Flats in 3-manifolds

Michael Kapovich

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Abstract

We prove that if a closed aspherical Riemannian 3-manifold M contains a 2-flat, then there exists a free Abelian subgroup of rank two in $\pi_1(M)$. Under some restrictions on topology of M we prove the existence of an immersed incompressible flat torus in M. This generalizes results which were previously known for manifolds of nonpositive curvature.

1 Introduction

In this paper we address the following conjecture which is a special case of Thurston's Geometrization Conjecture:

Conjecture 1.1. (Weak Hyperbolization Conjecture): *Suppose that M is a closed aspherical 3-manifold. Then either $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$ or $\pi_1(M)$ is word-hyperbolic.*

Note that according to the results [M1], [Tu], [Ga1], [CJ], [Sco] and [T], Thurston's Geometrization Conjecture is satisfied for any closed irreducible 3-manifold M whose fundamental group contains $\mathbb{Z} \times \mathbb{Z}$. Such manifold is either Haken or Seifert. On the other hand, if $\Gamma = \pi_1(M)$ is word-hyperbolic then the ideal boundary $\partial_\infty \Gamma$ is a 2-dimensional sphere $\mathbb{S}^2$ (see [BM]). In the latter case, conjecturally, the ideal boundary of Γ is quasi-symmetric to the standard 2-sphere (see [Ca], [CS], [BK1], [BK2]). If this is trues, then Γ is isomorphic to a uniform lattice in $SO(3,1)$ and hence M is homotopy-equivalent to a closed hyperbolic manifold N. In the latter case the manifolds M and N are homeomorphic (see [Ga3]).

It is well-known that failure of a finitely-presented group $\Gamma = \pi_1(M)$ to be word-hyperbolic means that Γ doesn't have linear isoperimetric inequality. Moreover, according to Gromov ([Gro2], Assertion 6.8.S), $\pi_1(M)$ is word-hyperbolic iff there is no nonconstant conformal least area map $f : \mathbb{R}^2 \to M$. Stronger versions of this statement are proven in the works of Mosher & Oertel [MO] and Kleiner [Kl].

Thus, nonhyperbolicity of Γ implies the existence of a certain minimal surface S in M. In this paper we will prove Conjecture 1.1 under the assumption that S is a *flat*, Theorem 1.2.

Theorem 1.2. *Suppose that M is a closed aspherical Riemannian 3-manifold which contains a flat. Then $\pi_1(M)$ contains $\mathbb{Z}^2$.*

Interesting intermediate case between Theorem 1.2 and Conjecture 1.1 is when the universal cover of the manifold M contains a *quasi-flat*. Note however that the universal cover of any Sol-manifold does not contain quasi-flats since its asymptotic cone is 1-dimensional [Gro3].

It would be interesting to know if the manifold M in Theorem 1.2 contains an immersed flat incompressible torus. In Section 13 we prove that such torus exists under the assumption that the canonical decomposition of M contains no Seifert components, Theorem 13.1.

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Contents

2 Weak Hyperbolization Conjecture

In this section we describe several cases when the Weak Hyperbolization Conjecture is proven. Our first example is given by 3-manifolds of nonpositive curvature.

If M is a manifold then $\tilde{M}$ will always denote the universal cover of M. A *k-flat* in a Riemannian manifold M is an isometric immersion $f : \mathbb{R}^k \to M$ so that the lift $\tilde{f} : \mathbb{R}^k \to \tilde{M}$ is an isometric embedding (i.e. $d(x, y) = d(\tilde{f}(x), \tilde{f}(y))$). Abusing notations we will call by k-flat the image of a k-flat as well. 2-flats will be called *flats*. The image of a k-flat $F = \tilde{f}(\mathbb{R}^k)$ is totally-geodesic in $\tilde{M}$, i.e. for any points $x, y \in F$ the minimizing geodesic connecting x and y is in F. If $\tilde{f} : \mathbb{R}^2 \to \tilde{M}$ is a *quasi-isometric embedding* then the image of $\tilde{f}$ is called a *quasi-flat*. We refer the reader to [He1] for basic definitions of 3-dimensional topology.

We recall the following results:

Theorem 2.1. *(P. Eberlein [E]) Suppose that M is a closed n-manifold of nonpositive curvature. Then either M contains a flat or $\pi_1(M)$ is a word-hyperbolic group.*

Remark 2.2. *The hyperbolicity of the fundamental group was disguised in [E] as the "visibility" axiom. See [Gro2], [Br] for the case of general $CAT(0)$-metrics.*

Theorem 2.3. *(V. Schroeder [Sc]) Suppose that M is a n-manifold of nonpositive curvature and finite volume which contains a codimension 1 flat. Then M contains a compact $(n-1)$-flat. (In particular $\pi_1(M)$ contains $\mathbb{Z}^{n-1}$.)*

In the case of closed 3-manifolds of nonpositive curvature this theorem was first proven by S. Buyalo [B], see also [KK2] for a generalization of this result to CAT(0) Poincare duality groups. The present paper was motivated by the proofs of Schroeder and Buyalo.

Corollary 2.4. *Suppose that M is a closed 3-manifold of nonpositive curvature. Then M satisfies the Weak Hyperbolization Conjecture.*

Theorem 2.5. *(M. Kapovich and B. Leeb [KL]) Suppose that N is a closed Haken 3-manifold with nontrivial decomposition into geometric components and G is a torsion-free finitely-generated group quasi-isometric to $\pi_1(N)$. Then G is isomorphic to fundamental group of a Haken 3-manifold.*

Corollary 2.6. *Suppose that N is a manifold satisfying the Weak Hyperbolization Conjecture and N is not a Sol-manifold. If M is a closed 3-manifold with fundamental group quasi-isometric to $\pi_1(N)$, then M itself satisfies the Weak Hyperbolization Conjecture.*

This corollary shows that the Weak Hyperbolization Conjecture is a problem about some large-scale geometric properties of 3-manifold groups.

The deepest result in the direction of Thurston's Geometrization Conjecture is due to Thurston:

Theorem 2.7. *(W. Thurston [T], see also [Mor], [Mc], [O], [K]) Suppose that M is a Haken manifold. Then either $\pi_1(M)$ contains $\mathbb{Z}^2$ or M is hyperbolic.*

Corollary 2.8. *Suppose that M is finitely covered by a Haken manifold (i.e. M is "almost Haken"). Then either $\pi_1(M)$ contains $\mathbb{Z}^2$ or $\pi_1(M)$ is word hyperbolic.*

Note that in the last case $\tilde{M}$ can not contain quasi-flats. Hence the assertion of Theorem 1.2 is satisfied for all almost Haken manifolds.

Theorem 2.9. *(G. Mess [M1]) Suppose that M is a closed aspherical 3-manifold such that $\pi_1(M)$ contains an infinite cyclic normal subgroup. Then $\pi_1(M)$ contains $\mathbb{Z}^2$.*

Note that if the manifold M in Theorem 2.9 is *irreducible* then it must be a Seifert manifold (D. Gabai [Ga1], A. Casson & D. Jungreis [CJ]). If M is Haken then the assertion was first proven by Waldhausen, see [He1]. In our paper we will rely heavily on Theorems 2.7 and 2.9.

3 Outline of the proof

Notation: We say that $f(x) = O(x)$ if

$$0 < \liminf_{x\to\infty} f(x)/x \le \limsup_{x\to\infty} f(x)/x < \infty$$

Similarly $f(x) = o(x)$ if

$$\lim_{x\to\infty} f(x)/x = 0$$

The proof of Theorem 1.2 splits in three main cases:

Case I: the universal cover $X = \tilde{M}$ contains a "simple flat" F, i.e. a flat which doesn't intersect any other flats in the orbit ΓF (but F can have a nontrivial stabilizer).

Case II: the space X has no simple flats but contains a flat F with "double intersections", i.e. for any $g, h \in \Gamma$ we have: $F \cap gF \cap hF$ is not a point.

Case III (the case of "triple intersections"): the space X contains neither simple flats nor flats with double intersections.

We begin outline with the most interesting Case III. We first find a pair of flats F_1, F_2 whose intersection is a *recurrent* geodesic ℓ. Using Theorem 2.7 we conclude that unless Γ contains $\mathbb{Z}^2$ or X contains a simple flat, the path-connected component L of F_1 in ΓF_1 is the whole orbit ΓF_1 (to achieve this one may have to take a finite covering over M). Thus we assume the latter to be the case. Using parallel transport along flats we construct a "holonomy" representation ρ of Γ into $SO(3)$. If this representation has finite image then the family of lines parallel to ℓ in L is invariant under $Ker(\rho)$ and the discussion is similar to the Case II. If $\rho(\Gamma)$ has an invariant line and is infinite then a 2-fold cover over M has nonzero 1-st Betty number and the manifold M is homotopy-equivalent to an almost Haken manifold. Hence, in this case Theorem 1.2 follows from Thurston's Hyperbolization Theorem 2.7. Thus we can assume that $\rho(\Gamma)$ is dense in $SO(3)$. In particular this implies that the group Γ is not amenable.

Remark 3.1. *Instead of proving that first that Γ is not amenable one can use a Varopoulos' theorem (as it is done in [KK1]) to conclude in Case III that $\pi_1(M)$ has polynomial growth.*

We use recurrence of the geodesic ℓ to construct a family of "double simplices" D_n in X. Roughly speaking each D_n is the union of two adjacent simplices in X which have flat faces. We prove that the inscribed radii ι_{D_n} of D_n tend to infinity at the same rate as edges of the corresponding simplices. The area of ∂D_n grows as $O(\iota^2_{D_n})$. Since Γ is not amenable the growth rate of $Vol(D_n)$ is again $O(\iota^2_{D_n})$. This implies that the largest metric ball inscribed in D_n has radius ι_{D_n} and the volume at most $O(\iota^2_{D_n})$. Hence Γ has polynomial growth which contradicts the fact that this group is not amenable.

Remark 3.2. *It seems (however I cannot prove this statement) that more general set-up for the above argument is as follows. Suppose that M is a closed aspherical 3-manifold. Let X_ω be an asymptotic cone of X, assume that $H_2(Y,\mathbb{Z}) \neq 0$ for some compact $Y \subset X_\omega$ (where we consider singular homology theory). Then the fundamental group $\pi_1(M)$ is amenable. Indeed, in the Case III the sequence ∂D_n produces an embedded simplicial 2-sphere in X_ω.*

Now consider the Case II. In this case we repeat the construction of flats F_1, F_2 so that $\ell = F_1 \cap F_2$ is recurrent. Again we can assume that the orbit $L = \Gamma F_1$ is path-connected. Then L is foliated by lines which are "parallel" to ℓ and our goal is to show that this Γ-invariant foliation corresponds to the universal cover of a Seifert fibration of M.

We add to L the Γ-orbit of the flat F_2 and call the closure $\mathcal{L}$. The space $\mathcal{L}$ with the induced path metric fibers over a metric space Y with the fibers parallel to ℓ. We pass to an index 2 subgroup in Γ to guarantee that Γ preserves orientation of fibers of L. Zassenuhaus theorem implies that if G is a Lie group which fits into the exact sequence

$$1 \to \mathbb{R} \to G \to P \to 1$$

and Δ is a discrete finitely-generated subgroup of G then either the projection of Δ to P is discrete or Δ has an infinite normal cyclic subgroup. We generalize this fact to the case of the fibration $\mathcal{L} \to Y$. The group Γ does not act discretely on Y since the geodesic ℓ is recurrent. We conclude that the group Γ has a nontrivial center. Thus Γ contains $\mathbb{Z}^2$ according to Geoff Mess's Theorem 2.9.

Finally we discuss the Case I. We present two different proofs. One of them is a straight-forward application of the Rips Machine, another is more geometric and follows arguments of Buyalo and Schroeder.

The first proof is quite general and works for higher-dimensional manifolds as well. Consider the closure $\bar{L}$ of the Γ-orbit of a simple flat F. It projects to a lamination on M which admits a transversal-invariant measure since each leaf is amenable ([P1], [MO]). Thus the topological tree T dual to $\bar{L}$ is a metric tree and the group Γ acts on T by isometries. Therefore application of the Rips Machine to T will produce a simplicial Γ-tree $R(T)$ where edge-stabilizers are discrete subgroups of $Isom(\mathbb{R}^2)$. Hence Γ contains $\mathbb{Z}^2$. (Mosher and Oertel have very similar proof for laminations $\bar{L}/\Gamma$ of zero Euler characteristic, our proof was motivated by their approach.)

The second (geometric) proof goes as follows. We assume first that all simple flats in X have trivial stabilizers in Γ. We use Schroeder's trick to conclude that the dual tree T to the lamination $\bar{L}$ is a real line which implies that Γ is Abelian. Thus, there must be a simple flat F in X with nontrivial stabilizer. We assume that this stabilizer is a cyclic group $\langle\gamma\rangle$. Denote by G the maximal subgroup of Γ whose elements commute with $\langle\gamma\rangle$ (apriori it could be an infinitely generated locally cyclic group). Denote by $\bar{L}_F$ the closure of the G-orbit of F. We use Schroeder's arguments to prove that the quotient $\bar{L}_F/G$ is compact. Still this doesn't imply apriori that G is finitely generated since $\bar{L}_F$ is highly disconnected. However we prove that G has a finitely-generated subgroup $G_0 \supset \langle\gamma\rangle$ whose Cayley graph contains a quasi-flat (Lemma 6.6). Hence this subgroup is not $\mathbb{Z}$ and has infinite center. Therefore it must contain $\mathbb{Z}^2$ by the Mess's theorem as in the Case II.

4 Amenability

Recall that a finitely-generated group G acting cocompactly on a Riemannian manifold X is *amenable* if X contains an exhausting *Folner sequence* of codimension zero compact submanifolds Φ_n with piecewise-smooth boundary. This means that

$$\lim_{n\to\infty} Area(\partial\Phi_n)/Vol(\Phi_n) = 0$$

Examples of amenable and nonamenable groups:
(a) Any group which contains a free nonabelian subgroup is nonamenable.
(b) Any virtually solvable group is amenable.
(c) If G is a finitely-generated amenable subgroup of a linear group then G is almost solvable. (This follows directly from the Tits's alternative.)
(d) The class of amenable groups is closed under the operations of taking subgroups, direct limits, quotients and extensions.

All known examples of finitely presented amenable groups are *elementary*, i.e. they are built from finite and cyclic groups via operations (d). Grigorchuk [Gri] constructed examples of finitely generated amenable groups which are not elementary.

Lemma 4.1. *Suppose that M is a closed 3-manifold with amenable fundamental group. Then any 2-generated subgroup F of $\pi_1(M)$ is either Abelian or has finite index in $\pi_1(M)$.*

Proof: If the index of F is infinite then $\tilde{M}/F$ is a noncompact manifold. If F is freely decomposable then F is not amenable. Otherwise it is either cyclic or the compact core of $\tilde{M}/F$ is a Haken manifold which implies that F contains $\mathbb{Z} * \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. $\square$

It is easy to see that all elementary amenable 3-manifold groups are almost solvable.

G. Mess [M2] proved that if fundamental group of a closed 3-manifold M contains no free nonabelian subgroups then either $\pi_1(M)$ is almost solvable or it contains a *simple* finite-index subgroup.

Note that a particular case of Conjecture 1.1 is that any closed 3-manifold with amenable fundamental group has almost solvable fundamental group. However it is still unknown if a group quasi-isometric to Sol is almost solvable.

5 Geometric constraints

Let X be the universal cover of the compact Riemannian 3-manifold M, through the whole paper we shall denote by $\langle , \rangle$ the Riemannian metric on X. Propositions in this section follow directly from the compactness of M and we omit their proofs.

Suppose that F_1, F_2, F_1', F_2' are flats in X so that $F_1 \cap F_2 = \ell$, $F_1' \cap F_2' = \ell'$ are geodesics with the dihedral angles $\alpha, \alpha' \neq 0$.

Proposition 5.1. *There are continuous functions $\theta(\alpha, \alpha'), \kappa(\alpha, \alpha', t)$ such that:*

(i) If $x \in \ell, x' \in \ell'$ are points within the distance at most $\theta(\alpha, \alpha')$ then there is $y \in (F_1 \cup F_2) \cap (F_1' \cup F_2')$ such that $d(x, y) \leq \kappa(\alpha, \alpha', d(x, x'))$.

(ii) $\lim_{t \to 0} \kappa(\alpha, \alpha', t) = 0$.

Proposition 5.2. *(Cf. [Sc], Sublemma 2.) There exists $\epsilon > 0$ with the following property:*

Suppose that F_1, F_2 are flats in X with empty intersection, $x \in F_1$, $d(x, F_2) < \epsilon$. Let $c : [0, a] \to X$ be the unit speed minimal geodesic from x to F_2 so that $c(0) = x$ and N_x be the unit normal vector to F at x with the angle

$$\angle(N_x, c'(0)) < \pi/2$$

Then

$$\angle(N_x, c'(0)) < \pi/4$$

and the geodesic ray emanating from x in the direction N_x intersects the flat F_2 at the arc-length distance at most δ, where δ is the injectivity radius of M.

Proposition 5.3. *There exist $\lambda > 0$ and a continuous function $u(x, y)$ so that for any $\xi > 0$ the following is true. Pick any complete geodesic $l \subset X$, flat F, point $z \in F$ such that $d(z, l) \leq \lambda$ and $w \in l$ is the nearest point to z. Connect w and z by the shortest geodesic segment I and let ν be the parallel transport along I of a unit normal vector to F at the point z. Let ϵ_w be the unit tangent vector to l emanating from w. Suppose that $|\angle(\nu, \epsilon_w) - \pi/2| > \xi$. Then the flat F intersects l in a point y such that $d(z, y) \leq u(\lambda, \xi)$.*

6 Some facts about dynamical systems

6.1 Recurrent points

Suppose that X is a compact topological metric space. Let G be an infinite topological semigroup acting on X. We recall that a point $x \in X$ is called *recurrent* if there exists a divergent sequence $g_n \in G$ such that

$$\lim_{n \to \infty} g_n(x) = x$$

Lemma 6.1. *Under the conditions above for any point $z \in X$ the closure of the orbit $G \cdot z$ contains a recurrent point.*

Proof: Consider the orbit $G \cdot z$ and its accumulation set $Z_1 = \Lambda(z)$, which is closed and therefore compact. If the point z is not a recurrent point itself then $G \cdot z - \Lambda(z)$ is nonempty. Pick a point $z_1 \in Z_1$ and repeat the procedure. If z_1 is not recurrent then the set $Z_2 = \Lambda(z_1)$ is a proper subset in $G \cdot z_1$. By repeating this process we get a decreasing sequence of compact sets Z_j such that each Z_j is contained in $\Lambda(z)$. If the process doesn't terminate after a finite number of steps we take the intersection $Z_\omega = \cap_{j=1}^{\infty} Z_j$. This intersection must be nonempty since all the sets are compact. Continue the process. As the result we get a decreasing sequence of compact subsets Z_j where the index j runs over the ordinals. On each finite step the sets under consideration loose at least one point, the original set has the cardinality of at most continuum. Thus the process must terminate after at most a continuum of steps. $\square$

Suppose that M is a closed Riemannian manifold, $F(M)$ is the orthonormal frame bundle of M. We define the *geodesic flow* on $F(M)$ as follows. Points of $F(M)$ are pairs: (x, f) where $x \in M$ and f is an orthonormal frame in $T_x(M)$. Choose the first vector f_1 in the frame f and let $\gamma(t) = \exp_x(tf_1)$ be the geodesic emanating from x tangent to f_1. Let $G_t(x, f)$ be the parallel transport of (x, f) along the geodesic γ to the point $\gamma(t)$. We call the $\mathbb{R}$-action on $F(M)$

$$(t, (x, f)) \mapsto G_t(f)$$

the *geodesic flow on* $F(M)$. It is clear that this action is continuous.

Thus, Lemma 6.1 implies the following

Corollary 6.2. *For any point $z = (x, f) \in F(M)$ the accumulation set of the orbit $G_t(z)$ $(t \in \mathbb{R}_+)$ contains a recurrent point of the geodesic flow.*

Suppose that (x, f) is a recurrent point in M, consider the geodesic $\gamma \in M$

$$\gamma = \{\exp_x(tf_1), t \in \mathbb{R}\}$$

The geodesic γ as well as its lifts to the universal cover $\tilde{M}$ will be also called *recurrent*. Note that if $M' \to M$ is a finite covering, then the lift of a recurrent geodesic from M to M' is again recurrent.

6.2 Groups acting on $\mathbb{R}$

Theorem 6.3. *(O. Holder–J. Plante, [P2]) Suppose that Γ is a group of homeomorphisms of $\mathbb{R}$ acting freely. Then Γ is Abelian.*

Proof: We recall idea of the proof. Pick a point $x \in \mathbb{R}$. Then the orbit $\Gamma \cdot x$ is a set with an Archimedian linear order. Since the action of Γ is free this linear order doesn't depend on choice of the point x. Therefore we get an invariant Archimedian linear order on the group Γ. Then a theorem of Holder implies that Γ has a monomorphism into $\mathbb{R}$, hence Γ is Abelian. $\square$

Corollary 6.4. *Suppose that L is a (topological) foliation of a compact 3-manifold M so that its lift to the universal cover of M consists of topological planes. Assume that $M \neq S^1 \times S^1 \times S^1$. Then at least one leaf of L is not simply-connected.*

Proof: Consider the action of $\Gamma = \pi_1(M)$ on the universal cover $\tilde{M}$. This action preserves the foliation $\tilde{L}$ of $\tilde{M}$ by planes. The real line $\mathbb{R}$ is dual to the foliation $\tilde{L}$, thus Γ acts on $\mathbb{R}$ by homeomorphisms. If L has only simply-connected leaves then Γ is Abelian. Hence $G \cong \mathbb{Z}^3$ and since $\tilde{M}$ is irreducible this implies that $M = S^1 \times S^1 \times S^1$. $\square$

6.3 Quasi-isometries and proper pairs

Let (X_j, d_j) $(j = 1, 2)$ be a pair of metric spaces. We recall that a map $f : (X_1, d_1) \to (X_2, d_2)$ is a *quasi-isometric embedding* if there are two constants $K > 0$ and C such that

$$K^{-1} d_1(x, y) - C \le d_2(f(x), f(y)) \le K d_1(x, y) + C$$

for each $x, y \in X_1$. If (X_1, d_1) is the Euclidean plane $\mathbb{R}^2$ then f above (and its image) is called a *quasi-flat* in X_2.

A map $f_1 : (X_1, d_1) \to (X_2, d_2)$ is a *quasi-isometry* if there are two constants C_1, C_2 and another map $f_2 : (X_2, d_2) \to (X_1, d_1)$ such that both f_1, f_2 are quasi-isometric embeddings and

$$d_1(f_2 f_1(x), x) \le C_1, d_2(f_1 f_2(y), y) \le C_2$$

for every $x \in X_1, y \in X_2$. Such spaces X_1, X_2 are called *quasi-isometric*. For example, two metric spaces which admit cocompact discrete actions by isometries of the same group are quasi-isometric.

The Cayley graph of a finitely generated group Γ with a fixed finite set of generators carries a canonical metric which is called the word metric. The quasi-isometry class of the word metric does not depend on the generating set.

Suppose that X is the universal cover of a closed Riemannian manifold M, Γ is the group of covering transformations. Suppose that $E \subset X$, G is a subgroup in Γ so that $G(E) = E$. We say that $g_n E$ *accumulates* to a point $x \in X$ if for some sequence $x_n \in E$, $\lim_{n\to\infty} g_n(x_n) = x$.

We call a pair (E, G) *proper* if the sequence of sets $\{gE : g \in G\}$ is locally finite in X. This means that for any infinite sequence $\{g_n\} \subset \Gamma$ such that $g_n E$ accumulates to a point $x \in X$ it follows that there exists $\gamma \in \Gamma$ and a subsequence $\{g_{n_k}\} \subset \{g_n\}$ so that $x \in \gamma cl(E)$ and $g_{n_k} \in \gamma G$. Note that if G has finite index in Γ, the (E, G) is a proper pair.

Proposition 6.5. *For any proper pair (E, G) the quotient $cl(E)/G$ is compact.*

Proof: Suppose that $x_n \in E$ is a sequence of points. Since M is compact there exists a sequence $g_n \in \Gamma$ so that $g_n x_n \to x \in X$. By definition of a proper pair g_{n_k} splits as $\gamma \circ \gamma_{n_k}$ where $\gamma_{n_k} \in G$. Therefore

$$\lim_{k\to\infty} \gamma_{n_k} x_{n_k} = \gamma^{-1} x$$

This implies compactness of $cl(E)/G$. $\square$

We suppose that E is a closed subset in X invariant under a subgroup $G < \Gamma$ so that the pair (E, G) is proper. Assume that E is the union of flats (which are not necessarily disjoint).

Lemma 6.6. *There exists a finitely-generated subgroup $G_0 < G$ such that a Cayley graph of G_0 contains a quasi-flat. In the case when E is path-connected we can take $G = G_0$.*

Proof: The problem is that E is not a geodesic metric space with the metric induced from X, otherwise the assertion would follow from [Gh], Proposition 10.9. Thus we have to *thicken up* the space E to a geodesic metric space. Choose sufficiently small number $\sigma > 0$ which is less than the half of the injectivity radius of M. The compact E/G is covered by a finite number of open σ-balls B_j with centers at points on E/G, denote the union of these balls by $V_\sigma(E)/G$. It is a manifold which has only a finite number of connected components. Pick one of these components V_0. A connected component U_0 of the lift of V_0 to X has the stabilizer $G_0 < G$ so that $U_0/G_0 = V_0$. The intersection $L_0 = U_0 \cap E/G$ is closed in E/G and thus compact. Note that in the case of connected E we get $G = G_0$. Introduce in U_0 the path-metric d_P via the Riemannian metric on X. This metric projects to a path metric on V_0 so that the diameter of V_0 is bounded. Consider the completion $\bar{U}_0$ of U_0 with respect to this metric. The group G_0 still operates on $\bar{U}_0$ by isometries and this action is properly discontinuous. Let $\tilde{B}_j \subset U_0$ be a lift of one of the σ-balls which cover E/G. Then the closure $cl\tilde{B}_j$ of $\tilde{B}_j$ is isometric to closure of the ball B_j in M. On the other hand each point of $\bar{U}_0$ belongs to one of the closed balls $cl\tilde{B}_j$ which is compact. Then finiteness of the number of balls B_j implies that $\bar{U}_0/G_0$ is compact with respect to the topology defined by the path-metric d_P. By construction (U_0, d_P) is a quasi-geodesic metric space, thus the same is true for its completion. Hence we can apply [Gh], Proposition 10.9, to conclude that G_0 is finitely generated. The compactness of $\bar{U}_0/G_0$ implies that $\bar{U}_0$ is quasi-isometric to a Cayley graph of G_0. Note however that U_0 must contain one of the flats in E. This flat remains a flat in (U_0, d_P), since $d_P \geq d$ where d is the original metric on X. It implies that the Cayley graph of G_0 contains a quasi-flat. □

Corollary 6.7. *The group G under the conditions above is not word-hyperbolic and is not locally cyclic.*

Proof: Cayley graphs of word-hyperbolic groups do not contain quasi-flats. If G is locally cyclic then G_0 is cyclic and hence word-hyperbolic. This contradicts the existence of a quasi-flat in a Cayley graph of G_0. □

7 Inscribed radius

Suppose that X is a metric space, $z \in X, S \subset X$ be a point and a subset. We define the distance $d(z, S)$ from z to S as

$$\inf_{x \in S} d(z, x)$$

Define the *inscribed radius* ι_S of S as

$$\iota_S = \sup\{r : B_r(x) \subset S, \quad \text{for some point} \quad x \in S\}$$

where $B_r(x)$ is the metric ball of radius r with the center at x. We shall denote by $S_r(x)$ the metric sphere of radius r with the center at x.

Suppose now that $X = \tilde{M}$ is a simply-connected complete Riemannian 3-manifold, $O \in \tilde{M}$, Π is a flat which contains O. This flat separates $\tilde{M}$ into "left" and "right" sides (otherwise $H_1(X, \mathbb{Z}) \neq 0$). Denote by Π^+ the right side. Consider a sequence of metric balls $B_r(O)$, $r \to \infty$. Boundary of the ball $B_r(O)$ is the metric sphere $S_r(O)$. Define S_r^+ to be $(\Pi^+ \cap S_r(O)) \cup (\Pi \cap B_r(O))$. (The set S_r^+ looks like a metric hemisphere with a flat disc attached to the equator.)

Lemma 7.1. *In the "right half" $B_r^+(O) = \Pi^+ \cap B_r(O)$ of each ball $B_r(O)$ we can choose a point x_r such that*

$$d(x_r\ , S_r^+) = O(r)$$

Thus $\iota_{B_r^+(O)} = O(r)$.

Proof: For each r consider the "metric hemisphere" $\Pi^+ \cap S_{r/2}(O) = \Sigma_r$. Clearly for every $x \in \Sigma_r$ we have

$$r \geq d(x, S_r(O)) \geq r/2$$

Now suppose that

$$\phi(r) = \max_{x \in \Sigma_r} \ d(x, \Pi \cap B_r(O)) = o(r)$$

The hemisphere Σ_r is a singular chain in $C_2(B_r^+(O), \mathbb{Z})$ with the boundary equal to the circle ℓ_r with center at O and radius $r/2$. This circle is a nontrivial element of the homology group $H_1(\Pi - O, \mathbb{Z})$. Triangulate this chain so that size of each simplex is at most 1. We construct a continuous map $f = f_r : \Sigma_r \to \Pi \cap B_r(O)$ as follows. For each vertex x of the triangulation we let $f(x)$ be the nearest-point projection of x to $\Pi \cap B_r(O)$. Extend the map f to a piecewise-linear map of the cycle Σ_r. It is clear that $[f(\ell_r)] = [\ell_r]$ in $H_1(\Pi - O, \mathbb{Z})$. Moreover, for each $x \in \Sigma_r$ we have: $d(x, f(x)) \leq 2\phi(r) + 1$. For sufficiently large r we have: $2\phi(r) + 1 \leq r/4$. Therefore O doesn't belong to the image of f. However the chain $f(\Sigma_r)$ bounds the nontrivial cycle $f(\ell_r)$ in $\Pi - O$. Contradiction. $\square$

Remark 7.2. *Our proof actually shows that x_r can be chosen so that*

$$d(x_r\ , S_r^+) \geq r/8 - 1/2$$

8 Riemannian simplices

Suppose that N is a compact domain in a Riemannian 3-manifold X so that N has piecewise-smooth boundary which is combinatorially equivalent to the boundary of a Euclidean 3-simplex. We assume that N is contractible and the boundary of N is a collection of absolutely totally-geodesic flat faces F_j, $j = 1, ..., 4$. Under these conditions N will be called a *Riemannian simplex*. We do not assume that N is homeomorphic to a 3-ball (it would follow from the Poincare Conjecture).

Now consider a sequence of Riemannian simplices N_r such that:

as $r \to \infty$ the lengths of all edges of N_r grow as $O(r)$.

By triangle inequalities, for each r there exists a Euclidean 3-simplex Δ_r in $\mathbb{R}^3$ so that faces of Δ_r are isometric to the corresponding faces of N_r , we choose a homeomorphism $h_r = h : \partial N_r \to \partial \Delta_r$ which is an isometry on each face. We can assume that one of the vertices of Δ_r is the origin $0 \in \mathbb{R}^3$. Denote the rest of the vertices by A_{1r} , A_{2r} , A_{3r} . Let $B_{jr} = h^{-1}(A_{jr})$, $B_{0r} = h^{-1}(0)$.

We call the sequence of simplices Δ_{r_n} *nondegenerate* if for any sequence $0 < \rho_n < r_n$ and any subsequence in r_n, the Gromov-Hausdorff limit of the rescaled tetrahedrons $Q_{r_n} = \frac{1}{\rho_n}\Delta_{r_n}$ is not contained in a Euclidean plane. It is easy to see that this property depends only on the vertex angles of Δ_r . Namely, for any vertex A_j with the planar angles x_r , y_r , z_r at this vertex we have:

$$\lim_{r\to\infty} x_r + y_r + z_r \neq 2\pi \ , \lim_{r\to\infty} x_r + y_r - z_r \neq 0$$

for any subsequence.

Suppose that Y_r is a sequence of points on the edges $[B_{0r}, B_{1r}]$ so that $d(Y_r, B_{0r}) = O(\rho(r))$ where $0 < \rho(r) < r$ is a function of r. Let $F_{1r} = [B_{0r} \ , B_{2r} \ , B_{3r}]$ be the face opposite to B_{1r}.

Lemma 8.1. *Under the conditions above $d(Y_r \ , F_{1r}) = O(\rho(r))$ as $r \to \infty$.*

Proof: Suppose that the assertion is wrong, E_r is a nearest point to Y_r on the face F_1 and $d(Y_r, E_r) = o(\rho)$. Then $|d(B_{0r}, E_r) - d(B_{0r}, Y_r)| = o(\rho)$, $d(B_{0r}, E_r) = O(\rho)$. It implies that we can choose points $C_{2r} \in [B_{0r}, B_{2r}], C_{3r} \in [B_{0r}, B_{3r}]$ so that E_r is contained inside of the triangle $[B_{0r}, C_{2r}, C_{3r}]$ and $d(B_{0r}, C_{2r}) = O(\rho)$, $d(B_{0r}, C_{2r}) = O(\rho)$. Similarly we get $|d(Y_r, C_{jr}) - d(C_r, E_r)| = o(\rho)$.

The sequence of rescaled simplices $\rho^{-1}\Delta_r$ is subconvergent either to a nondegenerate simplex (if $\rho = O(r)$) or to an infinite tetrahedral cone with the vertex at zero. The points $\rho^{-1}hC_{jr}$, $\rho^{-1}hY_r$, $\rho^{-1}hE_r$ are convergent to points $\hat{C}_j$, $\hat{Y}, \hat{E}$ on the boundary of this cone (or simplex), $j = 2, 3$; so that

$$d(\hat{Y}, \hat{C}_j) = d(\hat{E}, \hat{C}_j), d(\hat{Y}, 0) = d(\hat{E}, 0)$$

This implies that the point $\hat{Y}$ actually belongs to the same plane P as the points $0, \hat{C}_2, \hat{C}_3$. On the other hand $\hat{Y} \neq 0$ and belongs to an edge of

$$\lim_{r\to\infty} \rho^{-1}\Delta_r$$

which is not on P since the sequence of simplices is not degenerate. Contradiction. □

Now we choose two sequences of Riemannian simplices $N_{r'}, N_{r''} \subset X$ so that each sequence is nondegenerate and edges of $N_{r'}, N_{r''}$ are $O(r'), O(r'')$ respectively. We denote the vertices by $B_{jr'}$ and $B_{jr''}$. Assume that these simplices are embedded in X so that:

- The vertex $B_{0r'}$ is identified with $B_{0r''}$ and subsegments of the edges $[B_{0r'}, B_{2r'}]$, $[B_{0r''}, B_{2r''}]$ and $[B_{0r'}, B_{3r'}]$, $[B_{0r''}, B_{3r''}]$ are glued together.
- The faces $F_{3r'}, F_{3r''}$ belong to the same flat in X.
- The interiors of simplices are disjoint.

The union $N_{r'} \cup N_{r''} = D$ is called a *double simplex.*

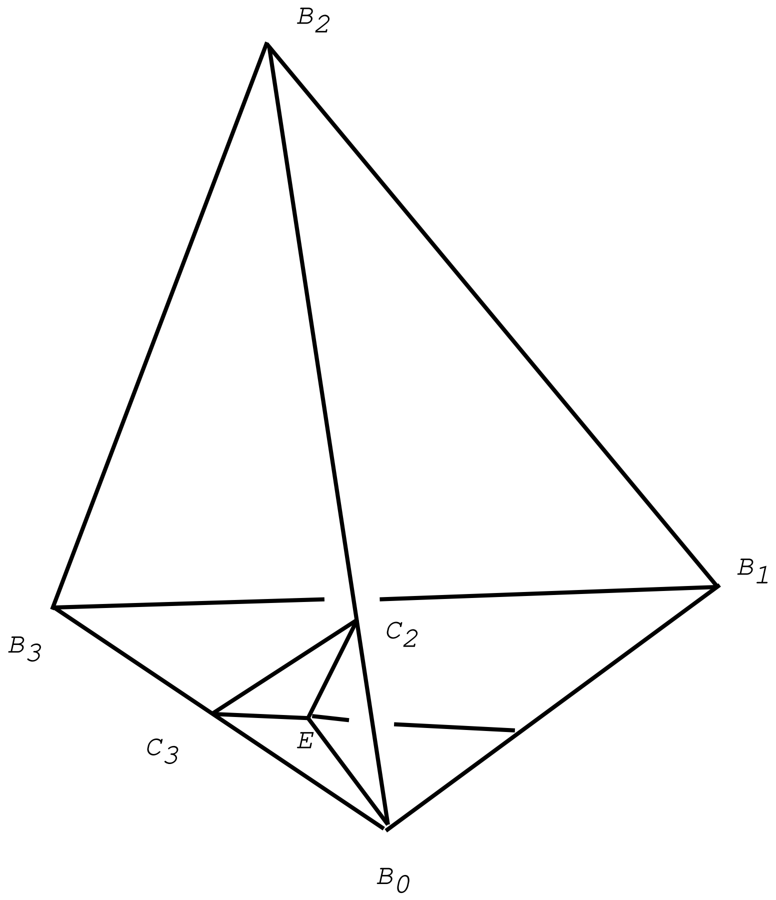

Figure 1:

Theorem 8.2. *If $r'' = O(r')$ then the inscribed radius of D is $O(r')$.*

Proof: Pick a point $Y_{r'} \in [B_{0r'}, B_{2r'}]$ so that $d(Y_{r'}, B_{0r'}) = O(r'), d(Y_{r'}, B_{2r'}) = O(r')$. Then $d(Y_{r'}, F_{0r''}) \geq O(r')$, $d(Y_{r'}, F_{0r'}) = O(r')$, $d(Y_{r'}, F_{2r'}) = O(r')$, $d(Y_{r'}, F_{2r''}) \geq O(r')$. It implies that a half-ball B^+ of radius $O(r')$ with center at $Y_{r'}$ is contained in D. Therefore according to Lemma 7.1 the inscribed radius of $B^+ \subset D$ is at least $O(r')$. □

Remark 8.3. *The assertion of Theorem fails if instead of a double simplex we consider an ordinary simplex. As a degenerate example of this possibility consider a regular Euclidean 3-simplex Σ, let P be the center of Σ. Now let N be the cone with the vertex P over the 1-dimensional skeleton Σ^1 of Σ. This is a degenerate simplex whose faces are cones over triangles in Σ^1. We give each face of N a path-metric isometric to the metric on a regular Euclidean triangle. Then the inscribed radius of N is zero. Such examples appear as ultralimits of sequences of nondegenerate Riemannian 3-simplices.*

9 Patterns of intersection

The proof of Theorem 1.2 splits in several cases according to the complexity of the pattern of intersections of flats in the manifold X. We will assume that the manifold

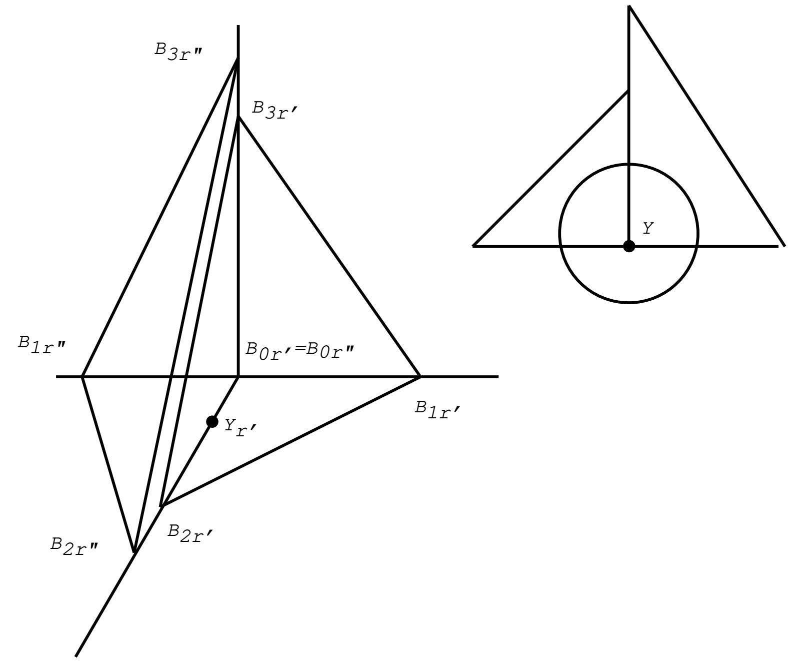

Figure 2:

M is orientable. The group $\Gamma = \pi_1(M)$ is torsion-free since M is aspherical [He1].

Case I: "Simple flats". There exists a flat F in X such that for each $g \in \Gamma$ the intersection $F \cap gF$ is either empty or $gF = F$, such flat is called *simple*.

Case II: "Double intersections". We assume that X contains no simple flats but there is a flat F so that for any elements $g, h \in \Gamma$ the intersection $F \cap gF \cap hF$ is different from a single point (i.e. the intersection is either empty or a complete geodesic or a flat). Such flat F is called *a flat with double intersections*.

Case III: "Triple intersections". We assume that the cases I, II do not occur (the space X contains neither simple flats nor flats with double intersections). Thus for any flat $F \subset X$ there are elements $g, h \in \Gamma$ so that $F \cap gF \cap hF$ is a single point in X.

We consider these cases in different sections.

Remark 9.1. *If g_1, g_2 are complete distance-minimizing geodesics which intersect at two distinct points x, y, then $g_1 = g_2$. This implies that in the Case II (and III) intersection of two (resp. three) flats must be connected.*

The discussion of the Cases II and III is considerably simplified by the following

Theorem 9.2. *Suppose that X contains no simple flats, $\Gamma = \pi_1(M)$ doesn't contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Let F be a flat in X. Define L_F to be the path-connected component of F in the orbit ΓF and let Γ_F denote the stabilizer of L_F in Γ. Then the subgroup Γ_F has finite index in Γ.*

Proof: It's clear that L_F is *precisely-invariant* under Γ_F in L, i.e. if $gL_F \cap L_F \neq \emptyset$ then $g \in \Gamma_F$. Let $\bar{L}_F$ denote the closure of L_F in X.

Lemma 9.3. *The pair (L_F, Γ_F) is* proper.

Proof: Suppose that g_n is a sequence so that $g_n F$ accumulates to a point $x \in X$. Taking if necessary a subsequence we can assume that there is a flat $F' \subset X$ which contains x so that $g_n F$ accumulates to F'. According to our assumptions X has no simple flats. Therefore there exists $\alpha \in \Gamma$ so that $\alpha F'$ intersects F' transversally. It follows that there is a number n_0 so that for all $n, m \geq n_0$, $\alpha g_n F \cap g_m F \neq \emptyset$. Let $\gamma = g_{n_0}$. Hence $\alpha g_n \gamma^{-1} \in \Gamma_{\gamma F}$ and $x \in \bar{L}_{\gamma F}$. Then $g_n \in \gamma \Gamma_F$. □

Remark 9.4. *Note that the same arguments as above prove that either X contains a simple flat (which is impossible) or $\bar{L}$ is path-connected.*

Thus Lemma 6.6 implies that the stabilizer Γ_F of L_F is a finitely-generated group whose Cayley graph contians a quasi-flat. Hence the group Γ_F is not word-hyperbolic. If Γ_F has infinite index in the group Γ then the Scott compact core M_F of X/Γ_F is an aspherical 3-manifold with nonempty boundary. Therefore Thurston's Hyperbolization Theorem can be applied to M_F and we conclude that since $\pi_1(M_F) \cong \Gamma_F$ contains no $\mathbb{Z} \times \mathbb{Z}$, the group Γ_F is isomorphic to a convex-cocompact subgroup of $PSL(2, \mathbb{C})$. This contradicts the fact that Γ_F is not word-hyperbolic. □

10 Case III: triple intersections

10.1 Parallel transport along flats

Choose any flat $F_1 \subset X$. We denote by L the path-connected component of $\Gamma(F_1)$ which contains the flat F_1. Let Γ_1 denote the stabilizer of L in Γ. Pick a PL path $\gamma \subset L$ which connects points y and x. We shall denote by Π_γ the parallel transport $T_y \to T_x$ along γ.

Lemma 10.1. *Let λ be a closed PL loop contained in the union of flats L. Then the parallel transport along λ is trivial.*

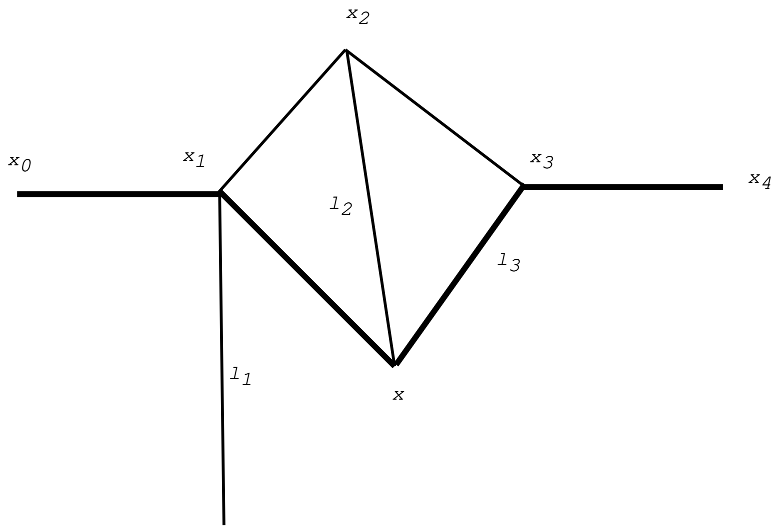

Figure 3:

Proof: We proceed by induction on the the *combinatorial length of* λ, i.e. the number n of its edges. If $n = 2$ then the assertion is obvious. Suppose that the statement is proven for all $k < n$. We consider 4 consecutive segments $[x_0, x_1], ..., [x_3, x_4]$ in λ as on Figures 3, 4.

Let F_j denote a flat in X which contains the segment $[x_j, x_{j+1}]$, let $l_j = F_{j-1} \cap F_j$ be a line through x_j. We first assume that the lines l_2, l_3 are not parallel and intersect in a point $x \in F_1 \cap F_2 \cap F_3$ (Figure 3). Substitute the PL path $[x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_4]$ in λ by $[x_1, x] \cup [x, x_4]$ to construct a new PL loop λ'. The move $\mu : \lambda \to \lambda'$ decreases the combinatorial length of the loop λ.

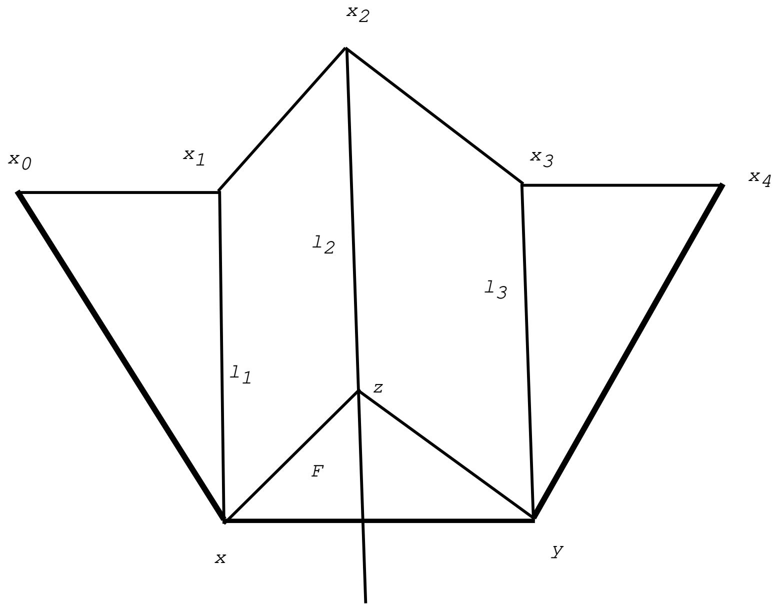

Figure 4:

Now we suppose that all three lines l_2, l_3, l_4 are parallel (otherwise we can apply the move μ). By the "triple intersection" assumption there exists a flat $F \subset L$ which is transversal to l_2 at the point z. Therefore it intersects l_1, l_3 at points x, y (see Figure 4). Hence we can substitute the PL path $[x_0, x_1] \cup ... \cup [x_3, x_4]$ by the path $[x_0, x] \cup [x, y] \cup [y, x_4]$. Denote the new PL curve by λ'. The move $\nu : \lambda \to \lambda'$ again decreases the combinatorial length of the path λ by 1. The parallel transport along λ' is trivial by the induction hypothesis.

Let us consider now only the case of the move ν, the other case is similar. All what we have to prove is that the parallel transport along the loop $[x_4, y] \cup [y, x] \cup [x, x_0] \cup [x_0, x_1] \cup ... \cup [x_3, x_4]$ is trivial. Using triviality of the parallel transport in the planes F_1, F_4 we reduce the problem to the curve $[x_3, y] \cup [y, x] \cup [x, x_1] \cup [x_1, x_2] \cup [x_2, x_3]$. Then we transform this loop to $[x_2, z] \cup [z, y] \cup [y, x] \cup [x, z] \cup [z, x_2]$ keeping the same parallel transport. The parallel transport along the last loop is obviously trivial. $\square$

Corollary 10.2. *If γ, γ' are two PL paths in L with the same initial point x and the final point y, then $\Pi_\gamma = \Pi_{\gamma'}$.*

Suppose that F', F'' are flats in L, $x \in F'$ and $y \in F''$. There are planes $P' \subset T_x(X), P'' \subset T_y(X)$ such that $\exp_x P' = F'$, $\exp_x P'' = F''$. We call the flats F', F'' "parallel" if for some (any) PL path $\gamma \subset L$ connecting $x \in F'$ and $y \in F''$ we have:

$$\Pi_\gamma P' = P''$$

Lemma 10.3. *If F', F'' are two nonparallel flats in L then they have nonempty intersection.*

Proof: Given two flats $F, F' \in L$ we define the "chain distance" $(F : F')$ between them to be the minimal number n such that there exists a chain of flats in L:

$$F_1 = F, F_2, ..., F_n = F'$$

so that $F_i \cap F_{i+1} \neq \emptyset$. We will prove Lemma by induction on the chain distance $n = (F'' : F')$. For $2 = (F'' : F')$ the assertion is obvious. Suppose that $3 = (F'' : F')$. Consider the chain

$$F_1 = F', F_2, F_3 = F''$$

If the line $F_2 \cap F_3 = l_2$ is not parallel to $l_1 = F_2 \cap F_1$ then $F'' \cap l_1 \neq \emptyset$ and we are done. Suppose that l_1 is parallel to l_2. By the assumption that we are in the Case III there exists another flat $F_2' \subset L$ such that $F_2' \cap F' = l_1'$ is a line in F' which is not parallel to l_1. It follows that $F_2' \cap F_2$ is a line which is not parallel to l_1. Thus it must intersect l_2 and (F_1, F_2', F_3) is another chain of flats. Again, if $l_2' = F_2' \cap F_3$ is not parallel to l_1' then we are done. Otherwise F_3 contains two nonparallel lines l_2, l_2' which are parallel to the flat F_1 via parallel transport in L. It implies that F'' is parallel to F' which contradicts our assumptions.

Now suppose that the assertion of Lemma is proven for all $k < n$ and $n = (F' : F'') > 3$. Consider a chain

$$F_1 = F', F_2, ..., F_n = F''$$

If F_2 is not parallel to F_n then by induction they must intersect which implies that $n = 3$ in which case the assertion is already proven. So we assume that F'' is parallel to F_2. Again as in the case $n = 3$ there exists a flat F_2' so that $F_2' \cap F_1$ is a line l_1' which is not parallel to $l_1 = F_2 \cap F_1$. The intersection $F_3 \cap F_2'$ is nonempty since otherwise $F_3 \cap F_1 \neq \emptyset$ and $(F' : F'') < n$. Thus

$$F_1, F_2', ..., F_n = F''$$

is again a chain of flats. Now F'' can't be parallel to F_2' which implies that $F'' \cap F' \neq \emptyset$. This means that $n \leq 3$. □

10.2 Holonomy representation

Pick a base-point $x \in \ell \subset F_1$. We define a representation $\rho : \Gamma_1 \to SO(T_xX) \cong SO(3)$ as follows. Let $g \in \Gamma_1$, $y = g(x)$. Choose a PL path $\gamma \subset L$ which connects y and x. Denote by Π_γ the parallel transport $T_y \to T_x$ along γ. The derivative of g is a map $Dg_x : T_x \to T_y$. Thus we let $\rho(g) = \Pi_\gamma \circ Dg_x : T_x \to T_x$, $\rho(g) \in SO(3)$. Corollary 10.2 implies that the map ρ is well-defined. We call ρ a *holonomy* representation of the group Γ.

Lemma 10.4. *The map ρ is a homomorphism.*

Proof: Take two elements $g, h \in \Gamma_1$, choose a PL curve $\alpha \subset L$ connecting gx to x, PL curve $\beta \subset L$ connecting hx to x and a PL curve $\gamma \subset L$ connecting $hg(x)$ to gx. We need to check that

$$\Pi_\alpha \circ \Pi_\gamma \circ D_{gx}(h) \circ D_x(g) = \Pi_\beta \circ D_x(h) \circ \Pi_\alpha \circ D_x(g)$$

However according to Corollary 10.2

$$\Pi_\beta^{-1} \circ \Pi_\alpha \circ \Pi_\gamma = \Pi_{h\alpha}$$

Since h is an isometry it commutes with the parallel translation which implies

$$\Pi_{h\alpha} \circ D_{gx}(h) = D_x(h) \circ \Pi_\alpha$$

□

10.3 Construction of a recurrent pair

Let $F_j, j = 1, 2, 3, 4$ be flats in X so that each three of them intersect transversally in a point and these four points of triple intersection are distinct. Since $\pi_2(X) = 0$, the points of triple intersection span a 3-simplex Δ in X whose faces are contained in the flats F_j. In this case we shall say that the flats F_j-s *generate* the simplex Δ.

Suppose that $F_1^0, F_2^0 \in \Gamma(F)$ are flats in $X = \tilde{M}$ which intersect along a geodesic ℓ^0. Corollary 6.2 implies that there exists a sequence of elements $g_n \in \Gamma = \pi_1(M)$ such that $\ell = \lim_{n\to\infty} g_n(\ell^0)$ is a recurrent geodesic. This geodesic is the intersection of the flats $F_j = \lim_{n\to\infty} g_n F_j^0$. (Here the convergence is understood in the Chabity topology.) The pair of flats (F_1, F_2) is a *recurrent pair*.

Since we consider the Case III, there exists an element $g \in \Gamma$ such that gF_i intersects ℓ transversally $(i = 1, 2)$.

For the flat F_1 we construct the connected components L_1 and the linear representation ρ of the stabilizer Γ_1 as in Sections 10.1, 10.2.

There are three cases to consider now:
(a) $\rho(\Gamma_1)$ is a finite subgroup of $SO(3)$.
(b) ρ is an infinite reducible representation.
(c) $\rho(\Gamma_1)$ dense in $SO(3)$.

Lemma 10.5. *In the case (c) it follows that the group Γ is not amenable.*

Proof: The homomorphic image of any amenable group is again amenable. Thus if Γ is amenable then so is $\rho(\Gamma_1)$. However it follows from the classification of amenable linear groups that the amenable group $\rho(\Gamma_1) \subset SO(3)$ must be almost Abelian. Hence in this case ρ is a finite or reducible representation which contradicts the property (c). □

By Theorem 9.2 we can assume that the group Γ_1 has finite index in Γ. Since it is enough to prove Theorem 1.2 for a finite-index subgroup we let $\Gamma := \Gamma_1$ so that the orbit ΓF_1 is path-connected.

10.4 Cases (a) and (b) of amenable holonomy

First we consider the Case (a). Denote by Γ_1' the kernel of ρ, which is a subgroup of finite index in Γ_1. In this case Γ_1' preserves the foliation of L by lines parallel to ℓ and the discussion reduces to the Case II.

Consider the Case (b): the representation ρ is infinite and reducible. It implies that a subgroup Γ' of index 2 in Γ admits an infinite representation in $U(1)$ and hence $H_1(\Gamma', \mathbb{R}) \neq 0$. Thus the 2-fold covering X/Γ' of the manifold M is homotopy-equivalent to a Haken manifold and we can apply Theorem 2.7 to conclude that $\Gamma \supset \mathbb{Z}^2$. This finishes the proof in the Case (b).

10.5 Generation of simplices: Case (c)

In what follows we shall consider the Case (c): the group $\rho(\Gamma)$ is dense in $SO(3)$. Note that according to a theorem of Bass [Ba] the group $\rho(\Gamma)$ either splits as an amalgamated free product, or HNN extension or (after conjugation in $SO(3)$) entries of the matrices in $\rho(\Gamma)$ belong to a ring of algebraic integers. First two cases imply that the manifold M is Haken which would finish the proof. Examples of representations such that entries of $\rho(\Gamma)$ belong to a ring of algebraic integers can be constructed using arithmetic subgroups of $PSL(2, \mathbb{C})$. In this case we do not see any algebraic arguments which can simplify our proof. Hence we will use geometry.

Proposition 10.6. *The orbits $\Gamma(F_1), \Gamma(F_2)$ contain three flats F_3, F_4, F_5 so that the flats $F_1, ..., F_5$ generate two distinct simplices T', T'' which form a* double simplex *in the sense of Section 8. These simplices have the properties:*

- *Their intersection is a triangle which is contained in the flat F_1;*
- *Both flats F_1, F_2 participate in generation of the simplices T', T'' (see Figure 5).*

Proof: Choose any flat F_3 which is transversal to ℓ and denote by x the point of intersection $\ell \cap F_3$. For the convenience we introduce in T_xX a metric $\langle\langle, \rangle\rangle$ where the lines of intersection $F_1 \cap F_2 = Span(e_3), F_2 \cap F_3 = Span(e_1), F_3 \cap F_1 = Span(e_2)$ are orthogonal. Since the group $\rho(\Gamma_1)$ is dense in $SO(3)$ there are elements g_4, g_5 in Γ_1 so that normal vectors n_4 , n_5 (with respect to $\langle\langle, \rangle\rangle$) of the planes $\rho g_4(F_1)$, $\rho g_5(F_1)$ have the properties:

(1) $\langle\langle n_j, e_3 \rangle\rangle > 0$, $j = 4, 5$;

(2) the points $P_4 = (\langle\langle n_4, e_1 \rangle\rangle, \langle\langle n_4, e_2 \rangle\rangle), P_5 = (\langle\langle n_5, e_1 \rangle\rangle, \langle\langle n_5, e_2 \rangle\rangle) \in \mathbb{R}^2$, do not lie on coordinate lines and belong to two different but adjacent open coordinate quadrants in $\mathbb{R}^2$.

Since the geodesic ℓ is recurrent, there exists a sequence $g_n \in \Gamma$ such that

$$\lim_{n \to \infty} \rho(g_n) = 1$$

$$\lim_{n \to \infty} g_n(\ell) = \ell$$

Thus for large n the flats $g_n g_4(F_1), g_n g_5(F_1)$ will intersect the line ℓ in points z, y which are not separated by the point x and the properties (1), (2) are still satisfied by the normal vectors to these flats.

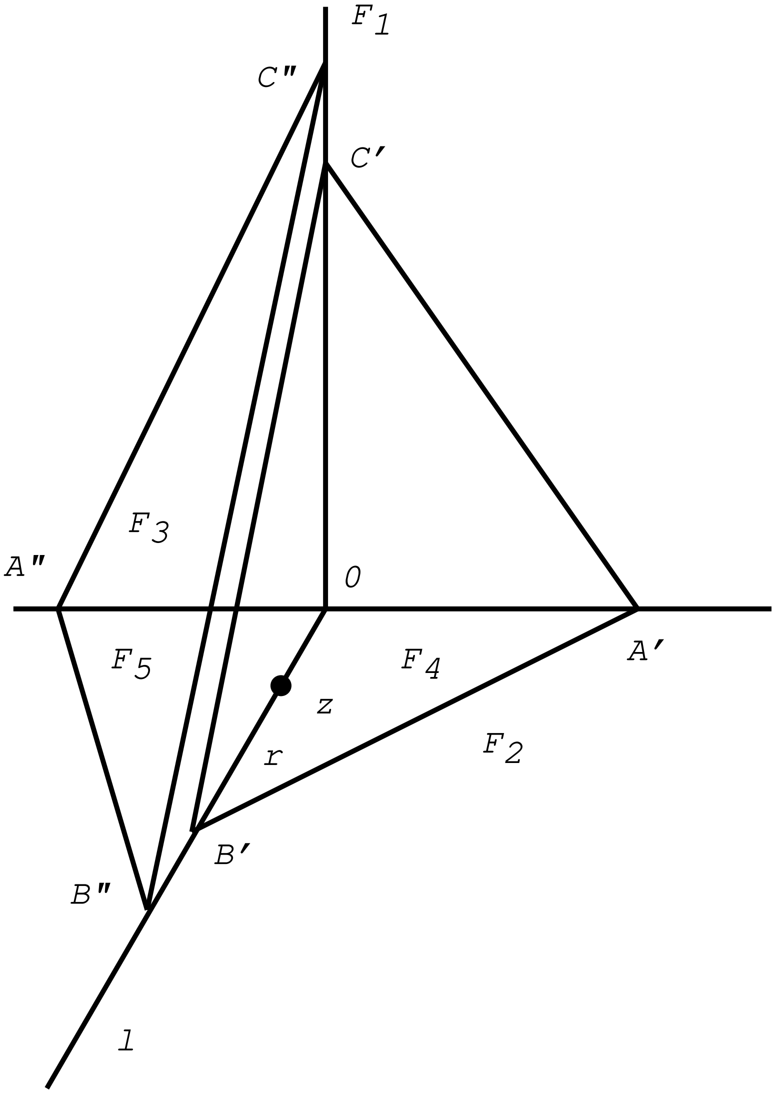

Figure 5:

It follows that $F_1, F_2, F_3, g_n g_4(F_1) = F_4, g_n g_5(F_1) = F_5$ form a configuration satisfying the assertions of Proposition 10.6. □

The arguments below are based on the following fact of Euclidean geometry. Suppose that T is a tetrahedron in $\mathbb{R}^3$ where we know dihedral angles at two vertices. Then we can find all dihedral angles at two other vertices as continuous functions of the known angles. Indeed, suppose T has vertices A, B, C, D and we know all the angles at A, B. Then we know dihedral angles at two edges emanating from C. The planar angle ACB between these two edges is $\pi - \angle CBA - \angle BAC$. Then we find the last dihedral angle at C from two known dihedral angles and ACB by the cosine formula of the spherical trigonometry. The same argument works for the vertex D.

Since the geodesic $\ell = F_1 \cap F_2$ is recurrent, there exist a sequence of elements $g_n \in \Gamma$ so that $g_n(\ell)$ is convergent to ℓ in the Chabity topology. Let $1 = g_0$. Now we fix the flats F_1, F_2, F_3 and apply the sequence of covering transformations $\{g_n\}$ to the flats F_4, F_5. Let $F_{j,n} = g_n(F_j), j = 4, 5$. Since $\rho(g_n) \to 1$ the flats $F_{j,n}$ intersect the line ℓ in F_1, F_2 by the angles $\alpha_{1,j,n}$, $\alpha_{2,j,n}$ which approximate the angles $\alpha_{1,j,0}$, $\alpha_{2,j,0}$.

Therefore the flats $F_1, F_2, F_3, F_{j,n}$ generate simplices $T_{j,n}$ in X. These simplices have flat faces and the angles at vertices of these simplices, which are continuous functions of the angles $\alpha_{1,j,n}$, $\alpha_{2,j,n}$, approximate the angles of the initial simplex $T_{j,0}$. The dihedral angles at the vertex $F_1 \cap F_2 \cap F_3$ of $T_{j,n}$ are fixed. Thus similarity classes of Euclidean models of the simplices $T_{j,n}$ do not degenerate as $n \to \infty$.

Denote $T_{4,n}$ by T'_n and $T_{5,n}$ by T''_n. We let O, A'_n, B'_n, C'_n be the vertices of T'_n and O, A''_n , B''_n , C''_n be the vertices of T''_n . It is clear that the simplices T'_n, T''_n form a double simplex D_n . Denote by r'_n the distance $d(B'_n, O)$ and by r''_n the distance $d(O, B''_n)$. Clearly $r''_n \to \infty$ and $r'_n \to \infty$. In Lemma 10.7 we will show that this convergence to infinity has the same rate.

Lemma 10.7. $r'_n = O(r''_n)$

Proof: By taking n sufficiently large we can guarantee that $d(g_n(B'), \ell) \leq \lambda$ and $d(g_n(B''), \ell) \leq \lambda$ where λ is given by Proposition 5.3. Connect $g_n(B')$ to ℓ by the shortest segment $I_n = [g_n(B'), w_n]$. Take the unit normal vector $\nu_{B''}$ to F_5 at the point B'' and the unit tangent vector $\epsilon_{B''}$ to ℓ at B''. Then $|\angle(\epsilon_{B''}, \nu_{B''}) - \pi/2| \geq \xi_1 > 0$. Similarly if $\nu_{B'}$ is a unit normal vector to F_4 at B' then $|\angle(\epsilon_{B'}, \nu_{B'}) - \pi/2| \geq \xi_2 > 0$. Let $\xi = \min(\xi_1, \xi_2)$.

Since g_n are isometries we get: $\langle \nu_{B'}, \epsilon_{B'} \rangle = \langle Dg(\nu_{B'}), Dg(\epsilon_{B'}) \rangle$. On the other hand, the geodesics $g_n \ell$ are convergent to ℓ thus there exists a number n_0 such that for all $n > n_0$ we have:

$$\angle(\epsilon_{w_n}, \Pi_I Dg(\epsilon_{B'})) \leq \xi/2$$

where ϵ_{w_n} is the unit tangent vector to ℓ at the point w_n obtained from $\epsilon_{B'}$ by parallel transport along ℓ. Thus

$$|\angle(\epsilon_{w_n}, \Pi_I Dg(\nu_{B'})) - \pi/2| \geq \xi/2$$

It follows from Lemma 5.3 that the point of intersection $x_n := \ell \cap g_n F_4$ is at the distance at most $u(\lambda, \xi/2)$ from $g_n(B')$ for all $n \geq n_0$. Similarly we can find n_1 so that for each $n \geq n_1$ the point of intersection $\ell \cap g_n F_5$ is at the distance at most $u(\lambda, \xi/2)$ from $g_n(B'')$ for all $n \geq n_1$. However $d(g_n(B'), g_n(B'')) = d(B', B'')$. Thus

$$d(g_n F_5 \cap \ell, g_n F_4 \cap \ell) \leq 2u(\lambda, \xi/2) + d(B'', B')$$

for all $n \geq \max(n_0, n_1)$. □

Lemma 10.8. *The group* Γ *has polynomial growth. (Actually the growth is at most quadratic.)*

Proof: According to Lemma 10.7, $r''_n = O(r'_n)$, so we let $r_n := r'_n$. Thus by Theorem 8.2 we get a sequence of double simplices $D_n = T'_n \cup T''_n$ such that their inscribed radius ι_n is $O(r_n)$. The area of each ∂D_n is at most $Area(\partial T''_n) + Area(\partial T'_n) = O(r_n^2)$ since these simplices have Euclidean boundary. However the group $\Gamma = \pi_1(M)$ is not amenable which implies that

$$Vol(D_n) = O(r_n^2)$$

Let B_{ι_n} be a sequence of metric balls of the radius ι_n inscribed in D_n. Then

$$Vol(B_{\iota_n}) \leq Vol(D_n) = O(r_n^2) = O(\iota_n^2)$$

□

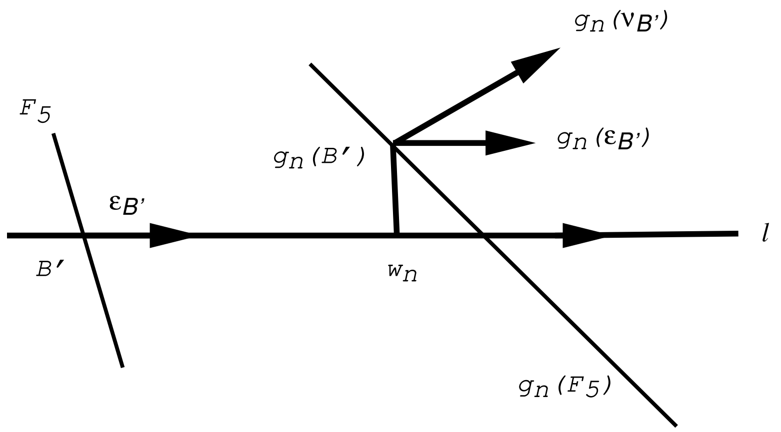

Figure 6:

Remark 10.9. *Formally speaking the group* Γ *has polynomial growth if for any sequence of balls* B_n *of radius* n *in* X *the volume of* B_n *grows slower than a polynomial function. However, a version of Gromov's theorem on groups of polynomial growth [VW] implies that it is enough to check the growth condition for a* sequence *of radii which tend to infinity.*

All the groups of polynomial growth are almost nilpotent [Gro1], [VW]. Thus Lemma 10.8 contradicts Corollary 10.5. It proves that the Case (c) actually can't occur which finishes the proof of Theorem 1.2 in the Case III. □

11 Case II: double intersections

Suppose that F is a flat in the space X which has only double intersections. We define $L = L_F$ to be the connected component of F in $\Gamma(F)$ and let Γ_F denote the stabilizer of L_F in Γ. Let $\bar{L}_F = \bar{L}$ be the closure of L. Again, each point of $\bar{L}$ is contained in a flat and intersection of any three flats from $\bar{L}$ is always different from a single point. The same arguments as in the Case III imply that $F = F_1$ can be chosen so that it contains a recurrent geodesic ℓ such that $\ell = F_1 \cap F_2$, where F_2 is another flat in X. By Theorem 9.2 we may assume that Γ_F is a finite-index subgroup in Γ, so we let $\Gamma := \Gamma_F$. Let $\mathcal{L}^0 = \Gamma(F_1 \cup F_2)$. It's clear that this is a path-connected set and its closure $\bar{\mathcal{L}}^0 = \mathcal{L}$ is also path-connected since X contains no simple flats (see Theorem 9.2 and Remark 9.4).

Foliate each flat in $\mathcal{L}$ by geodesics parallel to ℓ. This foliation is preserved under the action of Γ. By taking an index 2 subgroup in Γ we can guarantee that Γ preserves orientation on the fibers of the foliation. Denote by Y the quotient of $\mathcal{L}$ along this foliation and let $f : \mathcal{L} \to Y$ be the projection. We define a path-metric $d_Y(y_1, y_2)$ as

$$\inf\{d_{\mathcal{L}}(x_1, x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$$

where $d_{\mathcal{L}}$ is the path metric on $\mathcal{L}$. Each element $g \in \Gamma$ projects to an isometry $f_*(g)$ of the space Y via f. Note that $Isom(\mathcal{L})$ contains a normal subgroup H which consists of uniform vertical translations along fibers, thus $f_*(H) = \{1\}$.

Let $y_0 = f(\ell)$. We will identify the geodesic ℓ with the real line $\mathbb{R}$. Define $\varphi : \mathcal{L} \to \ell$ to be the nearest-point projection with respect to the path-metric $d_{\mathcal{L}}$. We define a function $v : \Gamma \times \mathcal{L} \to \mathbb{R}$ by

$$v(g, x) = \varphi(gx) - \varphi(x)$$

Clearly this function depends only on the pair $(g, f(x))$. The function v roughly speaking measures the "vertical displacement" of the isometry g.

Note that the space (Y, d_Y) is NOT locally compact. Nevertheless we have the following

Lemma 11.1. *Suppose that $q_n \in \Gamma$ is a sequence and $y_1 \in Y$ is a point such that $f^{-1}(y_1)$ is the intersection ℓ_1 of two flats in $\mathcal{L}$. Assume that $d_Y(f_* q_n(y_1), y_0) \leq const.$ Then $(f_* q_n(y_1))$ contains a convergent subsequence.*

Proof: The assumption that the distance $d_Y(f_* q_n(y_1), y_0)$ is bounded implies that the sequence $q_n \ell_1$ is subconvergent in the Chabity topology in X to a geodesic ℓ_∞. Let $\ell_1 = F' \cap F''$ and $\ell_\infty = F'_\infty \cap F''_\infty$. Fix a point $x \in \ell_\infty$. Denote by $x_n \in q_n(\ell_1)$ the nearest point to x. Let α be the angle between F', F''. Then for large n, $d(x, q_n(\ell_1)) \leq \theta(\alpha, \alpha)$ (see Proposition 5.1).

This implies that one of the flats $q_n(F'), q_n(F'')$ intersects $F'_\infty \cup F''_\infty$ at the distance at most $\kappa(\alpha, \alpha, d(x, x_n))$ from the both x_n, x (by Proposition 5.1). See Figure 7.

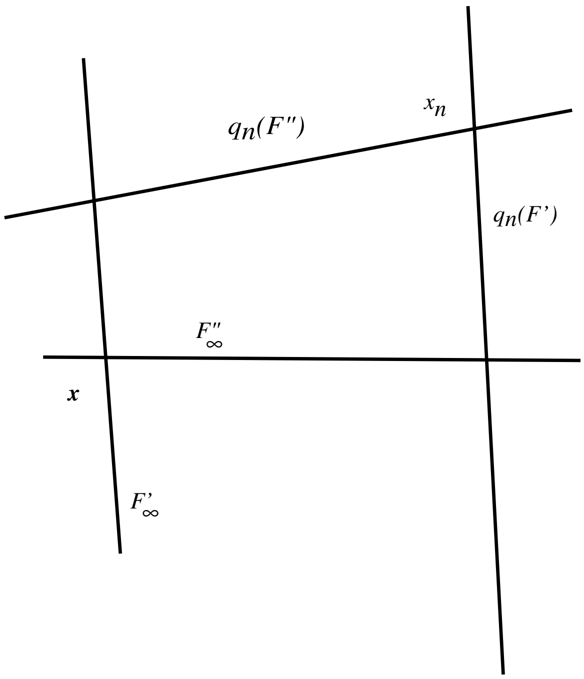

Figure 7:

This implies that $d_{\mathcal{L}}(x_n, x) \to 0$ as $n \to \infty$. $\square$

In particular Lemma 11.1 can be applied to the sequence $q_n = g_n$ and the geodesic $\ell_1 = \ell$. Thus $(f g_n(\ell))$ is convergent in Y to $f(\ell)$ since $\ell = \ell_\infty$. However apriori it is possible that the sequence $f_*(g_n)$ is not convergent to identity uniformly on compacts in Y. To deal with this problem choose any finite subset $K \subset Y$. Then $d(y_0, g_n K)$ remains bounded as $n \to \infty$. Therefore there exists a function $m = m(n) > n$ so that the elements $h_n = g_m^{-1} g_n$ have the property: the sequence $f_*(h_n)$ is convergent to the identity on K.

We choose the finite set K as follows. Denote by $\gamma_1, ..., \gamma_r$ the set of generators of the group Γ. Let y_1 be a point of $f(F_1)$ which is different from $f(\ell) = y_0$ and $f^{-1}(y_1)$ is the intersection of two flats in $\mathcal{L}$. We take

$$K = \{\gamma_1(y_0), \gamma_1(y_1), ..., \gamma_r(y_0), \gamma_r(y_1)\}$$

Suppose that n is sufficiently large and for any $y \in K$ we have $d(y, f_* h_n(y)) \le \zeta$. Direct calculation show that $d(y_j, f_*[\gamma_i, h_n](y_j)) \le 2\zeta$ for each $i = 1, ..., r$; $j = 0, 1$ and $n \in \mathbb{Z}$, where $[a, b] = a^{-1} b^{-1} ab$.

Theorem 11.2. *Suppose that h_n is a sequence as above, $\gamma = \gamma_j$ is one of the generators of Γ. Then there is a finite collection of elements $w_i \in \Gamma$ such that for sufficiently large n, $[h_n, \gamma] \in \{w_1, ..., w_l\}$ and all the elements w_i have trivial projection to Y.*

Proof: Choose elements t_n and $s \in H$ with the vertical displacement the same as $v(h_n, f(\ell))$ and $v(\gamma, f(\ell))$ respectively. Let $\hat{h}_n = t_n^{-1} h_n$, $\hat{\gamma} = s^{-1}\gamma$. Clearly $[\hat{\gamma}, \hat{h}_n] = [\gamma, h_n]$. For each compact $J \subset Y$ we have

$$|v(\hat{\gamma}, y)|, |v(\hat{h}_n, y)| \le c(J) < \infty$$

where $y \in J$ and the constant $c(J)$ depends only on J and not on n. Therefore

$$|v([\hat{h}_n, \hat{\gamma}], y)| \le c(J')$$

where $y \in J$ and $J' \supset J \ni y_0$ is a compact which contains

$$(\gamma(J)) \cup \cup_n h_n(\gamma(J)) \cup \cup_n \gamma^{-1} h_n \gamma(J) \cup$$

$$\cup_n h_n^{-1}(J \cup \gamma(J) \cup \cup_n h_n(\gamma(J)) \cup \cup_n \gamma^{-1} h_n \gamma(J))$$

On the other hand, the sequence $f_*([\gamma, h_n])$ is convergent to the identity on $\{y_0, y_1\}$. By discreteness of Γ we conclude that for large n all the elements $f_*[h_n, \gamma]$ act trivially on $f(F_1)$ and the commutators $[h_n, \gamma]$ belong to some fixed finite set $\{w_1, ..., w_l\} \subset \Gamma$. Since the group Γ preserves the orientation on X the elements $f_*[h_n, \gamma]$ act trivially on Y. Therefore $\{w_1, ..., w_l\} \subset H \cap \Gamma$. ☐

Corollary 11.3. *The group Γ has infinite center.*

Proof: Let $\gamma_1, ..., \gamma_r$ be the set of generators of Γ as before. There are two possible cases. First we suppose that for some $\gamma_i = \gamma$ in Theorem 11.2 the element $w = [h_n, \gamma] \in H$ is nontrivial. Then w belongs to the center of Γ. Otherwise we assume that all the elements $[h_n, \gamma_i] = 1$ for sufficiently large n. Hence $\langle h_n \rangle$ is in the center of Γ. ☐

Finally we apply Geoff Mess's theorem [M1] to conclude that Γ contains $\mathbb{Z}^2$. This finishes our proof in the Case II.

12 Case I: simple flats

We start with a construction, which (in general case) is due to Morgan and Shalen [MS2]. Suppose that $L \subset X$ is a closed Γ-invariant subset which is the union of disjoint 2-flats. The set L is called a *lamination* on X, flats in L are *leaves* of this lamination. We shall assume that none of the leaves F of L has stabilizer in Γ which acts cocompactly on F. It's clear then that L has uncountably many leaves. We eliminate from L all leaves which are boundary flats for more 2 components of $X - L$.

Construct a *dual tree* T to L as follows. If $D \subset X - L$ is a component with the closure $\bar{D}$, collapse $\bar{D}$ to a single point $q(\bar{D}) \in T$. If F is a 2-flat in L which is not a boundary flat for any component $D \subset X - L$, then collapse to a single point $q(F) \in T$. As the *set* T is the quotient of X described above. Let $F \subset L$ be a leaf. Pick a point $x \in F_z$. Then there is a sufficiently small number $\epsilon_0 > 0$ (which depends only on geometry of M) such that: if $[x', x''] \subset X$ is a geodesic segment orthogonal to F_z at x, $d(x, x') = d(x, x'') = \epsilon_0$, then each leaf $F \subset L$ and each connected component $D \subset X - L$ intersects $[x, y]$ by a convex subset. Let $z \in T$ be a point such that $q^{-1}(z)$ is a single leaf of L. Define N_z as an above segment $[x', x'']$ for some choice of $x \in F$, let $\tilde{z} = x$ in this case. Suppose that $z \in T$ is such that $q^{-1}(z)$ is the closure of a component $D \subset X - L$. For each boundary flat F_x of D we pick a point $x \in F_x$ and an orthogonal segment $[x', x]$ disjoint from D which has the length ϵ. Let N_z be the union of such segments over all boundary flats of D and $\tilde{z}$ be the collection of all their end-points x.

Then we define open neighborhoods of $z \in T$ to be subsets $E \subset T$ such that $q^{-1}(E) \cap N_z$ is an open neighborhood of the set $\tilde{z}$ in N_z. It is easy to see that the topological space T is Hausdorff and the group Γ acts on T by homeomorphisms. If none of the complementary regions D of L has more than 2 boundary flats, then T is a 1-dimensional manifold which is clearly a real line. In general the space T is a topological tree, i.e. any two points are connected by a embedded topological arc and this arc is unique. It L has a transversal invariant measure, then T is a metric tree and Γ acts on T by isometries.

12.1 Proof via the Rips Theory

Theorem 12.1. *Suppose that N is a closed aspherical manifold of dimension n. Then $\pi_1(N)$ is neither a nontrivial amalgamated free product nor HNN extension with the amalgamation over $\mathbb{Z}^k$ for any $k < n - 1$.*

Proof: We consider only the case of amalgamated free products, the case of HNN extensions is similar. Suppose that $\pi_1(N) = A *_C B$ where $C \cong \mathbb{Z}^k$. Since this decomposition is nontrivial we conclude that both groups A, B have infinite index in $\pi_1(N)$. This implies that $H_n(A, \mathbb{Z}/2) \cong H_n(X/A, \mathbb{Z}/2) = 0, H_n(B, \mathbb{Z}/2) \cong H_n(X/B, \mathbb{Z}/2) = 0$ where X is the universal cover of N. Since $H_n(C, \mathbb{Z}/2) = H_{n-1}(C, \mathbb{Z}/2) = 0$ we apply the Mayer-Vietoris sequence to the amalgamated free product $\pi_1(N) = A *_C B$ and conclude that $0 = H_n(\pi_1(N), \mathbb{Z}/2) = H_n(N, \mathbb{Z}/2)$. This contradict the assumption that the dimension of N is equal to n. $\square$

The following proof of Theorem 1.2 in the case of simple flats was motivated by discussion with Lee Mosher, who explained to me how to prove Conjecture 1.1 under assumption that the universal cover X contains a *simple* least area surface conformal to $\mathbb{R}^2$.

The closure $\bar{L}$ of the Γ-orbit of a simple flat F is foliated by flats. It projects to a lamination Λ on M which admits a transversal-invariant measure since each leaf of Λ is amenable [P1]. Thus the topological tree T dual to $\bar{L}$ is an metric tree and the group Γ acts on T by isometries. Therefore application of the Rips Theory [R], [BF] (or of a theorem of Morgan and Shalen [MS1]) to T will produce a nontrivial simplicial Γ-tree $R(T)$ where edge-stabilizers are discrete subgroups of $Isom(\mathbb{R}^2)$. This means that the group Γ admits a nontrivial splitting as amalgamated free product of HNN extension where amalgamated subgroups are discrete subgroups of $Isom(\mathbb{R}^2)$. The group Γ is torsion-free and the manifold M is aspherical. Thus none of the amalgamated subgroups can be $\{1\}$ or $\mathbb{Z}$ (Theorem 12.1). This implies that Γ must contain $\mathbb{Z} \times \mathbb{Z}$. $\square$

12.2 Geometric proof

Our arguments here are very similar to the Schroeder's proof in [Sc]. Suppose that F is a simple flat in X. We will assume that Γ contains no $\mathbb{Z}^2$.

Theorem 12.2. *The space X contains a simple flat with nontrivial stabilizer.*

Proof: Consider the closure $\bar{L}$ of the Γ-orbit of the simple flat F. It is foliated by flats. Thus we get a Γ-invariant *lamination* of X by flats. Denote by T the dual tree to this lamination. Our goal is to prove that either T is homeomorphic to $\mathbb{R}$ or there is a leaf of $\bar{L}$ with nontrivial stabilizer in Γ. If $\bar{L} = X$ then $\bar{L}$ is actually a foliation and $T \cong \mathbb{R}$. Suppose now that the complement $X - \bar{L}$ is nonempty. Choose a component W of this complement and let F_1 be a boundary flat of this component. This flat is still simple. Assume that F_1 has trivial stabilizer in Γ, let $F := F_1$ and define

$$Q := \{x \in W : d(x, F) < d(x, F') \text{ for all \quad other boundary flats } F' \text{ of } W\}$$

Then $\gamma Q \cap Q = \emptyset$ for each $\gamma \in \Gamma - \{1\}$. Pick a base-point $q \in F$. For $x \in F$ we define

$$\phi(x) = \inf\{d(x, F') : F' \neq F \text{ is a boundary flat of } W\}$$

Lemma 12.3. *The function $\phi(x)$ tends to zero as $d(x, q) \to \infty$.*

Proof: Suppose that there exists a sequence $x_n \in F$ so that $d(x_n, q) \to \infty$ and $\phi(x_n) \geq \sigma$ for some positive σ. We assume that $d(x_{n+1}, q) > d(x_n, q) + 1$. Then for $\theta < \sigma/2$ the intersection $B_n^+ = B_\delta(x_n) \cap W$ has volume at least $\theta^3/2$ and the Γ-orbits of these balls are disjoint since all B_n^+ are contained in Q. This implies that the manifold M has infinite volume which is impossible. $\square$

Thus there exists $R > 0$ so that for all $x \in F - B_R(q)$ we have $\phi(x) < \epsilon$ where ϵ is given by Proposition 5.2. The set $F - B_R(q)$ is connected. The normal geodesic $l = l_x$ emanating from x intersects the nearest flat F' at the distance at most $\delta =$ the injectivity radius of of M. Hence the geodesic segment of l between x and F' is disjoint from any other flat in ∂W. Indeed, if it intersects one of these flats F'' before meeting F' at the time t_0 then to intersect F' at the time $t_1 > t_0$, the geodesic must first intersect F'' again at some time $t_2 \in (t_0, t_1)$. This contradicts the assumption that F'' is a flat (since l is distance minimizing for all $t < t_1$).

As in [Sc] we conclude that the nearest flat $F' = F_x \subset \partial W$ doesn't vary as we vary x in $F - B_R(q)$. In particular $d(x, F') < \epsilon$ for all $x \in F - B_R(q)$. Denote by E the part of X contained between F, F'. Since F, F' are Hausdorff-close, there is no other flats in E. Therefore $W = E$ has only two boundary components: F, F' and the same is valid for all components W of $X - \bar{L}$. This implies that the tree T dual to the lamination $\bar{L}$ is a real line.

Hence we get an action of Γ on $\mathbb{R}$ by homeomorphisms. It follows that either Γ is Abelian or one of leaves of $\bar{L}$ has nontrivial stabilizer (Theorem 6.3). This concludes the proof of Theorem 12.2. $\square$

Remark 12.4. *Alternatively in the last argument one can appeal to Theorem of Imanishi [I].*

Now suppose that F is a simple flat in X with the nontrivial stabilizer Γ_o. This must be an Abelian group acting discretely and isometrically on $\mathbb{R}^2$. Since Γ contains no $\mathbb{Z}^2$ it implies that Γ_o is an infinite cyclic group acting by translations in F. Denote by $\ell \subset F$ an invariant line for $\Gamma_o = \langle\gamma\rangle$. Let G denote the centralizer of Γ_o in Γ. Since for each $g \in G$ the elements g, γ commute, the flat gF is also γ-invariant. The displacement number of γ in gF is the same as in F. Consider the orbit L_F of F under G and denote by $\bar{L}_F$ its closure in X.

Lemma 12.5. *The pair $(\bar{L}_F, G)$ is proper.*

Proof: Suppose that the pair is not proper and $x \in X$ is an accumulation point for $g_n F, g_n \in \Gamma$. Since F is simple $g_n F$ accumulates also to a flat F' which contains x. Denote by $x_n \in F$ a sequence such that $g_n x_n \to x$. The displacement of γ in F equals C, thus $d(\gamma x_n, x_n) = C < \infty$. Hence the displacements of $g_n \gamma g_n^{-1}$ are also bounded by C at $g_n x_n$. This implies that elements $g_n \gamma g_n^{-1}$ have displacement at x bounded by $C + 1$ for large n. Since Γ is a discrete group we (taking a subsequence if necessary) can assume that $g_n \gamma g_n^{-1} = g_m \gamma g_m^{-1}$ for all n, m. This means that all the elements $h_{nm} = g_n^{-1} g_m$ commute with γ. Thus all h_{nm} belong to the subgroup G and $(\bar{L}, G)$ is a proper pair. $\square$

Corollary 6.7 implies that G contains a finitely generated infinite noncyclic subgroup G_0 with nontrivial center $\langle\gamma\rangle$. Thus according to Mess's theorem [M1], G_0 contains $\mathbb{Z}^2$. This finishes the proof of Theorem 1.2. $\square$

13 Closing up Euclidean planes

In this Section we will prove that under some topological restrictions the existence of a flat in a 3-manifold M implies the existence of an immersed incompressible flat torus in M.

Suppose that M is a closed aspherical orientable Riemannian manifold which contains a flat. Then by Theorem 1.2 there exists a subgroup isomorphic to $\mathbb{Z}^2$ in M. Apriori the manifold M is not irreducible, however it can be represented as a connected sum $N \# \Sigma$ where Σ is a homotopy sphere [He1] and N is either

Haken or Seifert manifold. In any case N has a canonical (Jaco-Shalen-Johannson) decomposition into hyperbolic and Seifert components. We assume that N has no Seifert components at all, thus it is obtained by gluing hyperbolic manifolds along boundary tori and Klein bottles. These boundary surfaces separate N; since N is orientable they must be tori.

Theorem 13.1. *Under the conditions above M contains an immersed incompressible flat torus.*

Proof: Any flat F in $\tilde{M}$ is a quasi-flat in $\tilde{N}$. In the paper [KL] we classify quasi-flats in universal covers of Haken manifolds. Provided that M has no Seifert components, [KL] implies that there exists an i ncompressible torus T embedded in M and a number $r < \infty$ so that F is contained in an r-neighborhood of the universal cover $\tilde{T} \subset X = \tilde{M}$.

Remark 13.2. *If M is a hyperbolic 3-manifold with nonempty boundary of zero Euler characteristic, then the existence of such torus T was first proven by R. Schwarzt in [Sch].*

Denote by A the fundamental group of T operating on $\tilde{T} = S$. This group is a maximal Abelian subgroup of Γ.

The Hausdorff distance $d_H(gF, S)$ is bounded from above independently on $g \in A$. We let $\bar{L}$ denote closure of the orbit $A(F)$. The quotient $\bar{L}/A$ is compact in M.

Lemma 13.3. *There exists a subgroup Γ' of finite index in Γ which contains A so that $\bar{L}$ is precisely invariant under A in Γ'.*

Proof: Recall that Γ is residually finite [He2]. There is at most a finite number of elements $g_1, ..., g_k \in \Gamma - A$ such that $g_j\bar{L} \cap \bar{L} \neq \emptyset$ and A is a maximal Abelian subgroup of Γ. Thus by applying [L] we conclude that Γ contains a finite-index subgroup Γ' which contains A and doesn't intersect $\{g_1, ..., g_k\}$. $\square$

We let $\Gamma := \Gamma'$ and retain the notation M for X/Γ'. Now we will apply our analysis of flats in 3-manifolds to the flat F.

First we suppose that F is a simple flat. Let F' be one of the flats in $\bar{L}$ which is the most distant from S in the Hausdorff metric. There are at most two such flats since all the flats in $\bar{L}$ are disjoint. Hence F' is invariant under an index 2 subgroup in A which implies Theorem 13.1.

Suppose now that any flat in $\bar{L}$ has "triple intersections". By compactness of $\bar{L}/A$ we can assume that $F = F_1$ intersects a flat $F_2 \subset \bar{L}$ along a recurrent geodesic. The group Γ_1 (as in Section 10) is contained in A by Lemma 13.3. Then we have three possible cases (a), (b), (c) according to the holonomy representation $\rho : \Gamma_1 \to SO(3)$. In the Cases (b), (c) we get: $\Gamma = \Gamma_1$ which is impossible. Hence either we have the Case III-a or the Case II (flats with double intersections). Note that Γ_1 is either infinite cyclic or is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. However the quotient $cl(\Gamma_1(F_1 \cup F_2))/\Gamma_1$ is compact. Thus Γ_1 is not cyclic and it must have a finite index in A. In the both cases III-a and II we have a finite-index subgroup $A' \subset A$ which preserves a parallel family of Euclidean geodesics on the orbit $L' = A'(F_1)$. From now on we consider the only the subgroup A' and the orbit L' so that the Cases III-a and II become indistinguishable.

Each flat F_j in $\bar{L}'$ separates X into two components, we let F_j^+ denote the "right side" and F_j^- denote the "left side" of F_j. We let S^+ denote the union of the right sides and S^- the union of left sides. Their complements C^+, C^- are disjoint open convex subsets of X whose boundaries $B^\pm$ are foliated by parallel lines ℓ_x. Both $B^\pm$ are Hausdorff close to the surface S and invariant under A', so the quotients $B^\pm/A'$ are tori. Each flat in L' separates C^+ from C^-. Now let a, b be generators of the group A and I be a shortest geodesic segment in X connecting B^+ and B^-. Hence each $F \subset \bar{L}'$ intersects I and this intersection consists of a single point. We identify I with an interval $[-h, h] \subset \mathbb{R}$ (here $h \geq 0$) so that $\pm h$ correspond to points on $B^\pm$. The surface B^+ is identified with the plane $\mathbb{R}^2$ which is foliated by vertical lines $\ell_x, x \in \mathbb{R}$. If one of the lines ℓ_x is invariant under an element $g \in A' - \{1\}$ then g leaves invariant any flat in $\bar{L}'$ which contains ℓ_x (otherwise B^+ is not g-invariant). We pick a generator a of A' which doesn't keep (any) line ℓ_x invariant. Therefore a acts on $\mathbb{B}^\pm \cong \mathbb{R}^2$ as a translation $(x, y) \mapsto (x + \alpha, y + \beta)$, we shall assume that $\alpha > 0$. Identify 0 on the x-axis with the projection of the point $I \cap B^+$. Now we pick a flat $F \subset \bar{L}'$ which intersects B^+ along a line (or a strip) whose projection to the x-axis is positive. For each $n > 0$ we let $\{h_n\} = a^n(F) \cap I$. Denote by π the projection of B^+ to the x-axis along the lines ℓ_x.

Lemma 13.4. *The sequence $h_n \in [-h, h]$ is monotone.*

Proof: For $n \geq 0$ we let $[z_n^-, z_n^+]$ denote the projection of the intersection $a^n(F) \cap B^+$ to the x-axis; these intervals belong to the positive ray $\mathbb{R}^+$. If $n > m \geq 0$ then $z_n^+ > z_m^+ > 0$. Thus $\pi^{-1}(z_n^+) \subset a^m(F)^+$ (see Figure 8). Note that $[h_m, h]$ also lies in $a^m(F)^+$. On the other hand, z_m^+ separates 0 from z_n^+. Suppose now that $h_n < h_m$. Then flat $a^n(F)$ intersects $a^m(F)$ in a non-connected set which is impossible. $\square$

Therefore there exists a limit $I \ni h_\infty = \lim_{n\to\infty} h_n$. The points $a(h_n)$ are convergent to a point $a(h_\infty)$. Let F_∞ be the union of flats of accumulation for the sequence $a^n(F)$.

Each flat in F_∞ must pass through the points $h_\infty, a(h_\infty)$. Recall however that any pair of flats in X intersect by a connected set, thus F_∞ consist of a single flat which must be invariant under the element a. Suppose that F_∞ is not b-invariant. Then we repeat the same argument as above by applying the sequence b^n to F_∞. The limiting flat Φ must be invariant under the both generators a, b. Hence Φ/A' is a torus. This finishes the proof of Theorem 13.1. $\square$

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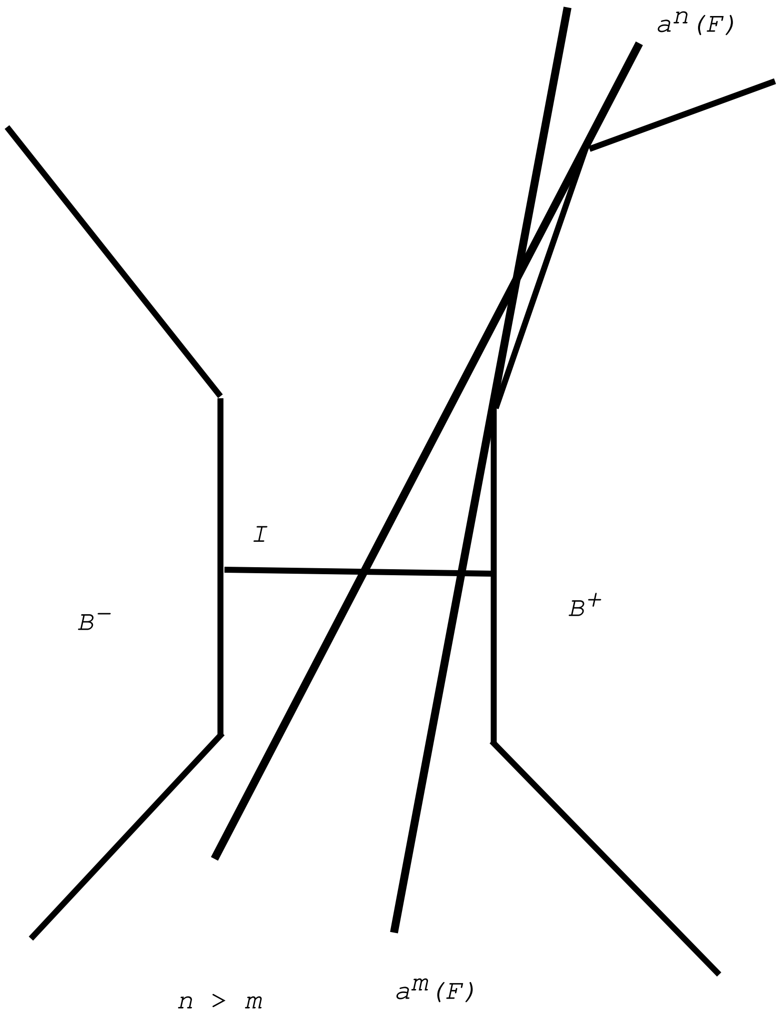

Figure 8:

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Michael Kapovich: Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA ; kapovich@math.utah.edu