) > 0$ for each $j = 1, ..., n$. According to Theorem 2.3 the character $\chi$ belongs to the subset $\Sigma \subset X$ of characters of abelian differentials.

# 6. The genus 2 case

In this section we consider the case when the surface $S$ has genus $n = 2$, $\chi = (u,v) \in X$ is a point such that $Span(u,v)$ is not a rational subspace of $\mathbb{R}^4$.

**Proposition 6.1.** *There exists $\gamma \in Sp(4,\mathbb{Z})$ such that $\gamma(u) = u', \gamma(v) = v'$ and $\omega_j(u', v') > 0$ for $j = 1, 2$.*

*Proof.* To prove this proposition it will be convenient to rescale the pair of vectors $(u, v)$ so that $\omega(u,v) > 0$ and $\omega_1(u,v) = 1$ (thus $\omega(u,v)$ is no longer equal to 1).

Let

$$u = (a_1, b_1, a_2, b_2), v = (c_1, d_1, c_2, d_2),$$

$$u_i := (a_i, b_i), v_i := (c_i, d_i), i = 1, 2.$$

Without loss of generality (using Lemmas 2.1, 2.6) we can assume that $u_1 = (1,0)$, $v_1 = (0,1)$ and $\omega_2(u,v) \neq 0$. Since the $Span(u,v)$ is not a rational subspace, either $u_2 \notin \mathbb{Q}^2$ or $v_2 \notin \mathbb{Q}^2$. We consider the former case since the latter is similar (in the matrix $\gamma$ below interchange the first two rows and columns). Assume $\omega_2(u,v) < 0$. For $p, q \in \mathbb{Z}$ the matrix

$$\gamma = \begin{bmatrix} 1 & 0 & -q & p \\ 0 & 1 & 0 & 0 \\ 0 & p & 1 & 0 \\ 0 & q & 0 & 1 \end{bmatrix}$$

belongs to $Sp(4,\mathbb{Z})$. Multiplying this matrix by the column vectors $u, v$ we get

$$(u', v') = (\gamma u, \gamma v) = \left[\begin{array}{c|c} * & * \\ * & * \\ a_2 & c_2 + p \\ b_2 & d_2 + q \end{array}\right]$$

and

$$\omega_2(u', v') = \omega_2(u,v) + \begin{vmatrix} a_2 & p \\ b_2 & q \end{vmatrix} = \omega_2(u,v) + \delta.$$

Since $\omega(u,v) = \omega(u', v')$ we also have $\omega_1(u', v') = \omega_1(u,v) - \delta$. Thus to have $\omega_j(u', v') > 0$ for $j = 1, 2$ we need:

$$-\omega_2(u,v) < \delta < \omega_1(u,v) = 1. \tag{6.2}$$

(We recall that $\omega(u,v) > 0$, hence $-\omega_2(u,v) < \omega_1(u,v)$.) The conditions (6.2) mean that the vector $(p,q)$ belongs to an open nonempty strip $\sigma$ in $\mathbb{R}^2$ which is parallel to the line $L = Span(u_2)$. Since the line $L$ is irrational, the strip $\sigma$ contains infinitely many points $(p,q)$ from the integer lattice $\mathbb{Z}^2$. This means that there exists a matrix $\gamma$ as above so that $\omega_j(u', v') > 0$ for $j = 1, 2$. □

According to Theorem 2.3, the above proposition implies that $\chi$ is the period character of an abelian differential and thus we are done. □

# 7. Meromorphic differentials

**Theorem 7.1.** *Suppose that $\chi$ is a nonzero character in $H^1(S, \mathbb{C})$ which does not satisfy either Obstruction 1 or Obstruction 2. Then there is a complex structure $\tau$ on $S$ and a meromorphic differential $\alpha$ with a single simple pole on $S_\tau$ so that $\chi$ is the character of $\alpha$.*

*Proof.* Case A. The vectors $u$ and $v$ are linearly independent. The group $Sp(2n, \mathbb{R})$ acts transitively on the collection $Y$ of pairs of vectors $u, v \in \mathbb{R}^{2n}$ so that $\omega(u,v) = 0$ and $u \wedge v \neq 0$. Thus (since $\Gamma = Sp(2n, \mathbb{Z})$ is Zariski dense in $Sp(2n, \mathbb{R})$) there exists $\gamma \in \Gamma$ such that $\chi' = \gamma\chi$ satisfies: $\omega_j(\chi') \neq 0$ for each $j = 1, ..., n$. If each $\omega_j(\chi') > 0$ then $\chi$ is the character of an abelian differential and there is nothing to prove. Hence (after relabelling $j$'s) we get: $\omega_1(\chi') < 0$ and $\omega_j(\chi') \neq 0$, $j = 2, ..., n$. Set $\chi := \chi'$.

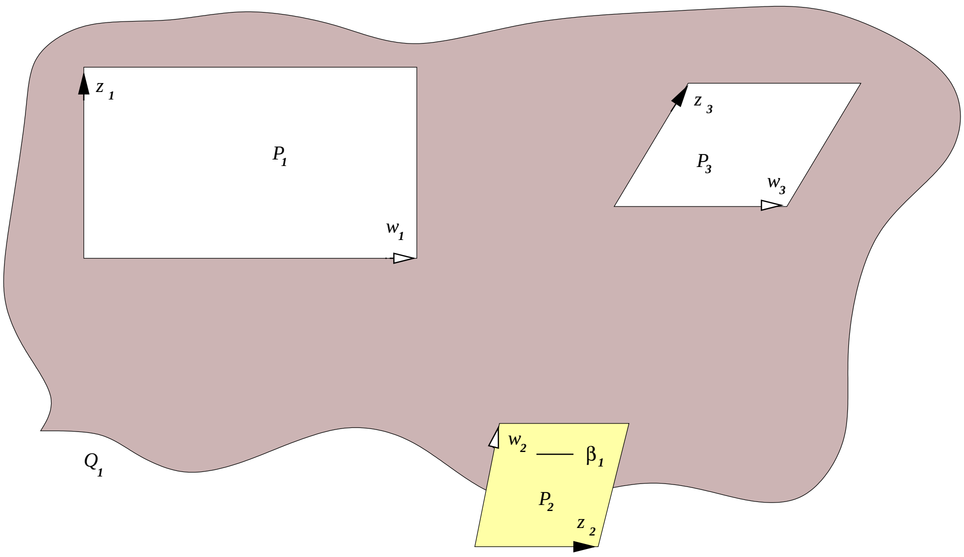


Figure 3:

We argue similarly to the proof of Theorem 2.3. Consider the fundamental parallelogram $P_1 \subset \mathbb{C}$ for the discrete group generated by the columns $z_1, w_1$ of the matrix $M_1(\chi')$. Let $Q_1$ denote the closure of the *exterior* of $P_1$ in $\mathbb{S}^2$. Note that topologically $Q_1$ is still a parallelogram: its edges are the edges of $P_1$. Identifying the opposite sides of $Q_1$ by $z_1, w_1$ we get a marked torus $T_1$ with a standard (symplectic) system of generators $x_1, y_1$, branched projective structure and an orientation-preserving developing mapping to $\mathbb{S}^2$ whose holonomy is the homomorphism $\chi_1$ which sends

$x_1 \to z_1, y_1 \to w_1$. (Here we identify a vector in $\mathbb{C}$ with the corresponding translation.) Taking pull-back of the form $dz$ on $\mathbb{C}$ we get a meromorphic differential on $T_1$ with the single simple pole (corresponding to the point $\infty \in Q_1$) and the period character $\chi_1$. We now extend this to the rest of the surface $S$. If $j \geq 2$ is such that $\omega_j(\chi) > 0$ then similarly to the proof of Theorem 2.3 we add to $T_1$ the flat torus $T_j$ obtained by identifying the sides of a fundamental parallelogram for the translation group generated by the columns of $M_j(\chi)$. If $\omega_j(\chi) < 0$ we pick a fundamental parallelogram $P_j$ so that it is disjoint from the $P_i$'s ($1 \leq i \leq n$, $i \neq j$). Remove the interior of $P_j$ from $Q_1$ and identify the opposite sides of $P_j$ via translations. See Figure 3.

As the result we get an oriented surface $S$, a developing map to $\mathbb{S}^2$ which is $\chi$-equivariant. The meromorphic differential on $S$ is obtained via pull-back of $dz$ from $\mathbb{C}$. Its only pole corresponds to the point on the torus $T_1$ which maps to $\infty$ under the developing map.

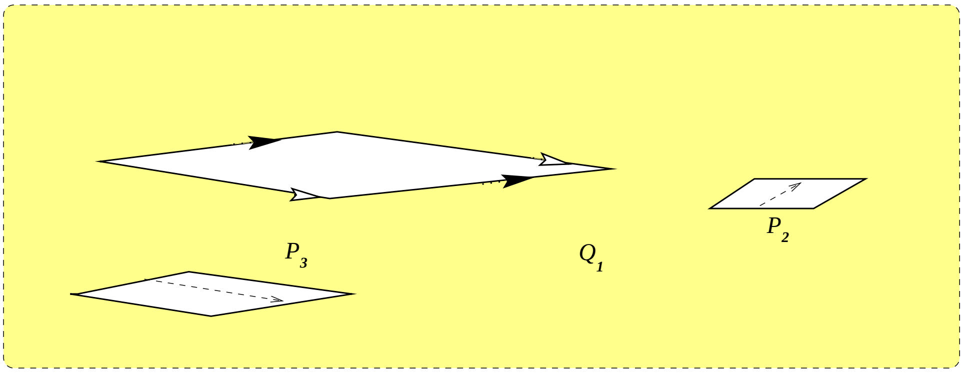


Figure 4:

Case B. Let $u$ and $v$ be linearly dependent. It suffices to consider the case $u \neq 0$ (otherwise replace $\chi$ by $\sqrt{-1}\chi$). Using Zariski density of $\Gamma$ in $Sp(2n, \mathbb{R})$ (the latter acts transitively on $\mathbb{R}^{2n} - 0$) choose $\gamma \in \Gamma$ so that no coordinate of $\gamma(u)$ is zero and let $\chi := \gamma\chi$. We now argue analogously to the Case A. Let $z_1, w_2$ denote the columns of the matrix $M_1(\chi)$. Let $P_1$ denote the convex hull of the set $0, z_1, w_1, z_1 + w_1$. We will think of $P_1$ as a degenerate parallelogram with the edges $[0, z_1], [0, w_1], [z_1, z_1 + w_1], [w_1, z_1 + w_1]$. Now cut $\mathbb{S}^2$ open along $P_1$ and denote the result $Q_1$, it is homeomorphic to a parallelogram, identification of the opposite edges via translations by $z_1, w_1$ yields the torus $T_1$. To reconstruct the rest of the surface $S$ we choose disjoint degenerate "fundamental parallelograms" $P_j$ for the groups generated by the translations $z_j, w_j$, cut $Q_1$ open along the $P_j$'s ($j \geq 2$) and get $S$ by identifying the opposite edges on each cut. See Figure 4. $\square$

**Remark 7.2.** *We note that the branched projective structures $\sigma$ associated to the meromorphic differentials constructed in the above theorem have the branching degree $deg(\sigma) = 2g$.*

We will now prove the upper bound on the degree of branching of the projective structures with the holonomy in the translation subgroup $\mathbb{C}$ of $PSL(2, \mathbb{C})$. Suppose that $\sigma$ is a branched projective structure with the holonomy $\rho : \pi_1(S) \to \mathbb{C} \subset PSL(2, \mathbb{C})$. We will assume that $\rho$ is nontrivial, otherwise clearly $deg(\sigma) \geq 2g + 2$

by the Riemann-Hurwitz formula. The representation $\rho$ lifts to a representation $\theta : \pi_1(S) \to SL(2, \mathbb{C})$ (with the image in the group of unipotent upper triangular matrices $U$). Let $V$ denote the holomorphic $\mathbb{C}^2$–bundle over $S$ associated with the representation $\theta$. The structure $\sigma$ gives rise to a holomorphic line subbundle $L \subset V$ so that

$$deg(L) = g - 1 - \frac{deg(\sigma)}{2} \tag{7.3}$$

where $deg(\sigma)$ is the degree of branching of $\sigma$ (see [GKM00, Chapter C]). The bundle $V$ fits into short exact sequence

$$0 \to \Lambda \to V \xrightarrow{p} \Lambda \to 0$$

where $\Lambda$ is the trivial bundle; the fibers of $\Lambda = ker(p)$ correspond to the line in $\mathbb{C}$ fixed by the group $U$. Under the projectivization $\mathbb{C}^2 \to \mathbb{CP}^1$ this line projects to the point $\infty \in \mathbb{CP}^1$. Hence the developing mapping of $\sigma$ does not cover $\infty$ iff $L \cap ker(p) = 0$. It also follows that $L \neq ker(p)$ (otherwise the developing mapping of $\sigma$ would be constant). Therefore we get a nonzero map $p : L \to \Lambda$ by restricting the projection $p : V \to \Lambda$ to $L$. By Riemann-Roch, $deg(L) \leq 0$ with equality iff $p : L \to \Lambda$ is injective; (7.3) then implies that $deg(\sigma) \geq 2g - 2$. The equality here is attained only if the developing map of $\sigma$ takes values in $\mathbb{C}$, i.e. $\sigma$ is a singular Euclidean structure. In other words, if $deg(\sigma) = 2g - 2$ then the developing mapping of $\sigma$ is obtained by integrating an abelian differential on $S$. If $\rho$ is not the holonomy of any singular Euclidean structure then $deg(\sigma) \geq 2g + 1$. However, since $\rho$ lifts to $SL(2, \mathbb{C})$, $deg(\sigma)$ has to be even (see [GKM00, Chapter C]). We conclude that in this case $deg(\sigma) \geq 2g$. Recall that for a representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$, $d(\rho)$ is the least degree of branching of all projective structures on $S$ (consistent with the orientation) with the monodromy $\rho$. We thus proved

**Proposition 7.4.** *Suppose that $\rho$ is a representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ whose image is contained the translation subgroup $\mathbb{C}$ of $PSL(2, \mathbb{C})$. Then $d(\rho) \geq 2g - 2$ and $d(\rho) \geq 2g$ provided that the corresponding character $\chi \in H^1(S, \mathbb{C})$ is not the period character of any abelian differential.*

Combining this proposition with Theorems 1.2 and 7.1 we get Corollary 1.4.

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