COARSE FIBRATIONS AND A GENERALIZATION OF THE SEIFERT FIBERED SPACE CONJECTURE

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Abstract. We prove a version of the Seifert fibered space conjecture for coarsely fibered 3-manifold groups. As an application of our technique we prove that all 2-dimensional Poincare duality groups over commutative rings are commensurable with surface groups.

1. Introduction

The main goal of this paper is to establish a coarse analogue *Seifert fibered space conjecture* for 3-manifolds. We recall that this conjecture (proved as the result of collective efforts of a large number of mathematicians in early 1990-s) assertes that:

If M is a 3-dimensional irreducible manifold whose fundamental group has infinite center, then M is a Seifert manifold.

Note, that if γ is a smooth loop in M representing a nontrivial element of the center of $G = \pi_1(M)$, then the inverse image of γ in the universal cover X of M is a family of proper (noncompact) curves L (not necessarily disjoint or embedded) which is invariant under the action of the fundamental group G. The above curves are not just proper in X, but are *uniformly proper*, i.e., there exists a *distortion function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each curve L,

$$\eta(d_L(x, y)) \leq d(x, y)$$

where $\lim_{t\to\infty} \eta(t) = \infty$ and d_L is the distance measured along L and d is the distance in X with respect to a G-invariant Riemannian metric. Moreover, the distance between curves L, L' in this family cannot oscillate arbitrarily: if $x \in L, x' \in L'$ are within distance $\leq r$ then any point $y \in L$ is within distance $\leq \psi(r)$ from a point in L', where the function ψ does not depend on L, L', x, x', and y.

Similar examples of families of curves in Riemannian manifolds X appear as fibers of Riemannian submersions. One such example is given by Solv-manifolds:

Let $h : T^2 \to T^2$ be an Anosov affine mapping; recall there are two h-invariant foliations $\mathcal{A}$ of T^2 by affine geodesics. Let M be the mapping torus of h. Then each foliation $\mathcal{A}$ yields a 1-dimensional lamination $\mathcal{L}$ on M, whose lift to the universal cover X gives rise to a family of curves L satisfying all the above properties.

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Note however that in the latter example, no curve L is invariant under a nontrivial element of G. Moreover, no curve L is Hausdorff-close to a curve invariant under $g \in G \setminus \{1\}$. More examples of the same kind are given by irrational affine foliations on 3-dimensional tori. Combining the above examples we arrive to the following

Definition 1.1. Let X be a metric cell complex, see section 2 for the definition. For the purposes of the introduction the reader can assume that X is a simplical complex, which admits a cocompact group action, where X is metrized so that each simplex is isometric to the standard simplex in the Euclidean space. We define a *coarse fibration* $\mathcal{F}$ of X as follows. Let $\{F_\alpha, \alpha \in A\}$ be a collection of finite-dimensional metric cell complexes ("fibers"); suppose that for each $\alpha \in A$ we are given a map $f_\alpha : F_\alpha \to \bar{F}_\alpha \subset X$ which is uniformly proper with the distortion function η independent on α. In addition we assume that

(1) The union $\cup_{\alpha \in A} \bar{F}_\alpha$ is Hausdorff-close to X, and

(2) There is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that if for $\alpha, \beta \in A$ there are points $x \in \bar{F}_\alpha, y \in \bar{F}_\beta$ such that $d(x, y) \leq r$ then $d_H(\bar{F}_\alpha, \bar{F}_\beta) \leq \psi(r)$.

If $G \curvearrowright X$ is an (isometric) group action then we say that a coarse fibration $\mathcal{F}$ is G-invariant if for each $g \in G$ and $\alpha \in A$ we have: $g \circ f_\alpha \in \mathcal{F}$. We will say that $\mathcal{F}$ is a coarse fibration by lines if F_α is isometric to $\mathbb{R}$ for each $\alpha \in A$.

Example 1.2. Suppose that X is a Cayley graph of a group G, $C \subset G$ is a central subgroup. Let F be a Cayley graph of C. There is a natural map $F \to \bar{F} \subset X$. Thus for each element $g \in G$ we have a map

$$f_g : F \to \overline{F_g} = g\bar{F} \subset X$$

induced by $C \subset gC \subset G$. Then the collection $\{(F, f_g), g \in G\}$ is a coarse fibration of X.

Observe that a coarse fibration is a coarse analogue of a Riemannian submersion between complete Riemannian manifolds: fibers of such submersion are equidistant from each other.

We note that coarse fibrations of the universal covers of 3-manifolds M appear in the recent work of D. Calegari [12]. Namely, he considers *quasi-homomorphisms* h of the fundamental group $G = \pi_1(M)$ into $\mathbb{R}$. The inverse images $h^{-1}(x), x \in \mathbb{R}$, define a coarse fibration of the group G and, therefore, of the universal cover X of M. Note however that such coarse fibrations are far from being *by lines*; rather, under an appropriate finiteness assumption, they are coarse fibrations by *quasi-planes*. Another instance where coarse fibrations appear in the geometric group theory is in relation to Bieri-Neumann-Strebel invariant for finitely-generated groups [5].

The main result of our paper is:

Theorem 1.3. *Suppose that M is a closed 3-manifold whose universal cover admits an invariant coarse fibration by lines. Then either M is homotopy-equivalent to a Seifert manifold or to a* Solv*-manifold.*

In section 9.1 we will also classify all invariant coarse fibrations of universal covers of Seifert and Sol-manifolds.

As an application of our results we get an alternative proof of

Theorem 1.4. *(P. Tukia, G. Mess, D. Gabai, A. Casson and D. Jungreis.) Let M be a closed irreducible 3-manifold whose fundamental group has a nontrivial center. Then M is homeomorphic to a Seifert manifold.*

To be more precise, we are giving an alternative proof to the part of the above theorem which was proven by G. Mess in [30]. Our proof still relies upon the classification of uniform convergence groups acting on S^1, which was proven by Tukia, Gabai, Casson and Jungreis. We note that other alternative proofs of Geoff Mess' theorem were recently given by B. Bowditch [9] and S. Maillot [28].

Theorem 1.3 is used in our forthcoming work [25] to settle (under very strong extra assumptions) the *weak hyperbolization conjecture* for 3-dimensional manifolds:

The fundamental group of a compact 3-manifold M is either Gromov-hyperbolic or contains a rank 2 abelian subgroup.

A coarse fibration on the universal cover X of M in [25] appears as a family of intersection curves between leaves of certain immersed 2-dimensional laminations in X.

This paper is organized as follows.

In section 2 we review basics of the geometric group theory and controlled topology. We also present a proof of a "local-to-global" criterion for hyperbolicity of finitely-presented groups abd coarsely simply-connected metric spaces. As an application we prove a theorem of independent interest:

Theorem 1.5. *Suppose that G is a finitely presented group such that* some[1] *asymptotic cone of G is a metric tree. Then G is Gromov-hyperbolic.*

This contrasts with the fact that there are examples [36], [15], of finitely generated groups G for which *some* asymptotic cone of G is a metric tree but G is *not Gromov-hyperbolic.*

In section 3 we introduce coarse fibrations and discuss their basic properties. We define metric cell complexes which serve as *coarse bases* of coarse fibrations. In the case of a coarse fibration as in Example 1.2, the base is quasi-isometric to the quotient group G/C.

[1]It is easy to see that if *all* asymptotic cones of a finitely-generated group G are metric trees, then G is Gromov-hyperbolic.

In section 4 we show that coarse fibrations (under appropriate uniform contractibility assumptions on the total space and the coarse fibers) behave homotopy-theoretically as products of the coarse base and a coarse fiber.

In section 5 we introduce the concept of *coarse Poincare duality* and review some basic properties of metric cell complexes which satisfy this duality, this subject was discussed in much greater details in our previous paper [26]. For metric cell complexes which satisfy coarse Poincare duality one proves coarse analogues of Jordan separation; for instance, if L is a geodesic in $Y^{(1)}$, where Y satisfies 2-dimensional coarse Poincare duality, then some neighborhood of L separates $Y^{(1)}$ into exactly two components none of which is within finite distance from L. A similar property holds if instead of a geodesic one consider a *fat geodesic triangle* T in $Y^{(1)}$: one can define coarse interior and exterior of T.

We then prove one of the critical results of this paper: If X is a coarsely fibered uniformly contractible metric cell complex which is homeomorphic to an n-manifold, so that the fibers have locally compact cohomology of $\mathbb{R}^i$, then the coarse base Y of the fibration satisfies $n-i$-dimensional coarse Poincare duality. The reader familiar with the paper of G. Mess [30] will note that this part of the paper is parallel to the part of [30], where it is proven that the group $\pi_1(M^3)/C$ is quasi-isometric to a planar surface, where C is an infinite cyclic central subgroup of $\pi_1(M^3)$. Although, 2-dimensional coarse Poincare duality[2] for a metric cell complex Y *a priori* is not the same as Y being quasi-isometric to a planar surface, *coarse separation properties* of the coarse base Y suffice for our arguments.

Section 6 is, in a sense, the heart of the paper: we introduce the concept of a *surrounding function* for 1-ended metric cell complexes Y. Roughly speaking, the surrounding function $\mathrm{Sur}(r)$ measures the length of the shortest connected subgraph which *coarsely separates* an r-ball $B(y,r)$ from "infinity". We then use the local-to-global characterization of Gromov-hyperbolic spaces to establish the basic dichotomy for quasi-homogeneous quasi-planes Y (Corollary 6.12):

1. Either the surrounding function is superlinear, in which case Y is Gromov-hyperbolic.

2. Or the surrounding function is at most linear, in which case Y has polynomial growth.

To put this into prospective, suppose that Y appears as a base of a coarse fibration $\mathcal{F}$ of a contractible 3-manifold X, so that $\mathcal{F}$ is invariant under G. Then, in the case of the group-theoretic example 1.2 of a coarse fibration (as in G. Mess' theorem), it immediately follows that Y is quasi-isometric to the group G/C, which is therefore, is either Gromov-hyperbolic with topological circle as the ideal boundary, or is virtually nilpotent. This is our proof of G. Mess' theorem. The reader interested only in an alternative proof of [30] can stop reading the paper at this point.

[2]We refer to this as Y being a *quasi-plane*.

In our situation it is not a priori clear that the base Y is quasi-isometric to a group: the projection of a quasi-action of G from X to Y may not be discrete. Such examples are provided by Solv-manifolds.

In section 7 we discuss properties of the ideal boundaries of Gromov-hyperbolic spaces Y satisfying coarse n-dimensional Poincare duality, including the case where the coefficients for the (co)homology groups are not necessarily integers, but a commutative ring $\mathcal{R}$. Our main result is that the ideal boundary $\partial_\infty Y$ of Y is an $n-1$-dimensional Steenrod $\mathcal{R}$-homology sphere, provided that the ring $\mathcal{R}$ is *hereditary.* In the case of arbitrary commutative rings, $\partial_\infty Y$ is an $n-1$-dimensional Čech *cohomology sphere.* In the case $n = 2$ we establish that $\partial_\infty Y$ is homeomorphic to the circle. This is used later in the paper for characterization of 2-dimensional Poincare duality groups over $\mathcal{R}$.

We then observe (using reults ot Tukia, Gabai, Casson and Jungreis) that, if $n = 2$ and $G \curvearrowright Y$ is a cobounded quasi-action, then Y is quasi-isometric to $\mathbb{H}^2$ and the quasi-action $G \curvearrowright Y$ is conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.

In section 8 we restrict somewhat general discussion of coarse fibrations in the preceeding sections to the case of coarse fibrations by lines. We introduce the key tool for dealing with the problem of nondiscreteness of the quasi-action of G on the coarse base Y of polynomial growth: the *expansion function* $E(F)$ for the fibers of the coarse fibration $\mathcal{F}$. This function is a coarse analogue of the curvature in the case of Riemannian foliations: it measures the amount of expansion of the metric on a fiber $\bar{F}$ under the nearest-point projection $\bar{F} \to \bar{F}_0$ to a distinguished fiber $\bar{F}_0$. We then assume that the base Y has polynomial growth, in which case we prove:

(a) The function $E : \mathcal{F} \to \mathbb{R}$ is bounded.

(b) Using (a) and amenability of Y we then show that the total space of the coarse fibration is also amenable, i.e. admits a Folner sequence.

(c) We then construct a homomorphism ψ from the group G to the group of quasi-isometries of the fiber $\bar{F}_0$.

(d) 1. In the case when ψ is trivial we use a commutator trick to prove that the fiber $\bar{F}_0$ is Hausdorff-close to a *central infinite cyclic subgroup* $Z \subset G$.

2. In the case when ψ is nontrivial, by taking an asymptotic cone of $\bar{F}_0$ we construct a nontrivial homomorphism $G \to \mathbb{Z}$ (or, rather, we get such a homomorphism from a finite-index subgroup in G). Thus, in the context of the universal cover of a closed 3-manifold M, we conclude that M is homotopy-equivalent to a virtually Haken manifold.

In section 9 we prove the main theorem of our paper. In the case when the coarse base Y has polynomial growth, the results of section 8 imply that M is (up to homotopy) virtually Haken with amenable fundamental group, which implies that M is (up to homotopy) either flat, Solv or Nil-manifold.

In the case when the coarse base Y is Gromov-hyperbolic, by considering various classes of cobounded subgroups of $\mathrm{Isom}(\mathbb{H}^2)$: discrete, solvable, dense, we conclude that M is (up to homotopy) either a Seifert manifold with hyperbolic base, or $\pi_1(M)$ is virtually solvable. In the latter case M is homotopy-equivalent to a Solv-manifold.

In section 10 we classify coarse fibrations by lines of Seifert and Solv-manifolds. We show that they are Hausdorff-close to *standard examples* of coarse fibrations, i.e. Seifert fibrations, foliations by affine lines in the case of flat manifolds, and the liminations by lines (described earlier in the introduiction) for Solv-manifolds.

The main result of section 11 is:

Theorem 1.6. *Suppose that $\mathcal{R}$ is a commutative ring with a unit, G is a 2-dimensional Poincare duality group over $\mathcal{R}$. Then G is commensurable to a surface group.*

This theorem extends earlier results of Eckmann, Müller and Linnel [17, 16], in the case $\mathcal{R} = \mathbb{Z}$, and results of Bowditch [9], in the case when $\mathcal{R}$ is a field. The key to the proof of this theorem is a construction of an action of G on a quasi-plane Y (over $\mathcal{R}$), which then allows us to apply our results in sections 6 and 7 to conclude that G is either virtually nilpotent or acts discretely cocompactly isometrically on the hyperbolic plane. The reader who is interested only in the proof of Theorem 1.6 can read only sections 2, 5, 6, 7 and 11.

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2. Notation and preliminaries

2.1. **Notation and conventions.** We let $\mathbb{Z}_+ := \{m \in \mathbb{Z} \mid m \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. We let $d_H(\cdot,\cdot)$ denote the Hausdorff distance between subsets of a metric space; the usual (infimal) distance will be denoted $d(\cdot,\cdot)$. All maps between cell complexes will be continuous unless otherwise specified. Given a map f we let $\mathrm{Im}(f)$ denote the image of f.

We will be using singular (co)homology with $\mathbb{Z}$ coefficients unless we indicate otherwise. For each negative integer k we set $H_k(\cdot) = 0$, $H_c^k(\cdot) = 0$, etc.

Let r_i, R_i be two sequences of positive real numbers. We will use the notation

$$R_i \gtrsim r_i$$

if there exist a pair of constants A, B (independent of i) such that for all but finitely many $i \in \mathbb{N}$ we have:

$$R_i \geq Ar_i + B.$$

We will use the notation $R_i \simeq r_i$ if $R_i \gtrsim r_i$ and $r_i \gtrsim R_i$.

A subset $S \subset Z$ of a metric space is called δ-*dense* if each point $z \in Z$ is within distance $\leq \delta$ from S. A subset in Z which is δ-dense from some $\delta < \infty$, is called a *net* in Z.

A metric space Z has *bounded geometry* if there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each metric ball $B(x,r) \subset Z$, one has:

$$Vol(B(x,r)) := |B(x,r)| \leq \phi(r),$$

where $|S|$ denotes the cardinality of a set S. A bounded geometry metric space Z has *polynomial growth* if there is a constant $c \in \mathbb{R}_+$ such that for each ball $B(x,r) \subset X$ one has $r^c \gtrsim Vol(B(x,r))$. The optimal constant c is called the *degree of the polynomial growth.* A space Z has *superpolynomial growth* if it does not have polynomial growth for any c.

Similarly, a bounded geometry metric space Z has *exponential growth* if there is a constant $a > 0$ such that for each $x \in Z$ one has:

$$Vol(B(x,R)) \gtrsim e^{aR}.$$

A space has *subexponential growth* if

$$e^{aR} \gtrsim Vol(B(x,R))$$

for all $a > 0$.

A metric space Z is *doubling* if there is a constant $N \in \mathbb{Z}_+$ such that each ball $B(x,2R) \subset Z$ (where $R \geq 1$) is contained in the union of $\leq N$ balls of radius R.

Lemma 2.1. *Suppose that Z is a bounded geometry doubling metric space. Then Z has polynomial growth.*

Proof. For each $r \geq 1$ choose $n \in \mathbb{Z}$ such that $n-1 \leq \log_2(r) \leq n$. Hence for each metric ball $B(x,r) \subset X$ we have:

$$B(x,r) \subset \bigcup_{i=1}^{N} B(x_i, \frac{r}{2}) \subset \bigcup_{j}^{N^2} B(x_j, \frac{r}{4}) ... \subset \bigcup_{\ell=1}^{N^{n-1}} B(x_\ell, r2^{-(n-1)}).$$

Let $M := \sup_{x \in Z} Vol(B(x,2))$ and $c := \frac{1}{\log_N(2)}$. Then

$$Vol(B(x,r)) \leq N^{n-1}M \leq N^{c \log_N(r)} M = r^c M. \quad \square$$

Theorem 2.2. *(M. Gromov, [22]) Suppose that G is a finitely-generated group of polynomial growth. Then G is virtually nilpotent.*

An improvement of this theorem was established by Wilkie and Van der Dries [40]:

Theorem 2.3. *Suppose that G is a finitely-generated group and there is a sequence of metric balls $B(x_i, R_i) \subset G$ and constant $c \geq 0$ such that $R_i^c \gtrsim Vol(B(x_i, R_i))$. Then G is virtually nilpotent and has polynomial growth of degree $\leq c$.*

There is another closely related concept, amenability. Assume that Z has bounded geometry metric space. For a subset $D \subset Z$ define the *c-frontier* $\partial_c D$ as $\{x \in Z \setminus D : d(x, D) \leq c\}$. A sequence $D_j \subset Z$ is called a *c-Folner sequence* if

$$\lim_{j\to\infty} Vol(D_j) = \infty, \text{ and } \lim_{j\to\infty} \frac{Vol(\partial_c D_j)}{Vol(D_j)} = 0.$$

A metric space Z is called *amenable* if it admits a c-Folner sequence for each c. It is clear that amenability is a quasi-isometry invariant property for bounded geometry metric spaces. A finitely-generated group is amenable iff it has an invariant mean. Free nonabelian groups are nonamenable. If a group G contains a nonamenable subgroup or maps onto a nonamenable group, then G is nonamenable.

We note that for the metric spaces Z which are vertex sets of connected graphs with unit edges (e.g., finitely generated groups with their word metrics), to test amenability of Z it suffices to consider the 1-frontiers $\partial D := \partial_1 D$ in the definition of Folner sequences.

Lemma 2.4. *If the space Z has subexponential growth then is amenable.*

Proof. Pick a point $x \in Z$ and consider the seqence $D_j := B(x,j) \subset Z$, $j \in \mathbb{N}$. Suppose that the sequence (D_j) contains no c-Folner subsequence. Then there is a constant $C > 0$ such that for each j, $Vol(\partial_c D_j) \geq C\, Vol(D_j)$, i.e.

$$Vol(B(x, j+c)) \geq (C+1) Vol(B(x,j)).$$

Then $Vol(B(n)) \geq (C+1)^n Const$ and hence Z has exponential growth. $\square$

Let Z be a geodesic metric space. A geodesic triangle $\Delta \subset Z$ is called *R-thin* if every side of Δ is contained in the R-neighborhoodof the union of two other sides. An

R-fat triangle is a geodesic triangle which is not R-thin. A geodesic metric space Z is called *δ-hyperbolic in the sense of Rips* (Rips was the first to introduce this definition) if each geodesic triangle in Z is δ-thin.

Let X be a metric space (which is no longer required to be geodesic). For each base-point $p \in X$ define a number $\delta_p \in [0, \infty]$ as follows. For each $x \in X$ set $|x|_p := d(x, p)$ and

$$(x, y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x, y)).$$

Then

$$\delta_p := \inf_{\delta \in [0,\infty]} \{\delta | \forall x, y, z \in X, (x, y)_p \geq \min((x, z)_p, (y, z)_p) - \delta\}.$$

We say that X is *δ-hyperbolic in the sense of Gromov*, if $\infty > \delta \geq \delta_p$ for some $p \in X$. We note that if X a geodesic metric space which is δ-hyperbolic in Gromov'ssense then X is 4δ-hyperbolic in the sense of Rips and vice-versa (see [23, 6.3C]).

A metric space Z is *Gromov-hyperbolic* if it is δ-hyperbolic for some $\delta < \infty$.

2.2. **Maps and actions.** Let Y be a subset of a metric space X. A map $f : Y \to X$ has *bounded displacement* if there is $C \in \mathbb{R}$ such that $d(f, i_Y) \leq C$ where $i_Y : Y \to X$ is the inclusion.

A map $f : X \to X'$ between metric spaces is *uniformly proper* if there are constants L, A, and a continuous strictly increasing *distortion function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \eta(t) = \infty$ such that

$$\eta(d(x_1, x_2)) \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A \tag{2.5}$$

for all x_1, $x_2 \in X$. A family of maps $\{f_i\}_{i\in\mathcal{J}}$ is *uniformly proper* if one can choose L, A, and η as above so that (2.5) holds for all the maps in the family.

A map $f : X \to X'$ between metric spaces X and X' is an *(L, A)-quasi-isometry* if for all $x_1, x_2 \in X$ we have

$$\frac{1}{L}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$

and $d(x', \mathrm{Im}(f)) < A$ for all $x' \in X'$. We let $\widehat{\mathrm{QI}}(X, X')$ denote the collection of all quasi-isometries from X to X'. Two quasi-isometries $f_1, f_2 : X \to X'$ are *equivalent* if $d(f_1, f_2) < \infty$; we let $\mathrm{QI}(X, X')$ denote the set of equivalent classes of quasi-isometries, and use $\mathrm{QI}(X)$ (resp. $\widehat{\mathrm{QI}}(X)$) in place of $\mathrm{QI}(X, X)$ (resp. $\widehat{\mathrm{QI}}(X, X)$). Composition of quasi-isometries induces a group structure on $\mathrm{QI}(X)$.

A *quasi-action* of a group G on a metric space X, denoted $G \overset{\rho}{\curvearrowright} X$, is a map $\rho : G \to \widehat{\mathrm{QI}}(X)$ such that for suitable constants L, A,

1. $\rho(g)$ is an (L, A) quasi-isometry for all $g \in G$,
2. $d(\rho(e), \mathrm{id}_X) < A$, and

3. $d(\rho(g_1g_2), \rho(g_1)\rho(g_2)) < A$ for all g_1, $g_2 \in G$.

We will usually write $g(x)$ rather than $\rho(g)(x)$, suppressing the name of the quasi-action when it is understood. A quasi-action is *discrete* if for all $x \in X$, $R > 0$, the set $\{g \in G \mid d(g(x), x) < R\}$ is finite. A quasi-action $G \overset{\rho}{\curvearrowright} X$ is called *cobounded* if for some (for every) point $x \in X$ the *quasi-orbit* $G \cdot x = \{g(x) : g \in G\}$ has finite Hausdorff distance to X. We say that a metric space X is *quasi-homogeneous* if there are constants L, A such that for all x, $x' \in X$ there is an (L, A) quasi-isometry $f : X \to X'$ with $f(x) = x'$. Note that if there exists a cobounded quasi-action $G \curvearrowright X$ then X is quasi-homogeneous.

A *degenerating quasi-action* of a finitely-generated group G on a metric space X is a homomorphism $\rho : G \to \mathrm{QI}(X)$. Note that each quasi-action of $G \curvearrowright X$ defines a degenerating quasi-action. The *kernel* of a (degenerating) quasi-action is the kernel of the homomorphism $\rho : G \to \mathrm{QI}(X)$; it consists of the elements of G which act on X with bounded displacement.

Suppose that $G \overset{\phi}{\curvearrowright} Y$ is a quasi-action with the kernel K. Note that a priori there is no uniform bound on the displacement functions $d_g = d(y, \phi(g)y)$ (independent on $g \in K$). For instance, for any quasi-action $G \curvearrowright \mathbb{R}^n$, where G *acts* by translations, the kernel is the entire group G. However, if Y is a Gromov-hyperbolic geodesic metric space whose ideal boundary consists of at least 3 points, then there is a uniform upper bound on the functions $d_g, g \in K$.

Let G, H be groups where H is given a left-invariant metric. A map $\rho : G \to H$ is called a *quasi-homomorphism* if there is a constant C such that for each $g_1, g_2 \in G$ we have:

$$d(\rho(g_1g_2), \rho(g_1)\rho(g_2)) \le C.$$

In this paper we will need only quasi-homomorphisms to $\mathbb{R}$.

2.3. **Bounded geometry cell complexes.** We will be working with CW complexes endowed with an extra structure. Let X be a CW complex, and $X^{(m)}$ denote its m-skeleton, $m \in \mathbb{Z}_+$. Recall that a subcomplex Y of X is a closed subset which is a union of open cells, such that the boundary of each open cell $\sigma \subset Y$ is contained in Y.

A *control map* for X is a function $p : X \to X^{(0)}$ such that

1. $p\big|_{X^{(0)}} = \mathrm{id}_{X^{(0)}}$,
2. p is constant on open cells in X,
3. $p(x)$ belongs to the smallest subcomplex containing x, for all $x \in X$.

A *morphism* $(X, p) \to (X', p')$ is a skeleton preserving continuous map so that for each $i \in \mathbb{N}$ the diameter of the $p'(\sigma)$ is uniformly bounded (σ are i-cells in $X^{(i)}$).

A *bounded geometry metric cell complex* is a CW complex X equipped with a control map p, whose 1-skeleton $X^{(1)}$ is connected and equipped with a path metric with respect to which all edges have length 1, subject to the condition that every closed cell $\sigma \subset X^{(m)}$ intersects at most $D = D(m)$ closed cells in $X^{(m)}$.

Remark 2.6. Note that for such a complex, the metric space $X^{(0)}$ has bounded geometry in the sense of section 2.1.

To simplify the terminology we will refer to bounded geometry metric cell complexes as simply *metric cell complexes*: the bounded geometry will be assumed by default.

We say that a metric cell complex X is Gromov-hyperbolic if its zero-skeleton $X^{(0)}$ is Gromov-hyperbolic.

Let X be a metric cell complex. If $V \subset X^{(0)}$ and $R \in \mathbb{Z}_+$, we denote the closed metric R-neighborhood of V in the 0-skeleton by

$$N_R^{(0)}(V) := \{x \in X^{(0)} \mid d(x, V) \leq R\}.$$

Given $m \in \mathbb{Z}_+$, $R \in \mathbb{Z}_+$, and a subcomplex $Y \subset X$, we define the *R-neighborhood of Y in the m-skeleton*, $N_R^{(m)}(Y)$, to be the largest subcomplex of $X^{(m)}$ whose 0-skeleton is $N_R^{(0)}(Y^{(0)})$. Note that

1. $N_0^{(m)}(Y) \supset Y \cap X^{(m)}$,

2. If Y, $Y' \subset X$ are subcomplexes and $R + R' < d(Y^{(0)}, Y'^{(0)})$, then for all $m \in \mathbb{Z}_+$ we have $N_R^{(m)}(Y) \cap N_{R'}^{(m)}(Y') = \emptyset$.

If X is an m-dimensional metric cell complex then we will use the abbreviation $N_R(V) := N_R^{(m)}(V)$.

We will only use the notation $B(x, r)$ to denote closed metric balls in the 1-skeleton $X^{(1)}$, i.e. $B(x, r) := \{y \in X^{(1)} \mid d(x, y) \leq r\}$, where $x \in X^{(1)}$. If Y is a subset of X, we define its diameter to be $\mathrm{diam}(Y) := \mathrm{diam}(p(Y))$. Similarly, we define the distance between two functions $f, f' : S \to X$ to be the quantity $d(p \circ f, p \circ f')$.

A (continuous) map $f : (X, p) \to (X', p')$ is *uniformly proper* if for each $m \in \mathbb{Z}_+$ there are constants $L = L_m$, $A = A_m$, and a *distortion function* $\eta = \eta_m : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, strictly monotone, with $\lim_{t\to\infty} \eta(t) = \infty$, such that for all cells $\sigma_1, \sigma_2 \in X^{(m)}$,

$$\eta(d(p(\sigma_1), p(\sigma_2)) \leq d(p'f(\sigma_1), p'f(\sigma_2)) \leq Ld(p(\sigma_1), p(\sigma_2)) + A. \tag{2.7}$$

A family of maps $\{f_i\}_{i\in\mathcal{J}}$ is *uniformly proper* if one can choose L_m, A_m, and η_m as above so that (2.7) holds for all the maps in the family.

A (continuous) map $f : X \to X'$ between metric cell complexes (X, p) and (X', p') is an *(L_m, A_m)-quasi-isometry* ($m \in \mathbb{Z}_+$) if for all $\sigma_1, \sigma_2 \in X^{(m)}$ we have

$$\frac{1}{L_m} d(p(\sigma_1), p(\sigma_2)) - A_m \leq d(p'f(\sigma_1), p'f(\sigma_2)) \leq L_m d(p(\sigma_1), p(\sigma_2)) + A_m$$

and $d(x', \mathrm{Im}(f^{(m)})) < A_m$ for all $x' \in X'^{(m)}$.

A metric cell complex (X, p) is *uniformly k-connected* if there is a function $\theta = \theta_k : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that the inclusion $N_r^{(k)}(x) \to N_{\theta(r)}^{(k+1)}(x)$ is null-homotopic for all $x \in X^{(0)}$, $r \in \mathbb{Z}_+$. X is *uniformly contractible* if it is uniformly k-connected for all k.

If we are given a collection $\mathfrak{X}$ of bounded geometry cell complexes X, then we will say that $\mathfrak{X}$ has bounded geometry (resp. is uniformly k-connected, uniformly contractible) if each $X \in \mathfrak{X}$ has bounded geometry (resp. is uniformly k-connected, uniformly contractible) where the functions $D(m)$ (resp. θ_k) may be chosen independently of $X \in \mathfrak{X}$.

Remark 2.8. When X is a uniformly contractible cell complex, given two continuous maps f_1, f_2 from a finite dimensional cell complex Z into X where $d(f_1, f_2) < D$, there is a homotopy from f_1 to f_2 whose tracks have diameter $< D'$ where $D' = D'(D, \dim Z)$.

An important class of metric cell complexes is given by *Rips complexes* of bounded geometry metric spaces. Let Z be such a metric space and $D \in \mathbb{R}_+$. We define the *D-Rips complex* $\mathrm{Rips}_D(Z)$ to be the simplicial complex whose vertex set is Z, where $x_0, ..., x_n$ span an n-simplex in $\mathrm{Rips}_D(Z)$ iff $d(x_i, x_j) \leq D$ for all $0 \leq i, j \leq n$. Thus we get a direct system of Rips complexes $\mathrm{Rips}_D(Z)$ with the inclusion morphisms $\mathrm{Rips}_D(Z) \to \mathrm{Rips}_{D'}(Z)$ for $D \leq D'$.

We metrize R-Rips complexes by taking the largest metric for which all simplicial embeddings $\sigma \to \mathrm{Rips}_R(Z)$ are 1-Lipschitz, where σ is a regular Euclidean simplex with side length R. We define the control map $c : \mathrm{Rips}_R(Z) \to Z$ by sending each simplex to one of its vertices.

If Z has bounded geometry, so does $\mathrm{Rips}_R(Z)$ for each $R < \infty$. If Z is δ-hyperbolic then for each $R \geq 10\delta$, the complex $\mathrm{Rips}_R(Z)$ is uniformly contractible, [23, Lemma 17.A].

Proposition 2.9. *Suppose that $Y^{(2)}$ is simply-connected. Then $\mathrm{Rips}_d^{(2)}(Y^{(0)})$ is 1-connected provided d is sufficiently large.*

Proof. Let d_i denote the supremum of the diameters $p(\sigma^i)$ where the supremum is taken over all i-cells $\sigma^i \subset Y^{(i)}$. Since Y is connected, it is clear that $\mathrm{Rips}_d(Y^{(0)})$ is connected for each $d \geq d_1$.

Let $d \geq d_1$. Consider a loop $\gamma : S^1 \to \mathrm{Rips}_d^{(1)}(Y^{(0)})$. After homotoping γ if necessary, we may assume that it is a simplicial map with respect to some triangulation $\mathcal{T}$ of S^1. Define a map $\gamma_1 : S^1 \to Y^{(1)}$ as follows. For each vertex v of $\mathcal{T}$, let $\gamma_1(v) \in Y^{(0)} \equiv \mathrm{Rips}_d^{(0)}(Y^{(0)})$ be equal to $\gamma(v)$. For each edge $e = [v_1 v_2]$ of $\mathcal{T}$, let $\gamma_1\big|_e$ be a geodesic in $Y^{(1)}$ between $\gamma_1(v_1)$ and $\gamma_1(v_2)$. There is a natural map $Y^{(1)} \xrightarrow{i_1} \mathrm{Rips}_d^{(1)}(Y^{(0)})$ which takes each $v \in Y^{(0)}$ to the corresponding vertex of

$\mathrm{Rips}_d^{(0)}(Y^{(0)})$ and maps each edge of $Y^{(1)}$ at constant speed to the corresponding edge of $\mathrm{Rips}_d^{(1)}(Y^{(0)})$. Let $\gamma_2 := i_1 \circ \gamma_1$.

If $d \geq d_2$ then i_1 can be extended to a map $Y^{(2)} \xrightarrow{i_2} \mathrm{Rips}_d^{(2)}(Y^{(0)})$. This implies that γ_2 is null-homotopic in $\mathrm{Rips}_d^{(2)}(Y^{(0)})$.

On the other hand, we claim that γ_2 is homotopic to γ in $\mathrm{Rips}_d^{(2)}(Y^{(0)})$. To see this, for each edge $e = [v_1 v_2]$ of $\mathcal{T}$, let $y_0 = \gamma(v), y_1, ..., y_m = \gamma(w)$ be the vertices of $Y^{(1)}$ on $\gamma_1(e)$ so that $\gamma_2(e)$ is the concatenation of the edges

$$[y_0 y_1], ..., [y_{m-1} y_m] \subset \mathrm{Rips}_d^{(2)}(Y^{(0)}).$$

Since $\gamma_1(e)$ is a geodesic between y_0, y_m and $d_{Y^{(1)}}(y_0, y_m) \leq d$, we get:

$$d_{Y^{(1)}}(y_0, y_i) \leq d, i = 1, ..., m-1.$$

Hence each triple of vertices y_0, y_i, y_m spans a 2-simplex Δ_i in $\mathrm{Rips}_d^{(2)}(Y^{(0)})$. Together these simplices define a homotopy between $\gamma(e)$ and $\gamma_2(e)$ (rel. the end-points). Thus the loops γ and γ_2 are homotopic. □

2.4. **Category $\mathcal{C}_B$ of complexes with bounded control.** It is convenient to introduce a *category $\mathcal{C}_B$ of cell complexes with bounded control over a metric space B.* This category generalizes the category of metric cell complexes.

The objects of the category $\mathcal{C}_B$ are pairs (W, c) where W is a CW-complex and $c : W \to B$ is a not necessarily continuous *control* map. We note that B does not have to be equal to $W^{(0)}$.

Example 2.10. Let (W, p) be a metric cell complex. Set $B := W^{(0)}$, $c_B := p$.

A *morphism* between (W, c) and (W', c') is a continuous, skeleton-preserving map $\mu : W \to W'$ which is proper on each skeleton, and has (skeletonwise) bounded control over B, i.e.

$$d(c' \circ \mu\big|_{W^{(k)}}, c\big|_{W^{(k)}})$$

is finite for each k.

2.5. **Gromov's coarse version of the Cartan-Hadamard theorem.**

Theorem 2.11. *(Cf.* [23], [7, Theorem 8.1.2]*) There are constants d_0, C_1, C_2, and C_3 with the following property. Let X be a metric space of bounded geometry. Assume that for some δ, and $d \geq \max(C_1\delta, d_0)$, every ball of radius $C_2 d$ is δ-hyperbolic, and $\mathrm{Rips}_d^2(X)$ is 1-connected. Then X is $C_3 d$-hyperbolic.*

One can give a direct proof of this theorem modelled on the proof of the Cartan-Hadamard theorem. Instead of doing this, we will use 6.8M and 6.8N from [23].

Remark 2.12. We note that the δ-hyperbolicity in the statements of 6.8M and 6.8N is to be taken in Gromov's sense.

Taking $A'_0 = 500d^2$ in [23, 6.8M] we get:

Theorem 2.13 (6.8M, adapted version)**.** *Suppose that X is a metric space (of bounded geometry) such that for some $d \geq 0$ every simplicial circle S' in $P^1_d(X)$ with $500d^2 \leq A(S') \leq 64(500d^2)$ satisfies*

$$L(S') \geq d\sqrt{(4000)(64)(500)} \tag{2.14}$$

and $P^2_d(X)$ is 1-connected. Then $P^1_d(X)$ is $(400)\sqrt{500}d$-hyperbolic in the sense of Rips (see 6.8.J) and X is $(400)\sqrt{500}d$-hyperbolic in the sense of Gromov.

Here $P^m_d(X)$ denotes the simplicial complex $\mathrm{Rips}^{(m)}_d(X)$ endowed with a metric for which $X \to P^m_d(X)$ is an isometric embedding, and for which each simplex is path-isometric to a regular Euclidean simplex of side length d. The quantities $L(S')$ and $A(S')$ are the length of S' and the minimal area of a null-homotopy of S', where length and area are computed using the metric on $P^2_d(X)$ rather than the combinatorial length and area.
Theorem 6.8N from [23] states

Theorem 2.15 (6.8N)**.** *If X is δ-hyperbolic and $d \geq 8\delta$, then every simplicial circle $S' \subset P^1_d(X)$ satisfies $L(S') \geq \frac{d}{4\sqrt{3}}A(S')$.*

Proof of Theorem 2.11. Choose d_0 such that

$$\frac{500d_0^2}{4\sqrt{3}} \geq \sqrt{(4000)(64)(500)}, \tag{2.16}$$

and set $C_1 := 32$, $C_2 := 64 \cdot 500$. Let $S' \subset P^1_d(X)$ be a simplicial circle with

$$500d^2 \leq A(S') \leq 64(500d^2) \tag{2.17}$$

and let $f : D \to P^2_d(X)$ be a least area simplicial disk filling S'. There are at most (64)(500) triangles in the triangulated 2-disk D which are mapped isomorphically by f, by (2.17). Therefore, if we look at $\mathrm{Im}(f) \subset P^2_d(X)$, and let $W \subset \mathrm{Im}(f)$ be the closure of the 2-simplices contained in $\mathrm{Im}(f)$, then connected components of W have diameter $\leq (64)(500)d$. This means that we can decompose D along a collection $\{\alpha_1, \ldots, \alpha_k\}$ of disjoint arcs along each of which f is constant, and view D as a "concatenation" of minimal disks $D_1, \ldots, D_{k+1}$ each of whose images has diameter at most $(64)(500)d$. By assumption every ball of radius $C_2 d = 64 \cdot 500d$ is δ-hyperbolic and $d \geq C_1\delta = 32\delta$, so by applying Theorem 2.15 to $D_1, \ldots D_{k+1}$ and adding up the results, we obtain

$$L(S') \geq \frac{d}{4\sqrt{3}}A(S') \geq \frac{d}{4\sqrt{3}}500d^2 \geq \frac{d}{4\sqrt{3}}500d_0^2 \geq d\sqrt{(4000)(64)(500)}$$

where the last inequality comes from (2.16). By Theorem 2.13 we conclude that X is $C_3 d$-hyperbolic (in Gromov's sense) where $C_3 := (400)(\sqrt{500})$. $\square$

As a corollary we get:

Corollary 2.18. *There exist a constant $0 < c < \infty$ such that for each 1-connected 2-dimensional metric cell complex Y, there exists a constant $\rho = \rho(Y)$ with the property:*

If for some $R \geq \rho$, each ball $B_R(y) \subset Y$ is cR-hyperbolic (in the sense of Rips), then Y is Gromov-hyperbolic.

Proof. First of all, since Y is 1-connected, there exists a constant $D = D(Y)$ such that $\mathrm{Rips}^{(2)}_d(Y^{(0)})$ is 1-connected for each $d \geq D$, see Proposition 2.9.

Let C_1, C_2 and d_0 be the constants from Theorem 2.11. Choose ρ so that $\rho/C_2 \geq \max(D, d_0)$. Let $c := \frac{1}{4C_1C_2}$.

Set $d := R/C_2$ and $\delta := cR$. Then $\mathrm{Rips}^{(2)}_d(Y^{(0)})$ is 1-connected. If each ball $B_R(y) \subset Y$ is δ-hyperbolic (in the sense of Rips) then for each $x \in Y^{(0)}$, the ball $B_R(x) \subset Y^{(0)}$ is 4δ-hyperbolic (in Gromov's sense). Since (by our choice of the constant c)

$$d = \frac{R}{C_2} \geq 4C_1C_2cR$$

(in fact, the equality holds), and

$$d \geq \frac{\rho}{C_2} \geq \max(D, d_0),$$

Theorem 2.11 implies that Y is Gromov-hyperbolic. $\square$

Corollary 2.19. *Suppose that G is a finitely-presented group such that* some *asymptotic cone of G is a tree. Then G is Gromov-hyperbolic; in particular,* every *asymptotic cone of G is a tree. Thus, under the hypothesis of the corollary, all asymptotic cones of G are isometric.*

Proof. Let ω be a nonprincipal ultrafilter on $\mathbb{N}$, R_j is a sequence of positive real numbers such that $\omega\text{-}\lim R_j = \infty$. Let Y be a (simply-connected) Cayley complex for G, $y_j \in Y$ is a sequence of vertices. We give Y structure of a metric cell complex so that G acts on Y isometrically. By our assumption, $\omega\text{-}\lim \frac{1}{R_j}(Y, y_j)$ is a tree for some choice of $\omega, (R_j)$ (and the base-points y_j). Thus each geodesic triangle in an R_j-ball $B(y_j, R_j)$ is δ_j-thin, where

$$\omega\text{-}\lim \frac{\delta_j}{R_j} = 0.$$

Hence the same is true for each ball $B(y, R_j)$, $y \in Y^{(0)}$. For sufficiently large j, $R \geq \rho = \rho(Y)$ and $\frac{\delta_j}{R_j} < c$, where ρ, c are the constants from the previous corollary. Hence, by corollary 2.18, the complex Y is Gromov-hyperbolic and therefore G is too. $\square$

We note that the above corollary is false for finitely-generated groups, see [15], [36].

Question 2.20. Suppose that G is a finitely-presented group. Is it true that all asymptotic cones of G are homeomorphic?

2.6. **Finiteness properties of groups.** We recall that a group G is said to have type F_n ($n = 1, 2, ..., \infty$) if there exists an $n-1$-connected n-dimensional cell complex X and a discrete free action $G \curvearrowright X$ such that $X^{(i)}/G$ is compact for each $i < \infty$.

Proposition 2.21. *Suppose that there exists an $n-1$-connected n-dimensional cell complex G admits a discrete action $G \curvearrowright Y$ such that $Y^{(i)}/G$ is compact for each $i < \infty$. Then G is of type F_n.*

Proof. We note that if the action $G \curvearrowright Y$ were free, then this action would satisfy the properties stated in the definition of a group of type F_n and there would be nothing to prove. Our goal is to modify Y to make the action free. We do this by induction on skeleta. Let $i = 1$. Since $G \curvearrowright Y^{(1)}$ is cocompact and Y is connected, we conclude that G is finitely-generated. Hence we take $X^{(1)}$ to be a Cayley graph of G. Suppose that $2 \le i \le n$ and an $i-1$-connected complex $X^{(i)}$ together with a free discrete cocompact action $G \curvearrowright X^{(i)}$ was constructed. We convert X into a metric cell complex by taking a G-equivariant control map $p : X \to X^{(0)}$. Let $x_0 \in X^{(0)}$ be a base-point.

Lemma 2.22. *There are finitely many spherical i-cycles $\sigma_1, ..., \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_1(X^{(i)})$, in the sense that the normal closure of the cycles $\{\hat{\sigma}_j : j = 1, ..., k\}$ is $\pi_i(X^{(i)})$, where each $\hat{\sigma}_j$ is obtained from σ_j by attaching a "tail" from x_0.*

Proof. Without loss of generality we can assume that $X^{(i)}$ and Y are simplicial complexes. Let $f : X^{(i)} \to Y^{(i)}$ be a G-equivariant continuous map. Consider the embedding

$$X^{(i)} \hookrightarrow Cone(f)$$

where $Cone(f)$ is the mapping cone. Since $Cone(f)$ is i-connected, according to [26, Proof of Lemma 5.8], there are finitely many spherical i-cycles $\sigma_1, ..., \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_i(X^{(i)})$. Here is the brief outline of the construction of σ_j's:

Let $\tau_\alpha : S^i \to Y^{(i)}, \alpha \in \mathbb{N}$, denote the attaching maps of the $i+1$-cells in Y, these maps are just simplicial homeomorphic embeddings from the boundary S^i of the standard $i+1$-simplex into $Y^{(i)}$. Starting with a G-equivariant projection $Y^{(0)} \to X^{(0)}$ one inductively constructs a (non-equivariant) map $\bar{f} : Y^{(i)} \to X^{(i)}$ so that $f \circ \bar{f} : Y^{(i)} \to Y^{(i+1)}$ is homotopic to the identity inclusion $id : Y^{(i)} \hookrightarrow Y^{(i+1)}$ via a homotopy H whose tracks have "uniformly bounded complexity": the compositions

$$H \circ (\tau_\alpha \times id) : S^i \times I \to Y^{(i+1)}$$

are simplicial maps with a uniform upper bound on the number of simplices in a triangulation of $S^i \times I$. We let σ_j denote the composition $g_\alpha \circ \bar{f} \circ \tau_\alpha$ where $g_\alpha \in G$ are chosen so that the image of σ_j intersects a fixed fundamental domain for the action $G \curvearrowright X^{(i)}$. ☐

We now equivariantly attach $i+1$-cells along G-orbits of the cycles σ_j: for each j and $g \in G$ we attach an $i+1$-cell along $g(\sigma_j)$. Note that if σ_j is stabilized by a subgroup of order $m = m(j)$ in G, then we attach m copies of the $i+1$-dimensional cell along σ_j. We let $X^{(i+1)}$ denote the resulting complex and we extend the G-action to $X^{(i+1)}$ in obvious fashion. It is clear that $G \curvearrowright X^{(i+1)}$ is free, discrete and cocompact. ☐

Corollary 2.23. *Suppose that G is Gromov-hyperbolic. Then G is of type F_∞.*

Proof. Use the action of G on its contractible Rips complex Y. ☐

3. Coarse fibrations

3.1. **Coarse fibrations.** Let $\bar{\mathcal{F}}$ be a collection of subsets of a metric space Z. Then $\bar{\mathcal{F}}$ defines a *coarse fibration* of Z if there is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

1. For all $\bar{F}$, $\bar{F}' \in \bar{\mathcal{F}}$,

$$d_H(\bar{F}, \bar{F}') \le \psi(d(\bar{F}, \bar{F}')), \tag{3.1}$$

where d_H denotes Hausdorff distance.

2. $\cup_{\bar{F}} \bar{F}$ is a net in Z.

We will refer to elements of $\bar{\mathcal{F}}$ as *fibers* of the coarse fibration, and to the function η as a *oscillation function.* In case when Z is 0-skeleton of a metric cell complex X, by abusing notation we will also refer to $\bar{\mathcal{F}}$ as a coarse fibration of X.

Examples:

1. Let $h : Z \to B$ be a Riemannian submersion, where Z is a connected, complete Riemannian manifold. Then $\{h^{-1}(b)\}_{b\in B}$ is a coarse fibration of Z. More generally, if $h : Z \to B$ is a smooth map between complete connected Riemannian manifolds, and there is a constant C such that for all $x \in Z$, $\|Dh(x)\| < C$ and $Dh(x) : T_x Z \to T_{h(x)} B$ has a right inverse $T_{h(x)} B \to T_x Z$ with norm $< C$, then $\{h^{-1}(b)\}_{b\in B}$ is a coarse fibration. To see this, note that the conditions imply that h is Lipschitz, and that there is a smooth distribution transverse to the fibers of h (a "connection"), so that unit speed paths in B can be lifted horizontally to paths in Z with uniformly bounded speed. This leads easily to (3.1).

2. If $G \overset{\rho}{\curvearrowright} Z$ is a cobounded quasi-action on a metric space Z, then the collection of quasi-orbits $\{G \cdot x\}_{x\in Z}$, is a quasi-fibration of Z.

3. Suppose that we are given a finite collection of geodesic metric spaces $F_i, i \in \mathcal{J}$, and for each pair $i, j \in \mathcal{J}$, we are given an (L, A)-quasi-isometry $f_{ij} : F_i \to F_j$ satisfying

$$d(f_{ij} \circ f_{jk}, f_{ik}) \leq C.$$

Let Γ be a (connected) metric graph with edges of the unit length and suppose that $\phi : \Gamma^{(0)} \to \mathcal{J}$ is a map. Let Z denote the disjoint union of the metric spaces $\bar{F}_v = (F_{\phi(v)}, v)$, $v \in \Gamma^{(0)}$. We metrize Z as follows: For every edge $[v, w]$ in Γ we connect each $x \in \bar{F}_v$ to $f_{\phi(v)\phi(w)}(x)$ by an edge of the unit length. This, together with the geodesic metrics on F_i's determine a path metric on Z. The collection of subsets $\bar{F}_v$, $v \in \Gamma^{(0)}$, determine a coarse fibration of Z.

If $\bar{\mathcal{F}}$ is a coarse fibration of Z, we (pseudo)metrize $\bar{\mathcal{F}}$ using Hausdorff distance, and refer to the resulting metric space B as the *base* of the coarse fibration. When Z is a length space, $(\bar{\mathcal{F}}, d_H)$ is a quasi-isometric to a length space. To see this, observe that given two fibers $\bar{F}$, $\bar{F}' \in \bar{\mathcal{F}}$, and a 1-Lipschitz path $\gamma : [0, l] \to Z$ of length $l \in \mathbb{Z}$,

$$d(\bar{F}, \bar{F}') - 1 < l < d(\bar{F}, \bar{F}') + 1,$$

joining $\bar{F}$ and $\bar{F}'$, we can choose fibers $\bar{F}_i$ at uniform distance from $\gamma(i)$, and then we get

$$\sum_{i=1}^{l} d_H(\bar{F}_{i-1}, \bar{F}_i) < Cl < C d_H(\bar{F}, \bar{F}')$$

where C is independent of $\bar{F}$, $\bar{F}'$.

If $\bar{F}$, $\bar{F}' \in \bar{\mathcal{F}}$ are two fibers (given metrics induced from Z) and $\phi : \bar{F} \to Z$ is a bounded displacement map with $\operatorname{Im} \phi \subset \bar{F}'$ (e.g. any nearest point projection $\bar{F} \to \bar{F}'$), then ϕ is an (L, A)-quasi-isometry between $\bar{F}$ and $\bar{F}'$, where L and A depend only on the displacement of ϕ and the distortion function for $\bar{\mathcal{F}}$.

We say that two coarse fibrations $\bar{\mathcal{F}}$ and $\bar{\mathcal{L}}$ are Hausdorff-close if there exists a constant $C < \infty$ such that for each $\bar{F} \in \bar{\mathcal{F}}$ there exists $\bar{L} \in \bar{\mathcal{L}}$ so that $d_H(\bar{F}, \bar{L}) \leq C$, and vice-versa.

Definition 3.2. If $G \curvearrowright Z$ is a quasi-action on Z, then the coarse fibration $\bar{\mathcal{F}}$ is *G-invariant* if there is a constant C such that for all $g \in G$, $\bar{F} \in \bar{\mathcal{F}}$, there is an $\bar{F}' \in \bar{\mathcal{F}}$ such that $d_H(g(\bar{F}), \bar{F}') < C$.

For each coarse fibration $\bar{\mathcal{F}}$ which is invariant under a quasi-action $G \curvearrowright Z$, we obtain a natural quasi-action $G \curvearrowright (\bar{\mathcal{F}}, d_H)$ of G on the base: Given $g \in G$ and a point $x \in B$ represented by a fiber $\bar{F} \in \bar{\mathcal{F}}$, we let $g \cdot x$ to be a point in B represented by the fiber $\bar{F}' \in \bar{\mathcal{F}}$ such that $d_H(g(\bar{F}), \bar{F}') < C$. Thus we have:

Lemma 3.3. *Suppose that $G \curvearrowright Z$ is a quasi-action such that $\bar{\mathcal{F}}$ is invariant under G. Then the quasi-action $G \curvearrowright Z$ projects to a quasi-action $G \curvearrowright B$; the latter is cobounded provided that the quasi-action $G \curvearrowright Z$ is cobounded.*

We also get a *degenerating quasi-action* $G \overset{\psi}{\curvearrowright} \bar{F}_0$ on a fiber $\bar{F}_0 \in \bar{\mathcal{F}}$ as follows: Pick a nearest point projection $r_{\bar{F}_0} : Z \to \bar{F}_0$. Then we define a degenerating quasi-action ψ by composing $g\big|_{\bar{F}_0} : \bar{F}_0 \to Z$ with $r_{\bar{F}_0}$. The quasi-isometry constant of $\psi(g)$ depends only on $d_H(g(\bar{F}_0), \bar{F}_0)$. In particular, we obtain a homomorphism $G \to \mathrm{QI}(\bar{F}_0)$.

*************** MOVE:

In particular, we get a topological action of G on the space of ends of the fiber $\bar{F}_0$.

Definition 3.4. In case when $\bar{F}_0$ is 2-ended, we say that the degenerating quasi-action ψ *preserves orientation on* $\bar{F}_0$ if G acts trivially on the space of ends of $\bar{F}_0$.

Note that degenerating quasi-action of G preserves orientation on one fiber $\bar{F}_0$ iff it preserves orientation on any other one fiber $\bar{F}_i$.

****

Suppose $\mathcal{M}$ is a collection of metric spaces. A coarse fibration $\bar{\mathcal{F}} = \{\bar{F}_j\}_{j\in J}$ of a metric space Z is *by elements of* $\mathcal{M}$ if one can choose, for each j, an element $M_j \in \mathcal{M}$ and a map $f_j : M_j \to Z$ such that $\mathrm{Im}(f_j) = \bar{F}_j$ for all $j \in J$ and the family $\{f_j\}$ is uniformly proper.

If $\mathcal{A}$ is a collection of metric cell complexes, then a coarse fibration $\bar{F}$ of Z is *by elements of* $\mathcal{A}$ if $\bar{F}$ is a coarse fibration by elements of $\{A^{(0)} \mid A \in \mathcal{A}\}$.

3.2. **Parametrizing coarse fibrations.** In the sequel it will be convenient to specify parametrizations of the fibers of a coarse fibration. Suppose $\bar{\mathcal{F}} = \{\bar{F}_j\}_{j\in\mathcal{J}}$ is a coarse fibration of the 0-skeleton $X^{(0)}$ of a metric cell complex X. A *parametrization* of $\bar{\mathcal{F}}$ is a collection of pairs $\{(F_j, f_j)\}_{j\in\mathcal{J}}$ where

1. For each $j \in \mathcal{J}$, F_j is a bounded geometry cell complex, and the family $\{F_j\}_{j\in\mathcal{J}}$ has bounded geometry,
2. For each $j \in \mathcal{J}$, f_j is a morphism from F_j to X, where $f_j^{(0)} : F_j^{(0)} \to X^{(0)}$ satisfies $\mathrm{Im}(f_j^{(0)}) = \bar{F}_j$, and
3. The family of morphisms $\{f_j\}_{j\in\mathcal{J}}$ is uniformly proper.

Definition 3.5. We will refer to a coarse fibration $\bar{\mathcal{F}}$ together with a choice of parametrization $\{(F_j, f_j)\}_{j\in\mathcal{J}}$ as a *parametrized coarse fibration.* We denote the parametrized coarse fibration by $\mathcal{F}$. We will often refer to the F_j's as *fibers* of $\mathcal{F}$.

The next lemma shows that in some cases one can construct a parametrization for a given coarse fibration.

Lemma 3.6. *Suppose X is a metric cell complex, and $\bar{\mathcal{F}}$ is a coarse fibration of $X^{(0)}$ by elements of a collection $\mathcal{A}$ of cell complexes with uniformly bounded geometry. If X is uniformly $(k-1)$-connected for all $k \le \sup\{\dim(A) \mid A \in \mathcal{A}\}$, then $\bar{\mathcal{F}}$ admits a parametrization $\mathcal{F} = \{(F_j, f_j)\}_{j\in\mathcal{J}}$ where $F_j \in \mathcal{A}$ for all $j \in \mathcal{J}$. If in addition $\mathcal{A}$ is uniformly $(k-1)$-connected, then for each $(F, f) \in \mathcal{F}$, there is a (continuous) map*

$v_F^{(k)} : X^{(k)} \to F$ such that the composition $N_R^{(k)}(\bar{F}) \xrightarrow{v_F^{(k)}} F^{(k)} \xrightarrow{f} X^{(k)}$ is at distance $< D = D(k, R)$ from the inclusion map. In particular, $v_F^{(1)}$ is always defined since metric cell complexes are (linearly) uniformly connected.

Proof. One constructs the maps f_j and $v_F^{(k)}$ by induction on skeleta using the assumption of uniform $(k-1)$-connectedness and the nearest-point projection $f \circ v_F^{(0)} : X^{(0)} \to \bar{F}$ to begin the induction. $\square$

The following lemma is elementary and we leave it to the reader:

Lemma 3.7. *Suppose that $\mathcal{F}$ is a parameterized coarse fibration of X. Assume in addition that there is a uniform bound on the dimension of the fibers F_i and that X is uniformly contractible. Then there exists a parameterized coarse fibration $\mathcal{L}$ of X such that:*

(a) Distinct fibers of $\bar{\mathcal{L}}$ are disjoint.

(b) $\bar{\mathcal{L}}$ is Hausdorff-close to $\bar{\mathcal{F}}$.

(c) The union of fibers in $\bar{\mathcal{L}}$ equals $X^{(0)}$.

The following lemma is immediate from the definitions.

Lemma 3.8. *Let Z_0 be a metric space.*

1. If Z and Z' are length spaces, $f : Z \to Z_0$, $f' : Z' \to Z_0$ are uniformly proper maps, and $Z_0 \subset N_r(\mathrm{Im}(f)) \cap N_r(\mathrm{Im}(f'))$ for some $r \in \mathbb{R}_+$, then there is an (L, A) quasi-isometry $\phi : Z \to Z'$ so that $d(f' \circ \phi, f) < D$ where L, A, D depend only on r and the distortion data of f and f'.

2. If X and X' are uniformly contractible metric cell complexes, and $f : X^{(0)} \to Z_0$, $f' : X'^{(0)} \to Z_0$ are surjective uniformly proper maps, then there is an (L, A) quasi-isometry $\phi : X \to X'$ so that $d(f' \circ \phi^{(0)}, f) < D$ where L, A, D depend only on the distortion data of f and f'.

Proposition 3.9. *Let $\mathcal{F}$ be a parametrized coarse fibration of the zero skeleton $Z := X^{(0)}$ of a metric cell complex.*

1. If (F_1, f_1), $(F_2, f_2) \in \mathcal{F}$ and $\psi : \bar{F}_1 \to X^{(0)}$ is a bounded displacement map with image contained in $\bar{F}_2$, then there is a quasi-isometry $\phi : F_1^{(0)} \to F_2^{(0)}$ such that

$$d(\psi \circ f_1, f_2 \circ \phi) < D,$$

where D and the quasi-isometry constants depend only on the displacement of ψ.

2. Suppose $G \curvearrowright Z$ is a quasi-action preserving $\bar{\mathcal{F}}$. Then each $g \in G$ induces a quasi-isometry $(\bar{\mathcal{F}}, d_H) \xrightarrow{\bar{g}} (\bar{\mathcal{F}}, d_H)$. We also obtain, for each $(F, f) \in \mathcal{F}$, a degenerating quasi-action $G \overset{\psi}{\curvearrowright} F^{(0)}$ by sending $g \in G$ to the composition

$$F^{(0)} \xrightarrow{f} X^{(0)} \xrightarrow{g} X^{(0)} \xrightarrow{v_F^{(0)}} F^{(0)}.$$

Here the quasi-isometry constant of $\psi(g)$ depends only on $d_H(g(\bar{F}), \bar{F})$.

Proof. Let ψ be as in 1. Then $\mathrm{Im}(\psi \circ f_1^{(0)}$ is D_1-dense in $\mathrm{Im}(f_2^{(0)})$, where D_1 depends only on the displacement δ of ψ. We can perturb $f_2^{(0)}$ to a map h so that $\mathrm{Im}(h) = \mathrm{Im}(\psi \circ f_1^{(0)}$, and h is uniformly proper with distortion controlled by D_1 (and hence by δ). Now part 2 of Lemma 3.8 applies.

Part 2 follows from Lemma 3.8 and the corresponding assertions (see Lemma 3.3 and the discussion that follows it) for unparametrized coarse fibrations. □

As a corollary of the above lemma, we get a topological action of G on the space of ends of the fiber $\bar{F}_0$.

Definition 3.10. In case when F_0 is 2-ended, we say that the degenerating quasi-action ψ *preserves orientation on* F_0 if G acts trivially on the space of ends of F_0.

Note that degenerating quasi-action of G preserves orientation on one fiber F_0 iff it preserves orientation on any other one fiber F_i.

4. Coarse fibrations and product structure

In this section we will show that coarse fibration behave homotopy theoretically like products when the total space and the fibers are (the zero skeleta of) uniformly contractible metric cell complexes.

Let (X, p) be a uniformly contractible metric cell complex, and suppose $\bar{\mathcal{F}}$ is a coarse fibration of $X^{(0)}$ by a uniformly contractible, bounded geometry family $\mathcal{A}$ of metric cell complexes. Using Lemma 3.6, $\bar{\mathcal{F}}$ can be given a parametrization $\mathcal{F} = \{(F_j, f_j)\}_{j \in \mathcal{J}}$, where $F_j \in \mathcal{A}$.

Let $Y^{(0)}$ be a net in $(\bar{\mathcal{F}}, d_H)$ with the property that all R-balls in $Y^{(0)}$ have cardinality $< C = C(R)$. Let $c_X : X \to Y^{(0)}$ be a (discontinuous) projection map given by letting $c_X(x) \in Y^{(0)}$ be a fiber nearest to $p(x)$. The pair (X, c_X) defines an object in the category $\mathcal{C}_{Y^{(0)}}$ (see section 2 for the definition of bounded control categories). Pick $(F, f) \in \mathcal{F}$. If V is any CW-complex based on $Y^{(0)}$ with control map $c_V : V \to Y^{(0)}$, we get another object in $\mathcal{C}_{Y^{(0)}}$ by taking the pair $(V \times F, c_{V \times F})$ where $c_{V \times F}$ is the composition $V \times F \to V \xrightarrow{c_V} Y^{(0)}$.

Our main goal in this section is:

Proposition 4.1. *There is a bounded geometry, uniformly contractible CW complex Y based on $Y^{(0)}$ with control map $c_Y : Y \to Y^{(0)}$, and a morphism $\phi : (Y \times F, c_{Y\times F}) \to (X, c_X)$ which is a homotopy equivalence in the category $\mathcal{C}_{Y^{(0)}}$.*

Definition 4.2. We will refer to Y as a *coarse base* of the coarse fibration $\mathcal{F}$.

The proof of the Proposition 4.1 occupies the rest of this section. The proof is by induction on the skeleta of Y. We start by noting that since $Y^{(0)}$ is a quasi-isometric to a length space (see section 3.1), there is a metric graph $Y^{(1)}$ (based on $Y^{(0)}$) so that the inclusion map $(Y^{(0)}, d_H) \to Y^{(1)}$ is a quasi-isometry. The graph $Y^{(1)}$ has bounded geometry because each R-ball in $(Y^{(0)}, d_H)$ has cardinality $\le C(R)$. To continue with the induction we will need several auxiliary lemmas.

Lemma 4.3. *Pick $k \in \mathbb{Z}_+$. For every R there is an $R' = R'(R)$, and a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\bar{F} \in \mathcal{F}$, any map $\phi : S^k \to N^{(k)}_R(\bar{F})$ with* $\operatorname{diam}(\operatorname{Im}\phi) < D$, *admits a null-homotopy $\phi \cong const$ with image in $N^{(k+1)}_{R'}(\bar{F}) \cap N^{(k+1)}_{\alpha(D)}(\operatorname{Im}\phi)$.*

Proof. Let R be given, and let $\phi : S^k \to N^{(k)}_R(\bar{F})$ be a continuous map so that $\operatorname{diam}(\operatorname{Im}\phi) < D$. The image of the composition $v_F \circ \phi : S^k \to F$ has diameter $< D_1 = D_1(D, R)$. By the uniform contractibility of F, there is a null-homotopy $H : S^k \times I \to F$ of $v_F \circ \phi$ whose image $\operatorname{Im}(H)$ is contained in the $D_2 = D_2(D, R)$ neighborhood of $\operatorname{Im}(v_F \circ \phi)$ in the $(k+1)$-skeleton of X. By Remark 2.8, the map ϕ is homotopic to $f \circ v_F \circ \phi$ by a homotopy whose tracks have diameter $< D_3 = D_3(D, R)$. Combining this homotopy with the homotopy $f \circ H$, we obtained the desired null-homotopy. ☐

Let V be a metric cell complex based on $Y^{(0)}$ with a control map $c_V : V \to Y^{(0)}$. Pick $(F, f) \in \mathcal{F}$, and let $c_{V\times F}$ be as indicated above, so that $(V \times F, c_{V\times F})$ is an object in the category $\mathcal{C}_{Y^{(0)}}$.

Lemma 4.4. *There is a morphism $\Phi : (V \times F, c_{V\times F}) \to (X, c_X)$ in the category $\mathcal{C}_{Y^{(0)}}$ with the following property. For every R-ball $B \subset Y^{(0)}$, the restriction of $v_F \circ \Phi$ to $\left(c_V^{-1}(B) \cap V^{(k)}\right) \times F^{(l)}$ is at distance at most $D = D(k, l, R, d(B, c_X(\bar{F}))$ from the projection $\left(c_V^{-1}(B) \cap V^{(k)}\right) \times F^{(l)} \to F$.*

Proof. We construct Φ by induction on the skeleta of V. We define $\Phi^{(0)} : V^{(0)} \times F \to X$ so that for each $y = (F', f') \in V^{(0)} = Y^{(0)}$, the restriction of $\Phi^{(0)}$ to $y \times F$ agrees with the composition $F \overset{f}{\to} X \overset{v_{F'}}{\to} F' \overset{f'}{\to} X$.

We continue inductively. To proceed from $V^{(k)}$ to $V^{(k+1)}$, for each cell $\Delta^{(k+1)}$ in $V^{(k+1)}$ we construct a map $\Delta^{(k+1)} \times F \to X$ by extending the given map $(\partial\Delta^{(k+1)}) \times F \to X$. This extension is done by induction on skeleta in F via Lemma 4.3. ☐

Lemma 4.5. *For every $n \in \mathbb{Z}_+$, there is a bounded geometry cell complex $Y^{(n)}$ based on $Y^{(0)}$ with control map $c : Y^{(n)} \to Y^{(0)}$ so that $Y^{(n)}$ is uniformly $(n-1)$-connected.*

Proof. Assume inductively that a uniformly $(k-1)$-connected bounded geometry cell complex $Y^{(k)}$ has been constructed.

We note that given $D \in \mathbb{R}_+$, there exists a collection of "singular k-spheres" $h_j : S^k \to Y^{(k)}$ so that:

1. The diameters $\mathrm{diam}(\mathrm{Im}(h_j))$ are uniformly bounded.
2. The collection of sets $\mathrm{Im}(h_j), j \in \mathbb{N}$, is uniformly locally finite.
3. Any map $\phi : S^k \to Y^{(k)}$, whose image has diameter $< D$, is null-homotopic in $W^{(k+1)}(D) := Y^{(k)} \cup (\cup_{h_j} e_j)$.

We give $W^{(k+1)}(D)$ structure of a metric cell complex by extending the control map $p : Y^{(k)} \to Y^{(0)}$ to the $k+1$-cells e_j: send each point of $e_j \setminus \partial e_j$ to a point $y \in \mathrm{Im}(h_j) \cap Y^{(0)}$.

Remark 4.6. Arguing inductively, on the skeleta of the sphere S^k, using the above properties of $W^{(k+1)}$ one proves the following. Given D there is a $D_1 = D_1(D)$ such that any two maps $\phi, \phi' : S^k \to Y^{(k)}$ with $d(\phi, \phi') < D$ become homotopic in $W^{(k+1)}(D_1)$, where the tracks of the homotopy have diameter $< D_2 = D_2(D)$.

Using the uniform $(k-1)$-connectedness of $Y^{(k)}$ and induction on skeleta, we may construct a morphism $X^{(k)} \stackrel{\pi^{(k)}}{\to} Y^{(k)}$ (in the category $\mathcal{C}_{Y^{(0)}}$) which extends $c_X\big|_{X^{(0)}}$. Since X is a bounded geometry cell complex, the composition $X^{(k)} \stackrel{\pi^{(k)}}{\to} Y^{(k)} \to W^{(k+1)}(D)$ extends to a morphism $X^{(k+1)} \stackrel{\pi^{(k+1)}}{\longrightarrow} W^{(k+1)}(D)$ when $D > R_1 = R_1(k)$.

Pick $(F, f) \in \mathcal{F}$, and point $t \in F^{(0)}$. Let $\Phi : Y^{(k)} \times F \to X$ be the morphism provided by Lemma 4.4 when applied with $V = Y^{(k)}$. Let $\Psi : Y^{(k)} \to X^{(k)}$ be the composition of the inclusion $Y^{(k)} \to Y^{(k)} \times \{t\}$ with Φ; note that Ψ is a morphism in the category $\mathcal{C}_{Y^{(0)}}$.

Consider a map $\phi : S^k \to Y^{(k)}$, and the composition $\hat{\phi} := \Psi \circ \phi : S^k \to X^{(k)}$. By Lemma 4.3, the map $\hat{\phi}$ is null-homotopic via a null-homotopy $H : S^k \times I \to X^{(k+1)}$ such that $\pi^{(k+1)} \circ H$ has image diameter $< \theta(\mathrm{diam}(\mathrm{Im}(\phi)))$ where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ depends only on k. Since $d(\pi^{(k)} \circ \hat{\phi}, \phi) < R_2 = R_2(k)$, we conclude by Remark 4.6 that ϕ is homotopic to $\pi^{(k)} \circ \hat{\phi}$ within $W^{(k+1)}(D)$ whenever $D > D_1(R_2)$. Therefore ϕ is null-homotopic in $W^{(k+1)}(D)$ whenever $D > R_3(k)$; furthermore, the diameter of the homotopy is at most $\theta_1(\mathrm{diam}(\mathrm{Im}(\phi)))$ where θ_1 depends only on k. $\square$

By the preceding lemma, we may construct a uniformly contractible metric cell complex Y based on $Y^{(0)}$ with control map $c : Y \to Y^{(0)}$. Pick $(F, f) \in \mathcal{F}$. We apply Lemma 4.4 to construct a morphism $\Phi : Y \times F \to X$.

Lemma 4.7. *We let $\Phi : Y \times F \to X$ be as above. Then Φ admits a homotopy inverse $\tilde{\Phi}$ in the category $\mathcal{C}_{Y^{(0)}}$.*

Proof. Since the metric cell complex Y is uniformly contractible, we can construct a morphism $h:(X,c_X)\to(Y,c_Y)$ by induction on skeleta starting with the map $c_X:X^{(0)}\to Y^{(0)}$. We then let $\tilde\Phi:(X,c_X)\to(Y\times F,c_{Y\times F})$ be given by $\tilde\Phi:=(h,v_F)$. We verify that $\Phi,\tilde\Phi$ are homotopy-inverse to each other in the category $\mathcal{C}_{Y^{(0)}}$ using uniform contractibility of Y, F and Lemma 4.3.

□

Lemma 4.7 completes the proof of Proposition 4.1.

5. Coarse fibrations and coarse Poincare duality

Let X, $\mathcal{F}$, Φ, $\tilde\Phi$, and Y be as in the previous section. Assume in addition that X is homeomorphic to an n-manifold, and that every $(F,f)\in\mathcal{F}$ satisfies

$$H^*_c(F^{(i+1)})\cong H^*_c(\mathbb{R}^i).$$

We will show that under these assumptions the complex Y behaves coarsely like a manifold of dimension $n-i$.

We recall that X satisfies the Poincare duality:

Theorem 5.1. *For each closed subset $W\subset X$ and $k\in\mathbb{Z}$ there is an isomorphism*

$$P_{W,k}:\check H^k_c(W)\to H_{n-k}(X,X\setminus W)$$

which is local in the following sense: $\mathrm{Supp}(P_{W,k}(\tau))\subset N_{D_X}(\mathrm{Supp}(\tau))$ *for each* $\tau\in Z^k_c(W)$*; here $\check H^*_c(\cdot)$ denotes Čech cohomology. The constant D_X does not depend on W and τ. The family $\{P_{W,k}\}$ is compatible with homomorphisms induced by inclusions.*

Of course, the space Y will not satisfy Poincare duality, but we will show that it satisfies a coarse version of it. We first state the coarse Poincare duality in its general form, so that it is a *quasi-isometry invariant* and then we prove it for Y so that some of the constants appearing in the general definition vanish.

Definition 5.2. Let W be a uniformly acyclic metric cell complex. We say that W satisfies *coarse n-dimensional Poincare duality* if the following holds. For each $m\in\mathbb{N}$ there is a constant[3] $D=D_m\geq 0$ so that if $k\in\mathbb{Z}$ and $m\geq 1+\max(k,n-k)$, then:

There is a system of homomorphisms $\{P_K\}$ defined for subcomplexes $K\subset W^{(m)}$:

$$P_K:H^k_c(N^{(m)}_D(K))\to H_{n-k}(W^{(m)},W^{(m)}\setminus K),$$

which are compatible with homomorphisms induced by inclusions, and which determine an *approximate isomorphism*[4] in the sense that the homomorphisms α and β in

[3]If W is finite-dimensional, as in Section 7, we can of course take the constant D independent on m and replace $W^{(m)}$ with W, etc.

[4]Vanishing of α means the *approximate monomorphism* and vanishing of β means the *approximate epimorphism.*

the following commutative diagram are zero:

$$\begin{array}{ccccccc} ker(P_{N_D(K)}) & \to & H^k_c(N^{(m)}_{2D}(K)) & \stackrel{P_{N_D(K)}}{\longrightarrow} & H_{n-k}(W, V_D(K)) & \to & coker(P_{N_D(K)}) \\ \alpha\downarrow & & \downarrow & & \downarrow & & \beta\downarrow \\ ker(P_K) & \to & H^k_c(N^{(m)}_D(K)) & \stackrel{P_K}{\longrightarrow} & H_{n-k}(W, V_0(K)) & \to & coker(P_K). \end{array}$$

Here and in what follows $V^{(m)}_\rho(K) := W^{(m)} - N^{(m)}_\rho(K)$.

The homomorphisms P_K are local in the following sense: if $[\sigma] \in H^k_c(N^{(m)}_D(K))$ is represented by a cocycle $\sigma \in Z^k_c(N^{(m)}_D(K))$, then $P_K(\sigma)$ can be represented by a relative cycle τ supported in $N^{(m)}_D(\mathrm{Supp}(\sigma))$.

We say that a metric cell complex W satisfies *coarse n-dimensional Poincare duality in dimension j*, if the constant D and the family of operators $\{P_K\}$, as above, exists for $k = j$, and $m = 1 + \max(j, n-j)$ (we do not require W to be uniformly acyclic).

Definition 5.3. A 1-connected metric cell complex Y satisfying 2-dimensional coarse Poincare duality in dimension 1 will be called a *quasi-plane*. We remind the reader that by convention all our metriccell complexes have bounded geometry, see section 2.

We note that the above definitions make sense if instead of the integer coefficients for the (co)homology groups we use an arbitrary commutative ring $\mathcal{R}$ with a unit. Thus we can talk about *metric cell complexes satisfying n-dimensional coarse Poincare duality (over $\mathcal{R}$) in dimension j*. Accordingly, we can talk about *quasi-planes over $\mathcal{R}$*.

Lemma 5.4. *Suppose that Y is a quasi-plane over $\mathcal{R}$. Then $Y^{(1)}$ is 1-ended.*

Proof. Coarse Poincare duality implies that the we have an approximate isomorphism between the inverse systems of groups

$$\{H^1_c(B^{(2)}_R(y))\}_{R\in\mathbb{R}_+} \text{ and } \{H_1(Y^{(2)}, Y^{(2)} \setminus B_R(y))\}_{R\in\mathbb{R}_+}.$$

(Here and in the rest of the argument all (co)homologies are with coefficients in $\mathcal{R}$.) By taking inverse limit and using 1-connectedness of Y, we conclude that $H^1_c(Y^{(2)})$ is zero. Then, by using the long exact sequence of the end-cohomology groups of $Y^{(2)}$ we conlcude that $H^0(Ends(Y)^{(2)}) \cong \mathcal{R}$. This implies that the set of ends of Y is connected and thus $Y^{(1)}$ is 1-ended. □

We note that in [26] we have proven a number of coarse versions of Jordan separation theorem for complexes satisfying coarse Poincare duality. In particular:

Proposition 5.5. *Suppose that Z is a (finite-dimensional) metric cell complex satisfying n-dimensional Poincare duality, M is a bounded geometry uniformly acyclic cell complex which is homeomorphic to an n-manifold. Then for each uniform proper map $f : M \to Z$, the image $f(M^{(0)})$ is Hausdorff-close to $Z^{(0)}$.*

We next observe that (unlike the usual Poincare duality) coarse Poincare duality is a quasi-isometry invariant property:

Lemma 5.6. *Suppose Y and Y' are cell complexes with quasi-isometric 0-skeleta. Then Y satisfies coarse n-dimensional Poincare duality iff Y' does.*

Proof. We first note that if Y satisfies n-dimensional Poincare duality with the constants $D = D_m$ then it also satisfies n-dimensional Poincare duality with the constants $\tilde{D} = \tilde{D}_m$ for any $\tilde{D}_m \geq D_m$. Next, the (L, A)-quasi-isometry $f^{(0)} : Y^{(0)} \to Y'^{(0)}$ extends to a morphism $f : Y \to Y'$ which admits a homotopy-inverse $\tilde{f}$ such that:

1. For each m the maps $f^{(m)}, \tilde{f}^{(m)}$ are (L_m, A_m) quasi-isometric maps.

2. For each m the tracks of the homotopies $\tilde{f}^{(m)} \circ f^{(m)}$ and $f^{(m)} \circ \tilde{f}^{(m)}$ have diameter (measured in $Y^{(0)}$ and $Y'^{(0)}$) not exceeding certain constants A_m.

To construct the duality operators $P'_{K'}$ for $K' \subset Y'^{(m)}$ we take $K := f^{-1}(K') \cap Y^{(m)}$ and consider the composition $P' = f_* \circ P_{N_R(K)} \circ f^*$:

$$H_c^m(N_{D'}(K')) \xrightarrow{f^*} H_c^m(N_{R+D}(K)) \xrightarrow{P_{N_R(K)}}$$
$$\longrightarrow H_{n-k}(Y^{(m)}, V_0(N_R(K))) \xrightarrow{f_*} H_{n-k}(Y'^{(m)}, V_0(K'))$$

for appropriately chosen $D' = D'_m$ and $R = \frac{D'}{L_m} - A_m$. We have to choose D'_m so that P' would satisfy the *approximate isomorphism* property. We consider the approximate epimorphism property and leave the approximate monomorphism part to the reader. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & H_{n-k}(Y', V_{3\rho}(K')) \\
 & & & & & & \downarrow \\
 & & & & H_{n-k}(Y, V_{2d}(K)) & \xrightarrow{f_*} & H_{n-k}(Y', V_{2\rho}(K')) \\
 & & & & \downarrow & & \downarrow \\
 & & H_c^k(N_{2d}^{(m)}(K)) & \xrightarrow{P} & H_{n-k}(Y, V_d(K)) & \xrightarrow{f_*} & H_{n-k}(Y', V_\rho(K')) \\
 & & \downarrow & & \downarrow & & \downarrow \\
H_c^k(N_{3\rho}^{(m)}(K')) & \xrightarrow{f^*} & H_c^k(N_d^{(m)}(K)) & \xrightarrow{P} & H_{n-k}(Y, V_0(K)) & \xrightarrow{f_*} & H_{n-k}(Y', V_0(K'))
\end{array}$$

Here $d = d_m \geq D_m$ is chosen sufficiently large so that

$$A_m \leq \rho \leq \frac{d}{L_m} - A_m \text{ and } 3\rho \leq \frac{2}{L_m}(d - 2A_m).$$

Set $D' = D'_m = 3\rho$. Consider a homology class $\alpha \in H_{n-k}(Y', V_0(K'))$ which lifts into $\bar{\alpha} \in H_{n-k}(Y', V_{3\rho}(K'))$ with the intermediate lifts $\tilde{\alpha} \in H_{n-k}(Y', V_\rho(K'))$, and $\hat{\alpha} \in H_{n-k}(Y', V_{2\rho}(K'))$. Then using the map $\tilde{f}_*$ one can lift the clases $\tilde{\alpha}, \hat{\alpha}, \alpha$ to classes $\tilde{\beta}, \hat{\beta}, \beta$ which belong to $H_{n-k}(Y, V_{2d}(K))$, $H_{n-k}(Y, V_d(K)), H_{n-k}(Y, V_0(K))$ respectively. Next, use the *approximate epimorphism* property of P to lift $\hat{\beta}$ and β to classes $\hat\gamma, \gamma$ in the groups $H^k_c(N^{(m)}_{2d}(K))$, $H^k_c(N^{(m)}_d(K))$ respectively. Finally, using the projection of $\tilde f^*(\hat\gamma)$ to $H^k_c(N^{(m)}_r(K'))$ we lift γ to a class $\epsilon \in H^k_c(N^{(m)}_{3\rho}(K'))$. This establishes the *approximate epimorphism* property for P' with the constants $D' = D'_m$ above. $\square$

We now return to the discussion of the coarse fibrations. The following is an analog of the fact that if B is the base of a fibration whose total space and fibers are aspherical manifolds, then B is an aspherical manifold.

Recall that $\mathcal{F}$ is a parameterized coarse fibration on n-dimensional uniformly contractible manifold X so that every fiber $(F, f) \in \mathcal{F}$ satisfies

$$H^*_c(F^{(i+1)}) \cong H^*_c(\mathbb{R}^i),$$

and Y is a *coarse base* of the coarse fibration $\mathcal{F}$. Set $j := n - i$.

Theorem 5.7. *(Y satisfies j-dimensional coarse Poincare duality.) Pick $k \in \mathbb{Z}$, and set $m := 1 + \max(k, j-k)$. For every subcomplex $K \subset Y^{(m)}$, one can define a homomorphism*

$$P_K : H^k_c(K) \to H_{j-k}(Y^{(m)}, Y^{(m)} \setminus K)$$

so that the collection $\{P_K\}$ has the following properties for a suitable $D \in \mathbb{R}$:

1. *$\{P_K\}$ is compatible with inclusions $K_1 \subset K_2 \subset Y^{(m)}$.*
2. *The kernel of $P_{N^{(m)}_D(K)}$ is contained in the kernel of the restriction $H^k_c(N^{(m)}_D(K)) \to H^k_c(K)$ (i.e., P_K is an approximate monomorphism).*
3. *The image of the inclusion*

$$H_{j-k}(Y^{(m)}, Y^{(m)} \setminus N^{(m)}_D(K)) \to H_{j-k}(Y^{(m)}, Y^{(m)} \setminus K)$$

is contained in the image of P_K (i.e., P_K is an approximate epimorphism).

4. *P_K is local in the following sense: if $[\sigma] \in H^k_c(K)$ is represented by a cocycle $\sigma \in Z^k_c(K)$, then $P_K(\sigma)$ can be represented by a relative cycle τ supported in $N^{(m)}_D(\mathrm{Supp}(\sigma))$.*

Proof. We construct the maps P_K as follows. Let $[\omega]$ be a generator of $H^i_c(F^{(i+1)})$. First, send each $[\sigma] \in H^k_c(K)$ to the tensor product $[\sigma] \otimes [\omega] \in H^{k+i}_c((K \times F)^{(i+1)})$. Then pull back $[\sigma] \otimes [\omega]$ via $(\tilde\Phi^{(m+i)})^*$ to obtain $(\tilde\Phi^{(m+i)})^*([\sigma] \otimes [\omega]) \in H^{k+i}_c(\hat K)$ where $\hat K := (\tilde\Phi^{(m+i)})^{-1}(K \times F) \subset X^{(m+i)}$ and $\tilde\Phi$ is the homotopy-equivalence $\tilde\Phi : X \to Y \times F$ constructed in Lemma 4.7 of the previous section. Next, apply $P_{\hat K, k+i}$ given by Theorem 5.1:

$$[\sigma] \otimes [\omega] \mapsto (\tilde\Phi^{(m+i)})^*([\sigma] \otimes [\omega]) \mapsto P_{\hat K, k+i}((\tilde\Phi^{(m+i)})^*([\sigma] \otimes [\omega])) = [\eta].$$

The result is a homology class

$$[\eta] \in H_{n-(k+i)}(X^{(m+i)}, X^{(m+i)} \setminus \hat K) = H_{j-k}(X^{(m+i)}, X^{(m+i)} \setminus \hat K).$$

Then send $[\eta]$ to $H_{j-k}((Y\times F)^{(m+i)},(Y\times F)^{(m+i)}\setminus(K\times F)^{(m+i)})$ via $(\tilde{\Phi}^{(m+i)})_*$. Next we apply the isomorphism

$$H_{j-k}(Y^{(m)}\times F,(Y^{(m)}\setminus K)\times F)\simeq H_{j-k}(Y^{(m)},Y^{(m)}\setminus K).$$

given by projection. By composing with this isomorphism we get

$$[\eta]\mapsto(\tilde{\Phi}^{(m+i)})_*([\eta])\mapsto[\tau]\in H_{j-k}(Y^{(m)},Y^{(m)}\setminus K).$$

As the result we finally get the homomorphism

$$P_K:[\sigma]\mapsto[\eta]\mapsto[\tau].$$

It remains to verify that that the P_K's so defined satisfy conditions 1-4 above. Property 1 follows immediately from the fact that each of the factors in the composition defining P_K is compatible with inclusions. Property 4 follows readily from the locality of $P_{\hat{K},k+i}$ and the fact that $\tilde{\Phi}^{(m+i)}$ has bounded control over $Y^{(0)}$. We will only prove property 2, as the proof of 3 is similar. The map $P_K:[\sigma]\mapsto[\tau]$ is a composition of homomorphisms all but two of which are clearly isomorphisms, the exceptions being the pull-back and push-forward via $\tilde{\Phi}$. Since the map $\tilde{\Phi}$ preserves skeleta and X is an n-dimensional manifold, it suffices to restrict to

$$\Phi^{(n)}:Y^{(n)}\times F\to X,\ \tilde{\Phi}:X\to Y^{(n)}\times F\ .$$

We recall that (since Φ and $\tilde{\Phi}$ are homotopy inverses in $\mathcal{C}_{Y^{(0)}}$)

$$\tilde{\Phi}\circ\Phi^{(n)}:Y^{(n)}\times F\to Y^{(n)}\times F$$

is homotopic to $\mathrm{id}_{Y^{(n)}\times F}$ via a homotopy $H:Y^{(n)}\times F\times I\to Y^{(n+1)}\times F$ in the category $\mathcal{C}_{Y^{(0)}}$; projections to $Y^{(0)}$ of the tracks of this homotopy have diameter $\le D_Y$ for some D_Y. Similarly, (by enlarging D_Y if necessary) $\Phi^{(n)}\circ\tilde{\Phi}:X\to X$ is homotopic to id_X via homotopy $\tilde{H}:X\times I\to X$ in the category $\mathcal{C}_{Y^{(0)}}$; projections to $Y^{(0)}$ of the tracks of this homotopy have diameter $\le D_Y$. Finally, we assume that D_Y is chosen in such a way that $d(c_X\circ\Phi,c_X)\le D_Y$ and $d(c_{Y\times F}\circ\tilde{\Phi},c_{Y\times F})\le D_Y$.

Set $D:=D_X+4D_Y$: here D_X is the constant that appears in Theorem 5.1.

Suppose that $R\ge 3D_Y$. If $[\eta]\in H_{j-k}(X,X\setminus\widehat{N_R(K)})$ and $[\eta]\mapsto\tilde{\Phi}_*([\eta])=0$ then $[\eta]$ maps to zero under the inclusion induced map

$$H_{j-k}(X,X\setminus\tilde{\Phi}^{-1}(N_R^{(m)}(K)\times F))\to H_{j-k}(X,X\setminus\tilde{\Phi}^{-1}(N_{R-3D_Y}^{(m)}(K)\times F)).$$

(This follows by noticing that $\Phi_*\circ\tilde{\Phi}_*(\eta)-\eta=d(\theta)$ where the projection of the support of θ to $Y^{(0)}$ has diameter $\le D_Y$.)

The same argument applies to the map

$$[\sigma]\otimes[\omega]\mapsto\tilde{\Phi}^*[\sigma]\otimes[\omega],$$

where $[\sigma] \otimes [\omega] \in H_c^{k+i}(N_R^{(m)}(K) \times F^{(i+1)})$ for some $R \geq D_Y$. If $[\sigma] \otimes [\omega]$ maps to zero then (again using the homotopy H) we conclude that $[\sigma] \otimes [\omega]$ restricts to zero in $H_c^{i+k}(N_{R-D_Y}^{(m)}(K) \times F^{(i+1)})$.

Combining these two statements and locality of the usual Poincare duality map, it follows that if $[\sigma] \otimes [\omega] \in H_c^{j+k}(N_D^{(m)}(K) \times F)$ belongs to the kernel of the composition $\Phi_* \circ P \circ \tilde{\Phi}^*$, then $[\sigma] \otimes [\omega]$ belongs to the kernel of the inclusion

$$H_c^{i+k}(N_D^{(m)}(K) \times F) \to H_c^{i+k}(K \times F). \quad \square$$

Corollary 5.8. *For every k we have $H_c^k(Y^{(k+1)}) \cong H_c^k(\mathbb{R}^j)$. If $j \geq 2$ then Y is 1-ended in the sense that $Y^{(1)}$ is 1-ended.*

Corollary 5.9. *If in Theorem 5.7, $n = 3$ and $i = 1$ then the coarse base Y is a quasi-plane.*

6. The surrounding function

Let Y be a cell complex whose 1-skeleton $Y^{(1)}$ is 1-ended, connected, and equipped as usual with a path metric d where edges have length 1. We suppose further that $Y^{(0)}$ is *quasi-homogeneous*, i.e. there are constants L_0, $A_0 \in \mathbb{R}$ such that for all y, $y' \in Y^{(0)}$, there is an (L_0, A_0)-quasi-isometry $Y^{(0)} \to Y^{(0)}$ which maps y to y'.

Definition 6.1. A subgraph $\Gamma \subset Y^{(1)}$ *surrounds* a subset $\Sigma \subset Y^{(1)}$ if $\Gamma \cap \Sigma = \emptyset$, and Γ separates Σ from infinity, i.e. any proper path $\mathbb{R}_+ \to Y^{(1)}$ starting at Σ intersects Γ.

If $\Gamma \subset Y^{(1)}$ is a subgraph, we let $|\Gamma|$ denote the cardinality of the vertex set of Γ; we will refer to $|\Gamma|$ as the *size* of Γ.

Definition 6.2. Given $y \in Y^{(0)}$, $R \in \mathbb{R}_+$, let $\mathrm{Sur}(y, R)$ be the minimum of sizes of connected subgraphs $\Gamma \subset Y^{(1)}$ which surround $B(y, R) \subset Y^{(1)}$. We will refer to a graph Γ which realizes the minimum as a *smallest* graph which surrounds $B(y, R)$.

Lemma 6.3. *The function* $\mathrm{Sur}(\cdot, \cdot)$ *is "quasi-invariant": there are constants C and R_0 such that* $\mathrm{Sur}(y', \frac{R}{C}) \leq C\,\mathrm{Sur}(y, R)$ *for all y, $y' \in Y$, $R > R_0$.*

Proof. Pick y, $y' \in Y^{(0)}$. There are (L, A)-quasi-isometries $f : Y^{(0)} \to Y^{(0)}$ and $f' : Y^{(0)} \to Y^{(0)}$ such that $f(y) = y'$, $f'(y') = y$, and $d(f' \circ f, \mathrm{id}_{Y^{(0)}}) < A$, $d(f \circ f', \mathrm{id}_{Y^{(0)}}) < A$, where L, A are independent of y, y'.

Choose a smallest connected subgraph $\Gamma \subset Y^{(0)}$ which surrounds $B(y, R)$. Then $d(y', f(\Gamma^{(0)})) \geq \frac{1}{L}R - A$. If $x' \in Y^{(0)}$ and $Ld(y', x') + A \leq R$, then $f'(x') \in B(y, R)$, and consequently if $\rho : \mathbb{R}_+ \to Y^{(1)}$ is any proper path with $\rho(0) = x'$, then by the uniform connectedness of $Y^{(0)}$, the set $f'((\mathrm{Im}(\rho) \cap Y^{(0)})$ must pass within distance $D_1 = D_1(L, A, Y)$ of Γ, which means that ρ must pass within distance $D_2 = D_2(L, A, Y)$ of

$f(\Gamma^{(0)})$. Applying the uniform connectedness of $Y^{(1)}$ again, we can find a connected subgraph $\Gamma' \subset Y^{(1)}$ such that

$$N_{D_2}(f(\Gamma^{(0)})) \subset \Gamma' \subset N_{D_3}(f(\Gamma'))$$

for $D_3 = D_3(L, A, Y)$. If R' satisfies $R' + D_3 < \frac{1}{L}R - A$, then $B(y', R') \cap \Gamma' = \emptyset$, and hence Γ' surrounds $B(y', R)$. Since $Y^{(1)}$ has bounded geometry, there is a constant $c = c(Y)$ such that $|\Gamma'| \leq c|f(\Gamma^{(0)})| \leq c|\Gamma|$. Taking $C = \max(c, 2L)$, and R_0 sufficiently large, the lemma follows. $\square$

The quantity $\mathrm{Sur}(y, R)$ is finite since the space $Y^{(1)}$ is 1-ended. Note also that $\mathrm{Sur}(y, R) \geq cR$ for some $c > 0$ since there will be (uniform) quasi-geodesic lines passing through any $y \in Y^{(0)}$.

Lemma 6.4. *Suppose there is a constant C such that $\mathrm{Sur}(y, R) < CR$ for all $y \in Y^{(0)}$, and all $R \geq 1$. Then $Y^{(0)}$ is* doubling *(see section 2), and hence has polynomial growth.*

Proof. We first observe that there is a constant $C_1 \geq 1$ such that any connected subgraph with size at most $C_1 R$ which surrounds an R-ball $B(y, R)$ must lie in $B(y, C_1 R)$. This follows from the existence of a complete quasi-geodesic (with uniform constants) passing through any given point. Moreover, there is a constant $C_2 = 2C_1$ such that if Γ surrounds $B(y, R) \subset Y^{(1)}$ and has size $\leq C_1 R$, then the connected component of y in $Y^{(1)} \setminus \Gamma$ is contained in $B(y, C_2 R)$. Otherwise, if $q \in Y^{(1)} \setminus \Gamma$ lies in the same component of $Y^{(1)} \setminus \Gamma$ as y, and $d(q, y) > 2C_1 R = C_2 R$, then Γ surrounds $B(q, R)$, has size $\leq C_1 R$, and lies entirely outside $B(q, C_1 r)$, which is a contradiction.

Pick $y \in Y^{(0)}$, $R \geq 1$. Choose a connected graph $\Gamma_1 \subset Y^{(1)}$ with size at most CR which surrounds $B(y, R)$, and set $\mathcal{L}_1 = \{\Gamma_1\}$. Let $\mathcal{N}_1$ be an $\frac{R}{2}$-net in Γ_1. Let $\mathcal{L}_2$ be a collection of connected subgraphs of $Y^{(1)}$ surrounding the R-balls centered at points in $\mathcal{N}_1$. Proceed inductively in this fashion, building up k layers of surrounding connected subgraphs of $Y^{(1)}$. The union $V_k := \mathcal{N}_0 \cup \ldots \cup \mathcal{N}_k$ has cardinality bounded by an exponential function of k which depends only on C. We claim that the collection of $C_2 R$-balls centered at points in V_k covers $B(y, \frac{kR}{2})$. To see this, consider a path σ of length at most $\frac{kR}{2}$ starting at y. We inductively break σ into a concatenation of at most k subpaths of length at least $\frac{R}{2}$ as follows. Let σ_1 be the initial segment of σ until it arrives at Γ_1. The path σ_1 terminates within distance $\frac{R}{2}$ of a point $y_1 \in \mathcal{N}_1$. Let σ_2 be the initial segment of $\sigma \setminus \sigma_1$ until it arrives at the connected subgraph $\Gamma_2 \in \mathcal{L}_2$ surrounding $B(y_1, R)$. Et cetera. At each step, the segment σ_i has length at least $\frac{R}{2}$, and all of them are contained in $\cup_{q \in V_k} B(q, C_2 R)$.

Choosing k such that $\frac{k}{2} > 2C_2$, we see that any $2C_2 R$-ball can be covered by at most N balls of radius $C_2 R$, where N is independent of R. Hence $Y^{(0)}$ is doubling and thus has polynomial growth. $\square$

Below is an alternative argument for the polynomial growth of $Y^{(0)}$ in the case when it is quasi-isometric to a (finitely generated) group. Let $V(y,r)$ denote the number of points in $B(y,r)\cap Y^{(0)}$ for $y\in Y^{(0)}$. We will assume that the graph $Y^{(1)}$ is metrized so that each edges has unit length.

Proposition 6.5. *Suppose there is a constant C and a sequence $R_j\in\mathbb{R}_+$ diverging to ∞, such that $\operatorname{Sur}(y,R_j)<CR_j$ for all $y\in Y^{(0)}$, and each j. Assume in adition that $Y^{(1)}$ is the Cayley graph of a finitely generated group Q. Then Q is virtually nilpotent and has at most quadratic growth.*

Proof. We recall (N. Varopoulos, [41]) that if there are constants $C_0, a\in\mathbb{R}$ so that

$$V(y,r)\geq C_0 r^a,\quad \forall r\geq 1,\quad \forall y\in Y^{(0)}, \tag{6.6}$$

then there is $C_1\in\mathbb{R}$ so that

$$|\partial D|\geq C_1|D|^{(a-1)/a}$$

for all finite subsets $D\subset Y^{(0)}$. (Here $\partial D=\partial_1 D$ is the "boundary" of D, see section 2.) Let $\operatorname{Inrad}(D)$ denote the radius of the largest metric ball $B(z,r)\subset Y^{(1)}$ such that in $B(z,r)\cap Y^{(0)}$ is contained in D. Hence

$$|\partial D|\geq C'|D|^{(a-1)/a}\geq C_1[C_0\cdot \operatorname{Inrad}(D)^a]^{(a-1)/a}=C_2\operatorname{Inrad}(D)^{a-1}.$$

On the other hand, the hypothesis of the proposition implies that for each $y\in Y^{(0)}$ each ball $B(y,R_j)\subset Y^{(1)}$ is surrounded by a connected graph $\Gamma_{R_j}\subset Y^{(1)}$ of the size $\leq CR$ ($R\in\mathbb{N}$). Let D_R denote the vertex set of the component of y in $Y^{(1)}\setminus\Gamma_R$. Then

$$|\partial D_{R_j}|\leq C_3|D_{R_j}|\leq C_4R_j\leq C_4\operatorname{Inrad}(D_{R_j})$$

for all $j\in\mathbb{N}$. Thus (6.6) cannot hold in $Y^{(0)}$ for any $a>2$. Thus for each $a>2$ there are sequences $y_j\in Y^{(0)}, r_j\in\mathbb{N}$ so that

$$V(y_j,r_j)\leq C_0 r_j^a.$$

Hence, by an improvement of Gromov's theorem on groups of polynomial growth due to Van den Dries and Wilkie [40], the group Q is virtually nilpotent and moreover its growth $\leq a$ for each $a>2$. Thus Q has at most quadratic growth. $\square$

We now strengthen our assumptions on Y. We assume in addition that $Y^{(2)}$ has trivial 1-dimensional homology, and satisfies 2-dimensional coarse Poincare duality in dimension 1, i.e. the statement of Definition 5.2 holds for $n=2$ and $k=1=n-k=m-1$.

Lemma 6.7. *There are constants D and R_1 such that when $R\geq R_1$, the D-neighborhood of any R-fat geodesic triangle Δ in $Y^{(1)}$ will surround a ball of radius $\frac{R}{10}$.*

Proof. Let $\Delta \subset Y^{(1)}$ be an R-fat geodesic triangle, and let x be a point on one of the sides γ of Δ whose distance to the remaining two sides exceeds R. Let $\alpha \in C^1(\gamma)$ be a 1-cocycle supported in $B^{(1)}(x,1)\cap\gamma$, representing the fundamental class of the side γ relative to its boundary. Construct a map $f : Y^{(2)} \to \gamma$ by letting $f\big|_{Y^{(0)}} \to \gamma$ be a nearest point map, and extending f to $Y^{(2)}$ using the uniform contractibility of γ. For all $y \in Y^{(2)}$ we will have

$$d(p(f(y)), p(y)) \leq d(p(y), \gamma^{(0)}) + C_1 \tag{6.8}$$

where $p : Y \to Y^{(0)}$ is the control map, and C_1 is independent of γ. Using $f\big|_{N^{(2)}_{\frac{R}{3}}(\gamma)}$, pullback α to a 1-cocycle

$$\hat{\alpha} := \left(f\big|_{N^{(2)}_{\frac{R}{3}}(\gamma)} \right)^* (\alpha) \in Z^1(N^{(2)}_{\frac{R}{3}}(\gamma)).$$

Extending $\hat{\alpha}$ by zero, we get a cochain α' in $C^1(N^{(2)}_{\frac{R}{3}}(\Delta))$.

Recall that there is a constant C_2 depending only on $Y^{(2)}$ such that if σ is a 2-cell of $Y^{(2)}$ and τ is a 1-cell appearing in the boundary of σ, then $d(p(\tau), p(\sigma)) < C_2$. Using this and (6.8), it follows that if $R \geq C_3$ for $C_3 = C_3(Y^{(2)})$, σ is a 2-cell in $N^{(2)}_{\frac{R}{3}}(\Delta)$, τ is a 1-cell in the boundary of σ, and $\alpha'(\tau) \neq 0$, then the whole boundary of σ lies outside the $\frac{R}{3}$-neighborhoods of the other two sides of Δ; therefore the boundary of σ lies in $N^{(2)}_{\frac{R}{3}}(\gamma)$, which means that σ itself lies in $N^{(2)}_{\frac{R}{3}}(\gamma)$, and so $\alpha'(\partial\sigma) = \hat{\alpha}(\partial\sigma) = 0$. Thus α' is a 1-cocycle when $R \geq C_3$, which we henceforth assume. The restriction of α' to Δ is nontrivial in H^1 because of excision and the nontriviality of α.

Applying our coarse Poincaré duality assumption we get that α' is "dual" to an element

$$c := P_{N^{(2)}_{\frac{R}{3}}(\Delta)}(\alpha') \in H_1(Y^{(2)}, Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-D}(\Delta))$$

which maps nontrivially to $H_1(Y^{(2)}, Y^{(2)} \setminus N^{(2)}_D(\Delta))$ provided $\frac{R}{3} \geq 2D$, where D is the constant in the statement of coarse Poincaré duality. Since Y is 1-acyclic, this means that the 0-chain ∂c is a linear combination $\partial c = \sum a_i y_i$ with nonzero coefficients, where the y_i's lie outside $N^{(2)}_{\frac{R}{3}-D}(\Delta)$, and some pair y_k, y_l of the support points cannot be joined by a curve in $Y^{(2)} \setminus N^{(2)}_D(\Delta)$.

Set $y'_k := p(y_k)$, $y'_l := p(y_l)$. Then $y'_k, y'_l \in Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-r_1}(\Delta)$ for $r_1 = r_1(Y^{(2)})$, provided $\frac{R}{3} \geq r_1$. Also, the points y_k and y'_k (resp. y_l and y'_l) lie in the same component of $Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-r_2}(\Delta)$ for $r_2 = r_2(Y^{(2)}) > r_1$, provided $\frac{R}{3} \geq r_2$. Therefore $N^{(1)}_D(\Delta)$ separates one of the balls $B(y'_k, \frac{R}{10})$, $B(y'_l, \frac{R}{10})$ from infinity when R is sufficiently large. $\square$

We now add the assumption that $\pi_1(Y) \simeq \pi_1(Y^{(2)})$ is trivial.

Proposition 6.9. *There are constants R_2 and C_h such that if* $\mathrm{Sur}(p,r) > C_h r$ *for some $p \in Y$, $r \geq R_2$, then Y is Gromov hyperbolic.*

Proof. We will verify hyperbolicity by invoking a corollary of Gromov's coarse Cartan-Hadamard theorem, see Corollary 2.18. Let $\rho = \rho(Y)$ and c be constants from Corollary 2.18. According to Lemma 6.3,

$$\mathrm{Sur}(y,r) \leq C\,\mathrm{Sur}(y',Cr) \tag{6.10}$$

for all $y' \in Y^{(0)}$ provided $r > r_0/C$, which we will henceforth assume. Pick a number

$$C_h \geq \frac{40mC^2}{c}.$$

We assume that $r \geq \frac{4mC\rho(Y)}{C_h}$. Suppose $\mathrm{Sur}(y,r) \geq C_0 r$ for some $y \in Y^{(0)}$, $r \geq 0$. Then

$$C_h r \leq \mathrm{Sur}(y,r) \leq C\,\mathrm{Sur}(y',Cr) \tag{6.11}$$

for each $y' \in Y^{(0)}$. Consider a geodesic triangle $\Delta \subset Y^{(1)}$ which is $10Cr$-fat. If $r \geq r_1/C$, which we will assume from now on, then Lemma 6.7 implies that $N_D^{(1)}(\Delta)$, the D-neighborhood of Δ in $Y^{(1)}$, surrounds some Cr-ball $B(y',Cr)$, and so

$$\frac{C_h r}{C} \leq \mathrm{Sur}(y',Cr) \leq m\,\mathrm{length}(\Delta)$$

where m is the maximum number of vertices in a $(D+1)$-ball in $Y^{(1)}$. Therefore, any triangle contained in a ball of radius $R = \frac{C_h r}{4mC}$ is $\delta := 10Cr$-thin. According to our choice of C_h we have:

$$\frac{\delta}{R} = \frac{10Cr}{C_h r/(4mC)} = \frac{40mC^2}{C_h} = \frac{40mC^2}{C_h} \leq c.$$

Our assumption that $r \geq \frac{4mC\rho(Y)}{C_h}$ implies that $R \geq \rho(Y)$. Hence (since Y is 1-connected) we are in position to apply Corollary 2.18 and conclude that Y is Gromov-hyperbolic. ☐

Corollary 6.12. *Suppose that Y is a quasi-plane so that $Y^{(0)}$ is quasi-homogeneous. Then either:*

1. *The surrounding function of Y is superlinear, i.e. for all C_h there is a constant R_2 such that* $\mathrm{Sur}(y,R) > C_h R$ *for some $y \in Y^{(0)}$ and all $R \geq R_2$. In this case $Y^{(1)}$ is Gromov-hyperbolic.*

2. *The surrounding function of Y is sublinear, i.e. there is a constant C such that* $\mathrm{Sur}(y,R) < CR$ *for all $y \in Y^{(0)}$, $R \geq 1$. In this case $Y^{(1)}$ is doubling, and has polynomial growth.*

7. Gromov hyperbolic spaces satisfying coarse Poincare duality

In [2], Bestvina proposed a set of axioms for *boundaries* of groups, and showed that a boundary (in his sense) of an n-dimensional Poincare duality group is a (Steenrod) homology $(n-1)$-manifold with the homology of an $(n-1)$-sphere, [2, Theorem 2.8]. The main result of this section, Proposition 7.2, is similar to Bestvina's theorem, but differs in several respects. First, we stick to Gromov hyperbolic spaces, where there is a canonical notion of boundary, and where the boundary comes equipped with more structure than a mere topology – it has a canonical quasisymmetric structure. Second, we do not require a group action; in fact much of [2], including [2, Theorem 2.8] can probably be recast in group-less form. Third, we are able to quantify our topological conclusions about the boundary using the quasisymmetric structure.

We refer the reader to [31] and [20] for the detailed account of the Steenrod homology theory.

Let $\mathcal{R}$ be a commutative ring with unit. For the rest of this section all (co)homology groups will be with coefficients in $\mathcal{R}$. Let X be a cell complex satisfying coarse Poincare duality, Definition 5.2 with coefficients in $\mathcal{R}$, whose 1-skeleton is Gromov hyperbolic. Since coarse Poincare duality is quasi-isometry invariant we can replace X with $\mathrm{Rips}_s(X^{(0)})$ (for sufficiently large s) which has the advantage of being finite-dimensional. We will retain the notation X for this Rips complex, and let δ denote the hyperbolicity constant of $X^{(1)}$. For the remainder of this section we will denote neighborhoods $N_\rho^{(d)}(K)$, for $\rho \in \mathbb{Z}_+$, $d = \dim(X)$, $K \subset X$ a subcomplex, by $N_\rho(K)$. We endow $\partial_\infty X$ with a Gromov-type metric. This means that we can choose a basepoint $\star \in X$ and a constant c, so that whenever γ_1, γ_2 are unit speed geodesic rays starting at $\star$, and $R(\gamma_1, \gamma_2) \in (0, \infty]$ is their "overlap" distance (e.g. the maximum of the numbers r such that $B(\star, r) \cap \mathrm{Im}\,\gamma_1 \subset \overline{N_\delta(\mathrm{Im}\,\gamma_2)})$, then $d(\partial_\infty\gamma_1, \partial_\infty\gamma_2) \sim e^{-cR(\gamma_1,\gamma_2)}$. We remind the reader that metric neighborhoods of subsets in finite-dimensional metric cell complexes X are defined as metric neighborhoods within the top-dimensional skeleton:

$$N_R(S) := N_R^{(m)}(S), m = \dim(X).$$

Recall that a subset $Z \subset X^{(1)}$ is *C-quasi-convex* if every geodesic segment $\gamma \subset X^{(1)}$ with endpoints in Z is contained in $N_C(Z)$.

Remark 7.1. By an argument similar to the one that proves the contractibility Rips complexes of Gromov-hyperbolic spaces, one can show that for every C there is an $R_0 = R_0(C)$ such that every C-quasi-convex subset $Y^{(0)} \subset X^{(0)}$, the inclusion $Y \to N_{R_0}(Y)$ is a null-homotopic map and for all $R \geq R_0$ the complex $\mathrm{Rips}_R(Y^{(0)})$ is uniformly contractible with (linear) contractibility function depending only on C and R.

Given a subset $W \subset X^{(1)} \cup \partial_\infty X$, we define the *hull of* W, $\mathrm{Hull}(W)$, to be the union of the geodesics (segments/rays) with (ideal) endpoints in W. Hulls are 10δ-quasi-convex.

Proposition 7.2 (Linear local (co)acyclicity)**.** *Assume that the ring $\mathcal{R}$ is hereditary, see* [13]*. There is a constant $1 > \lambda > 0$ such that for all $\xi \in \partial_\infty X$ and all $r \leq \mathrm{diam}(\partial_\infty X)$, the inclusions $B(\xi, \lambda r) \to B(\xi, r)$ and $\overline{\partial_\infty X \setminus B(\xi, \lambda r)} \leftarrow \overline{\partial_\infty X \setminus B(\xi, r)}$ induce zero in reduced Steenrod homology. The boundary $\partial_\infty X$ is a linearly locally acyclic (Steenrod) homology $(n-1)$-sphere over $\mathcal{R}$.*

Remark 7.3. A commutative ring $\mathcal{R}$ is called *hereditary* if every ideal $I \subset \mathcal{R}$ is projective as an $\mathcal{R}$-module. Every PID and many other rings are hereditary. An example of a commutative ring which is not hereditary is a free Boolean ring on uncountable set of generators, see [1]. (A Boolean ring is a ring of characteristic 2.)

Following [2], using the universal coefficients theorem, for each C-quasi-convex subset $Y^{(0)} \subset X^{(0)}$ and $R \geq R_0(C)$, we may identify the (reduced) Steenrod homology $\tilde{H}^{st}_*(\partial_\infty Y^{(0)})$ with the locally finite homology $H^{LF}_{*+1}(\mathrm{Rips}_R(Y^{(0)}))$. This identification is compatible with the inclusions $Y^{(0)} \to Y'^{(0)}$ (i.e. the functors $\tilde{H}^{st}_*(\partial_\infty \cdot)$ and $H^{LF}_{*+1}(\mathrm{Rips}_\cdot(\cdot))$ on the category of quasi-convex subsets of $X^{(0)}$, are naturally equivalent).

Lemma 7.4. *For every C, and $R \geq R_0(C)$, there are constants $D_2 = D_2(C)$, $D_3 = D_3(C)$ such that if $Y^{(0)} \subset X$ is C-quasi-convex, then $H^{LF}_*(\mathrm{Rips}_R(Y^{(0)}))$ is canonically isomorphic to*

$$\mathrm{Im}(H^{LF}_*(N_{D_2}(Y)) \to H^{LF}_*(N_{D_3'}(Y)))$$

for any $D_3' \geq D_3$. These isomorphisms are compatible with inclusions, i.e. if $Y^{(0)} \subset Y'^{(0)}$ are both C-quasi-convex, then the isomorphisms are compatible with the induced homomorphisms $H^{LF}_*(\mathrm{Rips}_R(Y^{(0)})) \to H^{LF}_*(\mathrm{Rips}_R(Y'^{(0)}))$ and

$$\mathrm{Im}\left(H^{LF}_*(N_{D_2}(Y)) \to H^{LF}_*(N_{D_3'}(Y))\right) \to$$

$$\to \mathrm{Im}\left(H^{LF}_*(N_{D_2}(Y')) \to H^{LF}_*(N_{D_3'}(Y'))\right).$$

Proof. Using the uniform contractibility and the finite dimensionality of X, we get an L-Lipschitz map $i : \mathrm{Rips}_R(Y^{(0)}) \to X$ extending the inclusion $\mathrm{Rips}_R(Y^{(0)}) \supset Y^{(0)} \cap X^{(0)} \to X$, where $L = L(C, R)$. Similarly, extending a nearest point retraction $X^{(0)} \to Y^{(0)}$, we get a map $r : X \to \mathrm{Rips}_R(Y)$ which for each t restricts to an L'-Lipschitz map on $N_t(Y)$, where $L' = L'(C, R, t)$, and which satisfies $d(i \circ r(x), x) \leq C(1 + d(x, Y^{(0)}))$. Using uniform contractibility again, we find a homotopy from $r \circ i$ to $\mathrm{id}_{\mathrm{Rips}_R(Y^{(0)})}$ with tracks of diameter $\leq D = D(C, R)$ and a homotopy $i \circ r \sim \mathrm{id}_X$ whose track at $x \in X$ has diameter bounded by $\phi(d(x, Y^{(0)}))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ depends only on C and R. The lemma follows. MORE DETAILS?? $\square$

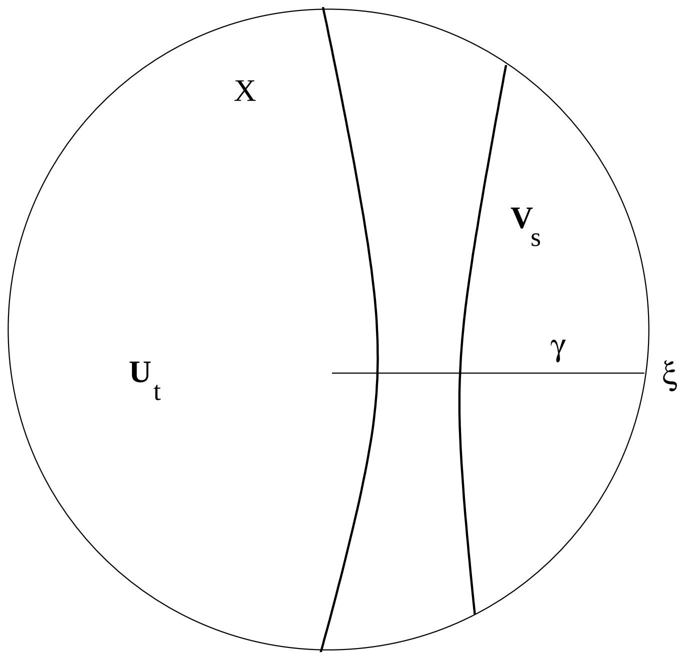

FIGURE 1. *Sets U_t and V_s.*

Suppose $\xi \in \partial_\infty X$, and $\gamma : \mathbb{R}_+ \to X^{(1)}$ is a unit speed geodesic ray asymptotic to ξ. Define $\rho : X \to \mathbb{R}_+$ so that $\gamma \circ \rho : X^{(1)} \to X^{(1)}$ is a nearest point projection to $\mathrm{Im}(\gamma)$.

For $t \in \mathbb{R}_+$ set $\hat{U}_t := \rho^{-1}([t, \infty))$, $\hat{V}_t := \rho^{-1}([0, t])$, $U_t := \overline{\mathrm{Hull}(\hat{U}_t)}$, and $V_t := \overline{\mathrm{Hull}(\hat{V}_t)}$.

Lemma 7.5. *There are constants a, b, c and a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ independent of ξ, such that $\lim_{s\to\infty} L(s) = \infty$ and for all $t, s \in \mathbb{R}_+$:*

1. *$d(U_{t+s}, V_t) > L(s)$.*
2. *$X \setminus V_{t+c} \subset U_t$ and $U_{t+c} \subset X \setminus V_t$.*
3. *If $\gamma(0) = \star$ (the basepoint in $X^{(1)}$ used to define the metric on $\partial_\infty X$), then*

$$B(\xi, \frac{1}{b}e^{-at}) \subset \partial_\infty U_t \subset B(\xi, be^{-at})$$

and

$$X \setminus B(\xi, be^{-at}) \subset \partial_\infty V_t \subset X \setminus B(\xi, \frac{1}{b}e^{-at}).$$

Proof. Pick u and v such that $100\delta \leq u \leq v \leq \infty$. Applying the δ-thinness condition repeatedly, one gets $\mathrm{Hull}(\rho^{-1}([u, v])) \subset \rho^{-1}([u - 100\delta, v + 100\delta])$. This implies 1 and 2. Claim 3 follows directly from the definition of the Gromov type metric on $\partial_\infty X$. $\square$

Proof of Proposition 7.2. In view of Lemma 7.5 (part 3), it suffices to prove that there is a constant c_0 such that for all unit speed rays γ and all $t \in \mathbb{R}_+$, the inclusions $\partial_\infty U_{t+c_0} \to \partial_\infty U_t$ and $\partial_\infty V_t \to \partial_\infty V_{t+c_0}$ induce zero on reduced Steenrod homology. Lemma 7.4 and the remark preceding it, show that we need only prove that when c_0 is sufficiently large, the inclusions $N_{D_3}(U_{t+c_0}) \to N_{D_3}(U_t)$ and $N_{D_3}(V_t) \to N_{D_3}(V_{t+c_0})$ induce zero on locally finite reduced homology. By the universal coefficient theorem applied to the compactly supported cochain complexes, (cf. [2, Remark 1.9], [13]), it suffices to show that when c_0 is sufficiently large these inclusions induce zero on compactly supported cohomology. DO WE WANT TO WRITE OUT THE MAP OF EXACT SEQUENCES, AND APPLY THE 5-LEMMA??

Recall that for $C = 10\delta$, the sets U_t and V_t are C-quasi-convex for all t; pick $D_3' \geq \max(D, D_3(C))$ where D is the constant from the Definition 5.2 and $D_3(C)$ is the function from Lemma 7.4. By Lemma 7.5 (parts 1 and 2), for $t \in \mathbb{R}_+$ and c' sufficiently large, we have the inclusions

$$N_{D_3+2D}(U_{t+c'}) \subset X \setminus (N_{R_0}(V_{t+c})) \subset X \setminus V_{t+c} \subset N_{D_3}(U_t).$$

The inclusion $V_{t+c} \to N_{R_0}(V_{t+c})$ induces zero on reduced homology since V_{t+c} is C-quasi-convex (see Remark 7.1). Therefore the composition of Poincare duality

$$P_{D_3,U_t} : H_c^*(N_{D_3}(U_t))) \to H_{n-*}(X, \overline{X \setminus U_t})$$

with the map

$$H_{n-*}(X, \overline{X \setminus U_t}) \overset{\approx}{\to} \tilde{H}_{n-*-1}(\overline{X \setminus U_t}) \to \tilde{H}_{n-*-1}(X \setminus N_{D_3+D}(U_{t+c'}))$$

is zero. But this composition is the same as the composition of the restriction

$$H_c^*(N_{D_3}(U_t))) \to H_c^*(N_{D_3+2D}(U_{t+c'}))$$

with the duality

$$H_c^*(N_{D_3+2D}(U_{t+c'})) \to H_{n-*}(X, \overline{X \setminus N_{D_3+D}(U_{t+c'})}) \overset{\approx}{\to} \tilde{H}_{n-*-1}(\overline{X \setminus N_{D_3+D}(U_{t+c'})}).$$

By the *approximate isomorphism* part of the Definition 5.2, we have:

$$\mathrm{Ker}\left(H_c^*(N_{D_3+2D}(U_{t+c'})) \to \tilde{H}_{n-*-1}(\overline{X \setminus N_{D_3+D}(U_{t+c'}))}\right)$$
$$\subset \mathrm{Ker}\left(H_c^*(N_{D_3+2D}(U_{t+c'})) \to H_c^*(N_{D_3}(U_{t+c'}))\right),$$

which implies that the composition of the restriction homomorphisms

$$H_c^*(N_{D_3}(U_t)) \to H_c^*(N_{D_3+2D}(U_{t+c'})) \to H_c^*(N_{D_3}(U_{t+c'}))$$

is zero. Thus the inclusions $N_{D_3}(U_{t+c'}) \to N_{D_3}(U_t)$ induce zero on the locally finite reduced homology groups and hence the maps $\tilde{H}_*^{st}(\partial_\infty U_{t+c'}) \to \tilde{H}_*^{st}(\partial_\infty U_t)$ are zero.

The proof that $\tilde{H}_*^{st}(\partial_\infty V_t) \to \tilde{H}_*^{st}(\partial_\infty V_{t+c'})$ is zero is similar. Thus we have proved that $\partial_\infty X$ is linearly locally acyclic (with respect to Steenrod homology).

We note now that $H_*^{st}(\partial_\infty X) \cong H_{*+1}^{LF}(X) \cong H_{*+1}(\mathbb{R}^n) \cong H_*(S^n)$. Hence $\partial_\infty X$ is a homology $(n-1)$-manifold over $\mathcal{R}$ which is also a homology $(n-1)$-sphere over $\mathcal{R}$. This concludes the proof of Proposition 7.2. $\square$

In the proof of Proposition 7.2, the assumption that $\mathcal{R}$ was hereditary was only used when we related locally finite homology with compactly supported cohomology. Hence we get the following variant of the proposition for arbitrary rings $\mathcal{R}$:

Proposition 7.6 (Linear local (co)acyclicity for Čech cohomology)**.** *There is a constant $1 > \lambda > 0$ such that for all $\xi \in \partial_\infty X$ and all $r \le \operatorname{diam}(\partial_\infty X)$, the inclusions $B(\xi, \lambda r) \to B(\xi, r)$ and $\overline{\partial_\infty X \setminus B(\xi, \lambda r)} \leftarrow \overline{\partial_\infty X \setminus B(\xi, r)}$ induce zero in reduced Čech cohomology. The boundary $\partial_\infty X$ has the same Čech cohomology over $\mathcal{R}$ as the $(n-1)$-sphere.*

Proof. Following [3], when $Y^{(0)} \subset X^{(0)}$ is C-quasi-convex and $R \ge R_0(C)$, we may identify the reduced Čech cohomology $\tilde{H}_c^*(\partial_\infty Y^{(0)})$ with the compactly supported cohomology $H_c^{*+1}(\mathrm{Rips}_R(Y^{(0)}))$. (This follows from tautness of Čech cohomology for the subset $\partial_\infty Y^{(0)} \subset \overline{\mathrm{Rips}_R(Y^{(0)})}$, and excision.) This identification is compatible with the inclusions $Y^{(0)} \to Y'^{(0)}$.

With this identification, we can repeat the argument of Proposition 7.2, replacing locally finite homology with compactly supported cohomology everywhere. $\square$

Corollary 7.7. *(cf. [6]) 1. For any ring $\mathcal{R}$, for each quasi-plane X over $\mathcal{R}$ with Gromov hyperbolic 1-skeleton, the graph $X^{(1)}$ is quasi-isometric to $\mathbb{H}^2$.*

2. If X is a cell complex satisfying 3-dimensional coarse Poincare duality over a hereditary ring $\mathcal{R}$, and $X^{(1)}$ is Gromov hyperbolic, then the boundary $\partial_\infty X$ is a linearly locally contractible 2-sphere.

Proof. 1. We first show that $\partial_\infty X$ is homeomorphic to S^1. We will do this by proving that $\partial_\infty X$ is connected, locally connected, has no cut points, and is separated by any pair of distinct points, see [42]. Note that $\partial_\infty X$ is connected since $\check{H}^0(\partial_\infty X, \mathcal{R}) \simeq H^0(S^1, \mathcal{R})$. Likewise, linear local connectedness follows from Proposition 7.6. In fact, the "co-acyclicity" implies that there are no cut points. (When X is quasi-isometric to a group, the fact that $\partial_\infty X$ is locally connected and has no cut points follows from [8, 35]; however their argument doesn't apply when there is nondiscrete cocompact quasi-action.) It remains to show that every pair of points separates.

First consider the case when $\mathcal{R}$ is hereditary. Then $\partial_\infty X \setminus \{p\}$ has trivial Steenrod homology for any $p \in \partial_\infty X$. Hence the Mayer-Vietoris sequence shows that each pair of points separates $\partial_\infty X$ into precisely two components. In the general case, let p, $p' \in \partial_\infty X$ be distinct points. By Proposition 7.6, we can exhaust $\partial_\infty X \setminus \{p\}$ (resp. $\partial_\infty X \setminus \{p'\}$ by a nested sequence of open sets $\{U_i\}$ (resp. $\{U_i'\}$) such that the inclusions $U_i \to U_{i+1}$ (resp. $U_i' \to U_{i+1}'$) induce zero on reduced Čech cohomology. We for large i, we obtain a sequence of coverings (U_i, U_i') of $\partial_\infty X$. The nested family

$\{U_i \cap U_i'\}$ of intersections exhausts $\partial_\infty X \setminus \{p, p'\}$. Applying Mayer-Vietoris, we obtain a compatible system of monomorphisms $\mathcal{R} \simeq \check{H}^1(\partial_\infty X) \to \widetilde{\check{H}^0}(U_i \cap U_i', \mathcal{R})$. Since $\cup_i U_i \cap U_i' = \partial_\infty X \setminus \{p, p'\}$ can be exhausted by compact sets, this implies that $\partial_\infty X \setminus \{p, p'\}$ is not connected.

Since $\partial_\infty X$ is linearly locally acyclic with respect to Čech cohomology over $\mathcal{R}$, $\partial_\infty X = \partial_\infty X^{(1)}$ is a topological circle which is linearly locally connected. Thus $\partial_\infty X$ is quasisymmetrically homeomorphic to the standard circle, see [39]. Then [33] implies that there is a quasiconvex subset $Z \subset X^{(0)} \subset X$ which is quasi-isometric to $\mathbb{H}^2$. It then follows from Proposition 5.5 that Z^0 is a net in $X^{(0)}$. This proves (1).

To prove 2, note that by Proposition 7.2, over each hereditary ring $\mathcal{R}$, the compact $\partial_\infty X$ is a homology 2-manifold with the homology of a 2-sphere (with respect to Steenrod homology). Thus, $\partial_\infty X$ is connected, locally connected and has no global cut-points. In particular, $\partial_\infty X$ is not a dendroid and therefore contains an embedded topological circle. To conclude that $\partial_\infty X$ is homeomorphic to $\mathcal{S}^2$ it remains to show that each embedded topological circle $S \subset \partial_\infty X$, separates $\partial_\infty X$, see [42], i.e. $\partial_\infty X$ satisies Jordan curve separation theorem. To prove the latter we repeat the proof of Jordan separation theorem given in [29, Ch. III, Corollary 6.4], using Steenrod homology with $\mathcal{R}$-coefficients instead of the singular homology with the integer coefficients. The inductive proof of [29, Ch. III, Corollary 6.4] is based on vanishing of $\tilde{H}_*(\partial_\infty X \setminus p)$ for each point $p \in \partial_\infty X$. In our case this property is satisfied since $\partial_\infty X$ is a Steenrod homology sphere over $\mathcal{R}$. ☐

Remark 7.8. In the case when $X^{(1)}$ is a planar graph, the part 1 of the above corollary was proven in [10].

Suppose now that X is a Gromov-hyperbolic quasi-plane. We recall that for a topological space Σ, $Trip(\Sigma)$ denotes the set of ordered triples $(x, y, z) \in \Sigma^2$ where all three coordinates are distinct. Let $G \curvearrowright \Sigma$ be a topological action and $\bar{G}$ be the closure of G in $Homeo(\Sigma)$ in the compact-open topology. We get the induced action $\bar{G} \curvearrowright Trip(\Sigma)$. Recall that the action $G \curvearrowright \Sigma$ is called a *convergence action* if the action $\bar{G} \curvearrowright Trip(\Sigma)$ is proper. The action $G \curvearrowright \Sigma$ is called *uniform* if then quotient $Trip(\Sigma)/G$ is compact.

Proposition 7.9. *Every cobounded quasi-action $G \curvearrowright \mathbb{H}^2$ is quasi-isometrically conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.*

Proof. Let $K \subset G$ be kernel of the quasi-action $G \curvearrowright \mathbb{H}^2$ and $G' := G/K$. By extending the quasi-action $G \curvearrowright \mathbb{H}^2$ to the circle at infinity we get an action by uniform quasi-similarities $G \curvearrowright S^1$. The kernel of this action of K and hence we get the action $G' \overset{\rho}{\curvearrowright} \mathcal{S}^1$ which is a *uniform convergence group action.*

It follows from the work of Tukia [37], Gabai [21], Casson and Jungreis [14] (see [38] for a survey of this work) that the action $G' \overset{\rho}{\curvearrowright} S^1$ is *topologically conjugate* to a Moebius action $G' \overset{\phi}{\curvearrowright} S^1$. (See [24, 27] for the discussion of the nondiscrete case.) Since the action ρ is uniform, it follows that the homeomorphism $h : S^1 \to S^1$ conjugating ρ and ϕ is itself a quasi-similarity. Finally, one extends the homeomorphism h to a quasi-isometry $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^2$ as follows: For each point $x \in \mathbb{H}^2$ pick an ideal triangle $T_x \subset \mathbb{H}^2$ so that x is the *center* of T_x, i.e., x is equidistant from the three sides of T_x. Let $a, b, c \in S^1$ be the ideal vertices of T_x. Let $y \in \mathbb{H}^2$ be the center of the ideal triangle with the vertices $h(a), h(b), h(c)$. Then set $\tilde{h}(x) = y$. We refer to [33] for more details. □

8. Coarse fibrations by lines

In this section $G \curvearrowright Z$ will be a cobounded quasi-action on the 0-skeleton Z of a metric cell complex X, which preserves a parameterized coarse fibration $\mathcal{F} = \{(F_i, f_i)\}_{i \in \mathcal{J}}$ of Z (see section 2). In this section we assume that the above coarse fibration is *by lines*, i.e. each F_i is isometric to the real line $\mathbb{R}$ with the usual metric and the cell complex structure given by the triangulation of $\mathbb{R}$ with vertices at the integer points. Let Y be a metric cell complex which is a *coarse base* of the fibration $\mathcal{F}$ (see section 3 for the definition).

For convenience, we will further assume that Z is the disjoint union of the fibers $\bar{F}_i$, $i \in \mathcal{J}$ and that for each $g \in G$, $i \in \mathcal{J}$, there is an $i' \in \mathcal{J}$ such that $g(\bar{F}_i) \subset \bar{F}_{i'}$. We will sometimes denote $F_{i'}$ by $g_* F_i$.

We recall (Lemma 3.9) that for each fiber $(F, f) \in \mathcal{F}$, we have an induced degenerating quasi-action $G \overset{\psi}{\curvearrowright} F$. We denote the image of an element $x \in F$ under $g \in G$ by $\psi_{F_1}(g)$ or $\psi(g)(x)$.

8.1. **The fiber expansion function.**

One interpretation of the curvature of a curve sitting in a Riemannian manifold is that it determines the rate of change length under normal variation. There is a coarse analog of curvature for the fibers of our parameterized coarse fibration $\mathcal{F}$ on a metric cell complex X. Consider a pair of fibers $\bar{F}_1, \bar{F}_2$. Given an interval $I \subset F_1$, we can measure the length of its image in F_2 under the mapping $v_{F_2} \circ f_1$ where $v_{F_2} : Z \to F_2$ has the property that the composition $f_2 \circ v_{F_2}$ is a nearest-point projection $Z \to \bar{F}_2$, see Lemma 3.6. By taking into account the action of the group, we find that the stretch factor of I under the mapping $v_{F_2} \circ f_1$ is independent of the choice of the interval up to a uniform multiplicative error, provided that I is sufficiently long.

Remark 8.1. This basic construction works more generally for coarse fibrations with uniformly doubling fibers. However instead of the "lengthwise" distortion discussed in this paper one would consider the distortion of volume for domains in the fibers which are quasi-balls with small isopetrimetric ratio. The analogs of Lemma 8.6, Definition 8.11, Corollaries 8.12, 8.13 hold in this context.

We begin by observing that there are many group elements which "almost preserve" any given pair of fibers:

Lemma 8.2. *Let F_1, F_2 be fibers, let $R := d(\bar{F}_1, \bar{F}_2)$, $R \geq 1$. There is a positive constant D (depending on the geometry of the quasi-action $G \curvearrowright Z$ and the coarse fibration $\mathcal{F}$) with the following property. Pick $x \in F_1$, let*

$$\mathcal{G} = \{g \in G \mid d(g(\bar{F}_i), \bar{F}_i) < D \textit{ for } i = 1, 2\},$$

and set $\mathcal{G}(x) := \{\psi g(x) \mid g \in \mathcal{G}\} \subset F_1$. Then $\mathcal{G}(x)$ is a net in F_1. If moreover the quasi-action $G \curvearrowright Z$ is discrete, then there are constants L, C (depending only on $G \curvearrowright Z$ and $\mathcal{F}$) such that if m is the number of fibers contained in $N_{LR}(\bar{F}_1)$, then the density of $\mathcal{G}(x)$ in F_1 is at least $\frac{C}{m}$, in the sense that if $I \subset F_1$ is an interval, then the quantity

$$\frac{|\mathcal{G}(x) \cap I|}{|I|} \tag{8.3}$$

is at least $\frac{C}{m}$ when I is sufficiently long.

Proof. The quasi-action $G \curvearrowright X$ is cobounded, so there is a D_0 such that for any x, $x' \in X$, there is a $g \in G$ such that $d(g(x), x') < D_0$. It follows that if

$$\mathcal{G}_0 := \{g \in G \mid d(g(\bar{F}_1), \bar{F}_1) < D_0\},$$

then for any $x \in F_1$, the set

$$\mathcal{G}_0 := \{\psi(g(x)) \mid g \in \mathcal{G}_0\}$$

is an r_1-net in F_1 where r_1 is independent of the fibers F_1, F_2 and the point $x \in F_1$.

For each $g \in \mathcal{G}_0$, the image of $\bar{F}_2$ under g^{-1} is contained in a fiber $\bar{F}_{i(g)}$ for some $i(g) \in I$, and $\bar{F}_{i(g)}$ lies in $N_{LR}(\bar{F}_1)$, where L depends only on the constants of the quasi-action $G \curvearrowright Z$. Let $I_0 \subset I$ denote the collection of $i(g) \in I$ as g ranges over $\mathcal{G}_0$, and form a set $\Sigma \subset G$ by choosing, for each $i \in I_0$, an $h \in G$ such that $h^{-1} \in \mathcal{G}_0$ and $h(\bar{F}_2) \subset \bar{F}_i$. Notice that for each pair $g \in \mathcal{G}_0$, $\sigma \in \Sigma$ such that $\sigma(\bar{F}_2)$ and $g^{-1}(\bar{F}_2)$ are contained in the same fiber, we have

$$d_H(g\sigma(\bar{F}_i), \bar{F}_i) < D \quad \text{for } i = 1, 2,$$

where D is independent of F_1, F_2, and x. Let

$$\mathcal{G} := \{g \in G \mid d_H(g(\bar{F}_i), \bar{F}_i) < D \text{ for } i = 1, 2\}.$$

Pick $x \in F_1$. Define a map $\pi : \mathcal{G}_0(x) \to \mathcal{G}(x)$ as follows. For each $y \in \mathcal{G}_0(x)$, pick $g_y \in \mathcal{G}_0$ such that $\psi(g_y)(x) = y$, choose $\sigma_y \in \Sigma$ such that $\sigma_y(\bar{F}_2)$ and $g^{-1}(\bar{F}_2)$ are contained in the same fiber, and set $\pi(y) := \psi(g\sigma_y)(x)$. The map π has bounded displacement, which implies that $\mathcal{G}(x)$ is a net in F_1. If $G \curvearrowright Z$ is discrete, then for each $x_1 \in \mathcal{G}(x)$, $\sigma \in \Sigma$, the set

$$\{y \in \mathcal{G}_0(x) \mid \pi(y) = x_1, \sigma_y = \sigma\}$$

has uniformly bounded cardinality. Hence π has multiplicity $< M|\Sigma| \leq Mm$ where M is independent of F_1, F_2, and x, and m is the number of fibers contained in $N_{RL}(\bar{F}_1)$. Since $\mathcal{G}_0(x)$ is an r_1-net in $\bar{F}_1$, the density estimate (8.3) follows.

□

Consider a pair of fibers (F_i, f_i), and choose a map $\rho : F_1 \to F_2$ which has "bounded displacement",

$$\delta := d(f_1, \rho \circ f_2) < \infty. \tag{8.4}$$

Definition 8.5. Given an interval $I = [a, b] \subset F_1$, we define the *expansion of I under ρ* to be the quantity $\frac{d(\rho(a),\rho(b))}{d(a,b)}$. We denote this quantity by $E(I, \rho)$.

Note that if $\rho' : F_1 \to F_2$ is another bounded displacement map, then the symmetric difference $[\rho(a), \rho(b)]\Delta[\rho'(a), \rho'(b)]$ has cardinality bounded independent of a, b, and hence the ratio of the quantities $E(I, \rho)$ and $E(I, \rho')$ tends to 1 as $|I|$ tends to infinity.

Let $(F_1, f_1), (F_2, f_2)$ be a pair of fibers, and let ρ be a map satisfying (8.4).

Lemma 8.6. *There is a constant C which is independent of the choice of fibers and the map ρ, such that when I and I' are sufficiently long intervals, we have*

$$\frac{1}{C}E(I, \rho) \leq E(I', \rho) \leq CE(I, \rho). \tag{8.7}$$

Proof. The idea is to use "translates" of an interval to get a controlled multiplicity covering of a longer interval; this allows one to compare expansion factors.

Consider the collection $\mathcal{G}$ of elements of G determined by the pair of fibers F_1, F_2 as in Lemma 8.2. Each $g \in \mathcal{G}$ determines a quasi-isometry $\psi_{F_i}(g) : F_i \to F_i$, $i = 1, 2$, and the quasi-isometry constants of these maps are bounded independent of g and the choice of fibers F_1, F_2. Note that

$$d(\rho \circ \psi_{F_1}(g), \psi_{F_2}(g) \circ \rho) \leq C_1 \tag{8.8}$$

where C_1 is independent of $g \in \mathcal{G}$.

Let $\mathcal{C}$ be the collection of intervals of the form $[\psi_{F_1}(g)(a), \psi_{F_1}(g)(b)] \subset F_1$, where g ranges over $\mathcal{G}$ and $I = [a, b]$. The collection of left endpoints of the intervals in $\mathcal{C}$ is a net in F_1 by Lemma 8.2. For each interval $\hat{I} = [\hat{a}, \hat{b}] \in \mathcal{C}$, we have

$$(8.9) \qquad |\hat{I}| \simeq |I| \quad \text{and} \quad |\rho(\hat{b}) - \rho(\hat{a})| \simeq |\rho(b) - \rho(a)| = E(I,\rho)|I|$$

provided $|I|$ is sufficiently big, since the family $\{\psi_{F_1}(g) : g \in \mathcal{G}\}$ has quasi-isometry constant bounded independent of F_1, F_2, and ρ is a quasi-isometry. Again using the fact that ρ is a quasi-isometry, we can find a constant C_2 such that if $\{x_1, x_2, x_3\} \subset F_1$ and $d(x_i, x_j) > C_2$ for $i \neq j$, then x_2 lies between x_1 and x_3 if and only if $\rho(x_2)$ lies between $\rho(x_1)$ and $\rho(x_3)$. When I is sufficiently long, we may choose a subcollection $\mathcal{C}_1 := \{[a_j, b_j]\}_{j\in\mathbb{Z}}$ of $\mathcal{C}$ (still covering $\mathbb{Z}$), such that for all $j \in \mathbb{Z}$,

$$(8.10) \qquad a_i + C_2 < b_{i-1} < a_{i+1} - C_2 \quad \text{and} \quad b_{i-1} + C_2 < a_{i+1} < b_i - C_2.$$

Pick an interval $J = [c, d] \subset F_1$. Let $j_1 := \max\{j \in \mathbb{Z} \mid a_j + C_2 < c\}$ and $j_2 := \min\{j \in \mathbb{Z} \mid b_j - C_2 > d\}$, and set $\mathcal{C}_J := \{[a_j, b_j]\}_{j=j_1}^{j_2}$. Then $\mathcal{C}_J$ covers J with multiplicity ≤ 2, and

$$\sum_{j=j_1}^{j_2} |b_j - a_j|$$

is comparable to $|J|$ when $|J|$ is large. Note that the collection

$$\rho_*(\mathcal{C}_J) := \{[\rho(a_j), \rho(b_j)] \mid [a_j, b_j] \in \mathcal{C}_J\}$$

covers $[\rho(c), \rho(d)]$ with multiplicity ≤ 2, by (8.10). Also,

$$\sum_{j=j_1}^{j_2} \rho(b_j) - \rho(a_j) \simeq E(I,\rho)|J|$$

is comparable to $\rho(d) - \rho(c)$ when $|J|$ is large. It follows that $E(I, \rho)$ is comparable to $E(J, \rho)$. Applying the same reasoning to I' yields that $E(I', \rho)$ is comparable to $E(J, \rho)$, hence (8.8) follows. □

Definition 8.11. Consider an ordered pair of fibers $((F_1, f_1), (F_2, f_2))$, which we will conflate with (F_1, F_2). The *fiber expansion factor* for (F_1, F_2) is defined to be

$$E(F_1, F_2) := \limsup_{|I|\to\infty} E(I, \rho)$$

where $\rho : F_1 \to F_2$ is any bounded displacement map. The remark after Definition 8.5 implies that this is independent of ρ.

Corollary 8.12. *1. Given three fibers F_1, F_2, F_3, we have*

$$\frac{1}{C^2} E(F_1, F_2)E(F_2, F_3) \leq E(F_1, F_3) \leq E(F_1, F_2)E(F_2, F_3).$$

2. There is a constant C' such that if $(F_1, f_1), (F_2, f_2) \in \mathcal{F}$, $g \in G$, then

$$\frac{1}{C'} E(F_1, F_2) \leq E(g_* F_1, g_* F_2) \leq C' E(F_1, F_2).$$

Corollary 8.13. *Pick a fiber F_0; for elements $g \in G$ consider the function*

$$l(g) = \log E(F_0, g_* F_0). \tag{8.14}$$

Then $l : G \to \mathbb{R}$ is a quasi-homomorphism.

8.2. **Coarse fibrations with amenable base.**

Recall that a bounded geometry metric space S is called *amenable* if it admits a c-Folner sequence for each c.

Proposition 8.15. *If the fiber expansion function of $\mathcal{F}$ is bounded and the base $Y^{(0)}$ is amenable, then Z is amenable as well.*

Proof. Let $p : Z \to Y^{(0)}$ be the projection (recall that Z is the disjoint union of the fibers $\bar{F} \in \bar{\mathcal{F}}$). Let $D_k \subset Y^{(0)}$ be a sequence with $\frac{|\partial_c D_k|}{|D_k|} \to 0$. (Recall that $\partial_c D_k$ consists of points in $Y^{(0)} \setminus D_k$ which are within distance $\leq c$ from D_k.)

Pick $(F, f) \in \mathcal{F}$. Choose $v_F : Z \to F$ so that $f \circ v_F : Z \to Z$ is a nearest point projection to $\bar{F}$. For each T, let $I_T \subset F$ be an interval of length T . For fixed k, consider the family of intersections $E_{k,T} := p^{-1}(D_k) \cap v_F^{-1}(I_T)$, where $T > 0$. Note that

$$\limsup_{T \to \infty} \frac{|\partial_c E_{k,T}|}{|E_{k.T}|} \leq C \frac{|\partial_c D_k|}{|D_k|},$$

where C is independent of k; so we can obtain a Folner sequence of the form E_{k,T_k} for an appropriately chosen sequence T_k. $\square$

Remark 8.16. It is tempting to try to prove that X has polynomial growth provided Y has polynomial growth or is doubling. However, it seems that to prove this directly one would need stronger control over the nearest point maps between fibers.

8.3. **Coarse fibrations with subexponentially growing base.**

Consider the foliation of $\mathbb{H}^2$ by horocycles asymptotic to $\xi \in \partial_\infty \mathbb{H}^2$. This fibration is invariant under the stabilizer of ξ in $PSL(2, \mathbb{R})$, which acts transitively on $\mathbb{H}^2$. If we discretize this example we get a coarse fibration of $\mathbb{H}^2$ by copies of $\mathbb{Z}$ whose base is quasi-isometric to $\mathbb{Z}$, with unbounded fiber expansion function. Examples like this are incompatible with a *discrete* quasi-action, however.

Proposition 8.17. *Assume that $G \curvearrowright Z$ is a discrete quasi-action, and $Y^{(0)}$ has subexponential growth. Then the fiber expansion function $E(\cdot, \cdot)$ of the coarse fibration $\mathcal{F}$ is bounded.*

Proof. By Corollary 8.13, the function l defined in (8.14) is a quasi-homomorphism. So unless it remains bounded, for suitable $g \in G$, $\alpha_0 > 0$, we will have

$$l(g^k) \leq -\alpha_0 k$$

for all $k > 0$. So there is an $\alpha > 0$ such that for all R_0, there are fibers F_1, $F_2 \in \mathcal{F}$ such that $d(\bar{F}_1, \bar{F}_2) = R \geq R_0$ and

$$E(F_1, F_2) \leq e^{-\alpha R}. \tag{8.18}$$

Let $\rho : F_1 \to F_2$ be map sfor which there is a bounded displacement map $\bar{\rho} : \bar{F}_1 \to \bar{F}_2$ such that $\bar{\rho} \circ f_1 = f_2 \circ \rho$.

Pick $x \in F_1$. Let $\mathcal{G}$, L be as in Lemma 8.2. The discreteness of $G \curvearrowright Z$ implies that the map $\mathcal{G} \to \mathcal{G}(x)$ (respectively $\mathcal{G} \to \mathcal{G}(\rho(x))$) given by $g \mapsto \psi_{F_1}(g)(x)$ (respectively $g \mapsto \psi_{F_2}(g)(\rho(x))$) has multiplicity bounded independent of F_1, F_2. Therefore, when $I = [a, b] \subset F_1$ is a long interval we have

$$|\mathcal{G}(x) \cap [a, b]| \simeq |\mathcal{G}(\rho(x)) \cap [\rho(a), \rho(b)]|. \tag{8.19}$$

By Lemma 8.2, when $I = [a, b]$ is long, we have

$$|\mathcal{G}(x) \cap [a, b]| \gtrsim \frac{(b-a)}{m} \tag{8.20}$$

where m is the number of fibers contained in $N_{LR}(\bar{F}_1)$. If we choose F_1, F_2 such that (8.18) holds, for long intervals $I = [a, b] \subset F_1$ we have

$$C_1|\mathcal{G}(\rho x) \cap [\rho(a), \rho(b)]| \leq \rho(b) - \rho(a) \leq 2(b-a)e^{-\alpha R} \leq C_2 m e^{-\alpha R}|\mathcal{G}(x) \cap [a, b]|. \tag{8.21}$$

When R is large, $m \ll e^{\alpha R}$ (recall that $Y^{(0)}$ has subexponential growth), so (8.21) contradicts the fact that the cardinalities $|\mathcal{G}(\rho x) \cap [\rho(a), \rho(b)]|$ and $|\mathcal{G}(x) \cap [a, b]|$ are comparable. ∎

By combining Propositions 8.15, 8.17 and Lemma 2.4 we obtain:

Corollary 8.22. *If $Y^{(0)}$ has polynomial growth and $G \curvearrowright Z$ is discrete then the fiber expansion function E is bounded and Z is amenable.*

8.4. **The homomorphism from G to $\mathrm{QI}(F)$.** Recall that for each $(F, f) \in \mathcal{F}$, we have a degenerating quasi-action $G \overset{\psi}{\curvearrowright} F$. For each $g \in G$, we obtain the map $\psi(g) : F \to F$ by post-composing the composition $F \xrightarrow{f} Z \xrightarrow{g} Z$ with $v_F : Z \to F$, see Lemma 3.9. Let $(g_*F, f_g) \in \mathcal{F}$ denote the fiber for which $\overline{g_*F} \supset g(\bar{F})$. Then there is a quasi-isometry $h : F \to g_*F$, with constants independent of F and g, such that $d(g \circ f, f_g \circ h) < C$ where C is independent of F and g (Lemma 3.8). Hence $\psi(g) := v_F \circ g \circ f$ is at bounded distance from the composition $v_F \circ f_g \circ h$. The composition $\rho := v_F \circ f_g : g_*F \to F$ satisfies (8.4). This means that when $x, y \in F$ are far apart, then

$$d(\psi(g)(x), \psi(g)(y)) \simeq d((v_F \circ f_g) \circ h(x), (v_F \circ f_g) \circ h(y))$$
$$\simeq E(g_*F, F) d(h(x), h(y)) \simeq E(g_*F, F) d(x, y).$$

Or, to put it another way, for every $g \in G$, there are constants L and A, where L is independent of g and F, so that

$$L^{-1}E(gF, F)d(x, y) - A \leq d(\psi(g)(x), \psi(g)(y)) \leq LE(gF, F)d(x, y) + A \tag{8.23}$$

for points $x, y \in F$ which are sufficiently far apart. So, modulo rescaling the metric on the target by the factor $E(g_*F, F)$, the map $\psi(g)$ defines a quasi-isometry with uniformly controlled Lipschitz constant (but uncontrolled additive constant). We recall that if ω is a nonprincipal ultrafilter on $\mathbb{N}$, $\phi : W \to W'$ is an (L, A) quasi-isometry between metric spaces, $w_k \in W$, $w'_k \in W'$, and $\lambda_k \in \mathbb{R}_+$ are sequences with $\lambda_k \to \infty$ where $\omega\text{-lim}_k \frac{1}{\lambda_k} d(\phi(w_k), w'_k) < \infty$ then ϕ induces an L-biLipschitz homeomorphism ϕ_ω between the asymptotic cones $\omega\text{-lim}(\frac{1}{\lambda_k} W, w_k)$ and $\omega\text{-lim}(\frac{1}{\lambda_k} W', w'_k)$. Hence if for some sequences $x_k \in F$ and $\lambda_k \in \mathbb{R}_+$ we have $\omega\text{-lim}_k \frac{1}{\lambda_k} d(\psi(g)(x_k), x_k) < \infty$ for each element g of a generating set for G, then we get an induced homomorphism ψ_ω from G to the group of biLipschitz homeomorphisms of $F_\omega \equiv \mathbb{R}$. Equation 8.23 implies that for such a homomorphism ψ_ω, and each $g \in G$, we have

$$L^{-1}E(gF, F)d(x, y) \leq d(\psi_\omega(g)(x), \psi_\omega(g)(y)) \leq LE(gF, F)d(x, y), \tag{8.24}$$

which means that ψ_ω is an action by homeomorphisms which all uniformly biLipschitz modulo rescaling of the target metric (by a factor which depends on the group element).

8.5. **The case when the homomorphism $G \to \mathrm{QI}(F)$ is trivial.** Assume the quasi-action $G \curvearrowright Z$ is discrete, and pick a fiber $F \in \mathcal{F}$.

Lemma 8.25. *Suppose the homomorphism $G \to \mathrm{QI}(F)$ induced by the quasi-action $G \curvearrowright F$ is trivial, or equivalently, suppose $d(\psi(g), \mathrm{id}_F) < \infty$ for every $g \in G$. Then for any $x \in \bar{F}$, there is an r such that $N_r(Z_G(x)) \supset \bar{F}$, where Z_G is the center of G. In particular Z_G is infinite.*

Proof. Let $g_1, ..., g_n$ be generators of G. For each $i = 1, ..., n$ the quasi-isometry $\psi(g_i) : F \to F$ has bounded displacement.

Let $H \subset G$ denote a collection of elements such that for some $x \in \bar{F}$, the sets $H(x)$ and $\bar{F}$ are at finite Hausdorff distance. Consider the set of commutators

$$K := \{[g_1, h], h \in H\}.$$

There is a constant $C_1 < \infty$ so that for each $k \in K$ moves $d(k(x), x) < C_1$. Hence there is an element $h_1 \in H$ and a subset $J_1 \subset H$ so that

1. $[g_1, h_1] = [g_1, h]$, for each $h \in J_1$.
2. The sets $J_1(x)$ and $\bar{F}$ are Hausdorff-close.

Set $H_1 := h_1^{-1} J_1$. The first property implies that $[g_1, h_1^{-1} h] = 1$ for each $h \in J_1$; the second property implies that $d_H(H_1(x), \bar{F}) < \infty$.

We now repeat the above argument using H_1 instead of H and proceed inductively. After n steps we get a subset $H_n \subset G$ so that:

3. $[g_i, h] = 1$, for each $h \in H_n$, $i = 1, ..., n$.
4. $d_H(H_n(x), \bar{F}) < \infty$.

Hence the center Z_G of the group G contains the infinite subset H_n and the assertion of the proposition follows. □

8.6. **Bounded expansion functions.** Suppose the expansion function $E(\cdot, \cdot)$ is bounded.

Lemma 8.26. *If the homomorphism $G \to \mathrm{QI}(F)$ is nontrivial, $F \in \mathcal{F}$, then there is a nontrivial isometric action $G \curvearrowright \mathbb{R}$. If moreover the image of $G \to \mathrm{QI}(F)$ consists of orientation preserving quasi-isometries, then there is a nontrivial homomorphism $G \to \mathbb{R}$.*

Let $\{g_1, \dots, g_n\} \subset G$ be a generating set. By assumption, the quasi-isometry $\psi(g_i) : F \to F$ has unbounded displacement for some $1 \le i \le n$. In this case there is a sequence of points $x_j \in F$ such that

$$\limsup_{j \to \infty} \max(d(x_j, \psi(g_i)(x_j)), i = 1, ..., n) = \infty.$$

Let ω be a nonprincipal ultrafilter. Then (after renumbering the generators g_i and passing to a subsequence of (x_j) if necessary) we may assume: $\max(d(x_j \psi(g_i)(x_j)) = d(x_j, \psi(g_1)(x_j)) =: \lambda_j$ for all j. Thus

$$\lim_{j \to \infty} \lambda_j = \infty.$$

We now consider the ultralimit

$$\omega\text{-}\lim_j (F, \frac{1}{\lambda_j} d_F, x_j) = F_\omega.$$

Since the fiber expansion function is bounded, the quasi-action $\psi : G \curvearrowright F$ has bounded multiplicative quasi-isometry constants, (8.23). Therefore after passing to the ultralimit we get an action $\psi_\omega : G \curvearrowright F_\omega \cong \mathbb{R}$ by uniformly bilipschitz homeomorphisms. Hence the action ψ_ω is conjugate to an action of G on $\mathbb{R}$ by isometries. The action of g_1 is nontrivial $(\psi_\omega(g_1)(x_\omega), x_\omega) = 1$ due to our choice of scale factors. When $G \to \mathrm{QI}(F)$ is orientation preserving, so is ψ_ω, and we get a nontrivial isometric action $G \curvearrowright \mathbb{R}$ by translations. □

We summarize our results in the following

Corollary 8.27. *Suppose that $\mathcal{F}$ is a parameterized coarse fibration of Z by lines so that the base $Y^{(0)}$ has polynomial growth and $G \curvearrowright Z$ is a discrete quasi-action which preserves the fibration $\bar{\mathcal{F}}$ and orientation on the fibers. Then:*

1. Z is amenable.

2. (a) Either G contains an infinite cyclic central subgroup Z whose orbit is Hausdorff-close to a fiber of $\bar{\mathcal{F}}$, or

(b) G admits a nontrivial homomorphism to $\mathbb{Z}$.

9. 3-manifolds coarsely fibered by lines

In this section we will prove the main theorem of our paper, Theorem 1.3. Suppose that M is a closed 3-manifold whose universal cover X admits a G-invariant parameterized coarse fibration

$$\mathcal{F} = \{(F_j, f_j)\}_{j\in\mathcal{J}}$$

by lines, where $G = \pi_1(M)$.

Suppose G has more than one end. Then there is a compact set $K \subset X$ and a pair C_1, C_2 of distinct unbounded components of $X \setminus K$. Suppose $\bar{F}_0 \in \bar{\mathcal{F}}$ is a fiber lying in a single component C of $X \setminus K$. Since $d_H(\bar{F}, \bar{F}_0) < \infty$ for every $\bar{F} \in \bar{\mathcal{F}}$, the portion of $\bar{F}$ lying outside C has bounded diameter. Using the fact that $\bar{F}$ is the image of a uniform embedding $\mathbb{R} \to X$ with controlled distortion, we conclude that $\bar{F} \setminus C$ has *uniformly* bounded diameter (for all coarse fibers $\bar{F}$ of $\bar{\mathcal{F}}$). Pick $i \in \{1, 2\}$ so that $C_i \neq C$. We then can find fibers passing through points $p \in C_i$ arbitrarily far from K, which is a contradiction. Therefore every fiber $\bar{F}$ must intersect at least two components of $X \setminus K$, which forces $\bar{F}$ to pass within uniform distance of K. It follows that X is at finite Hausdorff distance from every fiber, and so G is 2-ended. This implies that $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_2 \star \mathbb{Z}_2$, and in both cases M is homotopy-equivalent to a Seifert manifold.

Henceforth we will assume that G is 1-ended, which implies that X is contractible. Since we are interested in describing M up to homotopy-equivalence, we can assume that the manifold M is irreducible. If a manifold M is finitely covered by a Seifert manifold or a Solv-manifold, then M is itself Seifert or Solv-manifold (see [34] in Seifert case and [18] in the Solv-case); hence we a free to pass to finite index subgroups of G when convenient. Therefore, we pass to an index 2 subgroup in G that *preserves the orientation* of the fibers of $\mathcal{F}$, i.e. each element of G acts trivially on the set of ends of each fiber of $\mathcal{F}$.

We note that X has a natural structure of a uniformly contractible metric cell complex, the action $G \curvearrowright X$ is a properly discontinuous free[5] cocompact and preserves the structure of a metric cell complex on X. Hence (see Proposition 4.1) there exists a uniformly contractible metric cell complex Y, the *coarse base* of the coarse fibration

[5]Most of the discussion remains valid in the case when G has torsion.

$\mathcal{F}$. Note also that the (quasi)-action $G \curvearrowright X^{(0)}$ projects to a cobounded quasi-action $G \curvearrowright Y^{(0)}$, see Remark 3.3. According to Corollary 5.9, the complex Y is a quasi-plane.

Our goal is to analyze the group G and the geometry of the coarse fibration depending upon the coarse geometry of the space Y and the dynamics of the quasi-action $G \curvearrowright Y^{(0)}$. According to Corollary 6.12 one of the two cases can occur:

Case 1. $Y^{(1)}$ is Gromov-hyperbolic.

Case 2. $Y^{(1)}$ has polynomial growth.

9.1. **The hyperbolic case.** Assume that Y is a Gromov-hyperbolic quasi-plane. According to Corollary 7.7, in this case $Y^{(0)}$ is quasi-isometric to $\mathbb{H}^2$ and (by Proposition 7.9) the quasi-action $G \curvearrowright Y^{(0)}$ is quasi-isometrically conjugate to a cobounded isometric action of G on $\mathbb{H}^2$. The action $G \curvearrowright \mathbb{H}^2$ factors through a homomorphism

$$\phi : G \to \hat{G} \subset \mathrm{Isom}(\mathbb{H}^2).$$

Below is the discussion of the three possible types of isometric actions $\hat{G} \curvearrowright \mathbb{H}^2$.

Lemma 9.1. *Each cobounded isometric action $\Gamma \curvearrowright \mathbb{H}^2$ belongs to one of the following types:*

1) Γ is dense in $PSL(2, \mathbb{R})$.

2) Γ is a discrete cocompact lattice in $\mathrm{Isom}(\mathbb{H}^2)$.

3) Γ fixes a point $\xi \in \partial_\infty \mathbb{H}^2$ and the closure of Γ in $\mathrm{Isom}(\mathbb{H}^2)$ *is a solvable nonabelian group.*

Proof. Since the connected component $\overline{\Gamma}^0$ of the closure $\overline{\Gamma}$ in $\mathrm{Isom}(\mathbb{H}^2)$ is a connected Lie subgroup (which is trivial iff Γ is discrete and which acts transitively on $\mathbb{H}^2$ otherwise), $\overline{\Gamma}^0$ is either $\{1\}$, or a solvable nonabelian, or is equal to $PSL(2, \mathbb{R})$. $\square$

Case 1a. The action $\hat{G} \curvearrowright \mathbb{H}^2$ is discrete. Thus $\hat{G}$ is a cocompact Fuchsian group, which therefore has virtual cohomological dimension 2. Since the cohomological dimension of G is 3, the homomorphism ϕ has to have a nontrivial kernel. Moreover, each orbit of this kernel is Hausdorff-close to a fiber $\bar{F}$ in the coarse fibration $\bar{\mathcal{F}}$. Hence $Ker(\phi)$ is quasi-isometric to $\mathbb{Z}$, which in turn implies that $Ker(\phi)$ is isomorphic to $\mathbb{Z}$ (recall that G is torsion-free). Hence the fundamental group G of the 3-manifold M fits into a short exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(M) \to \hat{G} \to 1.$$

Thus M is a Seifert manifold with hyperbolic base-orbifold (whose fundamental group is isomorphic to $\hat{G}$), see [34], and the fibers of $\bar{\mathcal{F}}$ are uniformly close to the lifts of the fibers of Seifert fibration from M to X.

In the following two cases the action $\hat{G} \curvearrowright \mathbb{H}^2$ is nondiscrete; hence the quasi-action $G \curvearrowright Y^{(1)}$ is nondiscrete as well, i.e. there exists $p \in Y^{(0)}$ and sequences of $g_i \in G$ and numbers $R_i \to \infty$ such that

$$d(g_i|_{B_{R_i}(p)}, id) \le Const, \quad B_{R_i}(p) \subset Y^{(1)},$$

but each $g_i : Y^{(1)} \to Y^{(1)}$ does not have bounded displacement. In this case the *kernel* of the quasi-action $G \curvearrowright Y$ has to be trivial since the action $G \curvearrowright X$ is discrete and $Y^{(1)}$ is Gromov-hyperbolic (see section 2.1). In particular, the kernel of the homomorphism $\phi : G \to \hat{G}$ is trivial.

Case 1b. $\hat{G}$ is dense in $PSL(2, \mathbb{R})$. We claim that this case cannot occur. Since $\hat{G}$ is dense in $PSL(2, \mathbb{R})$, it contains a nontrivial elliptic element $\hat{g}$ and it also contains a sequence of elements $\hat{h}_i$ which converge to $1 \in PSL(2, \mathbb{R})$. Let $g, h_i \in G$ be elements which map (via ϕ) to $\hat{g}$ and $\hat{h}_i$ respectively. By taking conjugates $g_i := h_i g h_i^{-1}$, we get an infinite collection of distinct elements $\{g_i : i \in \mathbb{N}\}$ of G so that for each $n \in \mathbb{Z}$, $g_i(\bar{F})$ is contained in $N_R(\bar{F})$ where $R \in \mathbb{R}_+$ and $\bar{F} \in \bar{\mathcal{F}}$ are independent of i. We note that since all g_i are pariwise conjugate, there exists $C < \infty$ such that $d(x, g_i(x)) < C$ for each $x \in \bar{F}$ and $i \in \mathbb{N}$. This contradicts discreteness of the action of G on X.

Case 1c. $\hat{G}$ is solvable. Hence, since $ker(\phi : G \to \hat{G})$ is trivial, G is virtually solvable itself. Therefore, according to the classification of Haken manifolds with solvable fundamental groups (see [19]), M is modelled on one of the three geometries: $\mathbb{E}^3$, Nil, Solv.

9.2. **The case of doubling base. Case 2.** $Y^{(0)}$ has polynomial growth. Then according to Corollary 8.27, the group G is amenable and either

2a. G has infinite center Z_G, or

2b. M is Haken.

In the first case we can use the solution of Seifert conjecture, in the case at hand (i.e., when G is amenable) it follows from G. Mess' work [30] that M is a Seifert manifold modelled on $\mathbb{E}^3$ or Nil geometry. As an alternative we can use the following trick. Since the infinite group Z_G is torsion-free, it contains an infinite cyclic subgroup $Z \subset Z_G$. The Z subgroup determines a coarse fibration $\mathcal{L}$ of X by lines (apriori this coarse fibration is not Hausdorff-close to $\bar{\mathcal{F}}$).

We now argue as before: form the coarse base Y' of the coarse fibration $\mathcal{L}$. The quasi-action $G \curvearrowright X$ descends to a quasi-action $G \curvearrowright Y'$ and the complex Y' is a quasi-plane. Now however the space $Y'^{(0)}$ is quasi-isometric to a Cayley graph of the quotient group $G' = G/Z$ and the quasi-action $G' \curvearrowright Y'^{(0)}$ is discrete and cocompact.

If $Y'^{(0)}$ is quasi-isometric to the hyperbolic plane then we get a homomorphism $G' \to \mathrm{Isom}(\mathbb{H}^2)$ whose image is a discrete cocompact subgroup: this contradicts amenability of G. If $Y'^{(0)}$ has polynomial growth then the group G' has polynomial growth and therefore is virtually nilpotent. This implies that the group G is virtually nilpotent as well and hence the manifold M is modelled on $\mathbb{E}^3$ or Nil. $\square$

In the case 2b we use

Lemma 9.2. *Each closed Haken 3-manifold with amenable fundamental group is geometric, and hence must be modelled on* $\mathbb{E}^3$*,* Nil*, or* Solv*.*

Proof. We give two different proofs.

First proof. By Thurston's Geometrization theorem for Haken manifolds, our manifold M admits a decomposition along genus 1 surfaces into geometric components which are each diffeomorphic to quotients of model spaces by lattices. If there is more than one component in this decomposition, then all the lattices in question must be nonuniform. But nonuniform lattices occur only in the geometries $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL(2, \mathbb{R})}$, all of whose lattices are nonamenable. Subgroups of amenable groups are amenable, so this is impossible.

Second proof. We let $G := \pi_1(M)$. Consider a two-sided, embedded, incompressible surface $S \subset M$. Since G is amenable, S has genus 1. Let $G \curvearrowright T$ be the action on the Bass-Serre tree corresponding to the decomposition of G along $\pi_1(S)$. Since G is amenable, either T is a line, or G must fix a point in the ideal boundary of T (otherwise G will contain a free nonabelian subgroup).

In the first case, after passing to an index two subgroup if necessary, the action $G \curvearrowright T$ will be by translations, and hence we get an exact sequence $1 \to H \to G \to \mathbb{Z} \to 1$ where $H = \pi_1(S)$. Therefore, by the Stallings-Waldhausen theorem, M is homeomorphic to a surface bundle over a circle with fiber S. Thus the manifold M is modelled on $\mathbb{E}^3$, Nil, or Solv.

We claim that the second case is impossible unless T is a line. Let $v \in T$ be a vertex. The vertex stabilizer G_v must fix the incident edge e which points towards the fixed point at infinity. Hence the vertex group G_v coincides with $G_e \simeq \pi_1(S)$. Hence the stabilizers of all edges incident to v are of finite index in G_v (all groups are virtually $\mathbb{Z}^2$). If there were more than two edges incident to v, we could pass to a finite index subgroup G' of G_v which fixed at least two of them, and then the quotient of the vertex space by G' would have two boundary components whose inclusion is a homotopy equivalence, which implies that the difference of (the fundamental classes of the) two boundary components is null homologous, and hence there can be no other boundary components. This means that T is a line. $\square$

Thus in the both cases 2a, 2b, the manifold M is geometric modelled on $\mathbb{E}^3$, Nil, or Solv. This concludes the proof of the main theorem. $\square$

10. Classification of coarse fibrations of geometric 3-manifolds

In this section we classify invariant coarse fibrations of the universal covers of closed irreducible 3-manifolds M. By the main theorem we already know that every such M must be either Seifert or Solv-manifold. We recall several constructions of coarse fibrations:

1. Suppose that M is a Seifert manifold with noncompact universal cover X; consider the lift $\bar{\mathcal{L}}$ of the Seifert fibration on M to the universal cover X. Then $\bar{\mathcal{L}}$ is a coarse fibration of X by lines.

2. Let $h : T^2 \to T^2$ be an Anosov affine mapping. Then there exists two h-invariant foliations $\mathcal{G}$ of T^2 by affine geodesics. Let M be the mapping torus of h. Then each foliation $\mathcal{G}$ yields a 1-dimensional lamination $\hat{\mathcal{L}}$ on M. Let $\bar{\mathcal{L}}$ denote the lift of $\hat{\mathcal{L}}$ to the universal cover X of M. Then $\bar{\mathcal{L}}$ is a coarse fibration of X by lines.

3. Let M be a Euclidean 3-manifold, i.e. the quotient of $X = \mathbb{E}^3$ by a torsion-free lattice G. Consider a G-invariant family $\bar{\mathcal{L}}$ of parallel lines in X. Then $\bar{\mathcal{L}}$ is a coarse fibration of X by lines.

We will refer to these examples as *standard coarse fibrations by lines.* The main result of this section is

Theorem 10.1. *Suppose that M is a Seifert manifold or* Solv*-manifold with universal cover X and $\bar{\mathcal{F}}$ is an invariant coarse fibration of X by lines. Then $\bar{\mathcal{F}}$ is Hausdorff-close to a standard coarse fibration $\bar{\mathcal{L}}$ of X by lines.*

In the case when the coarse base Y of the coarse fibration $\bar{\mathcal{F}}$ is hyperbolic (which includes the Solv-manifolds and Seifert manifolds with hyperbolic base), the proof of the main theorem immediately implies that $\bar{\mathcal{F}}$ is within finite distance from a standard coarse fibration $\bar{\mathcal{L}}$ by lines.

Thus we are done in the case when M is a Seifert manifold with hyperbolic base. However even in the case of Solv-manifolds, there is a possibility that the coarse base Y of the coarse fibration $\bar{\mathcal{F}}$ has polynomial growth. Hence we are left with three classes of 3-manifolds M to consider:

(a) Euclidean manifolds.

(b) Nil-manifolds.

(c) Solv-manifolds.

Thus (after passing to a finite cover) we may assume that our manifold M is of the form K/G, where K is $\mathbb{R}^3$, the Heisenberg group, or the solvable connected Lie group $\mathrm{Isom}^o(\mathrm{Solv})$, and G is a torsion-free lattice in K.

The following lemma will be used repeatedly in the analysis.

Lemma 10.2. *Let $G \curvearrowright X$ be a cobounded quasi-action on a metric space, and let $\bar{\mathcal{F}}$ be a G-invariant coarse fibration. Then there is a constant D with the following* *property. For all $\bar{F} \in \bar{\mathcal{F}}$, $x, x' \in \bar{F}$, there are sequences $\{x_i\} \in \bar{F}$, $\{g_i\} \in G$ such that for all i,*

$$d_H(g_i\bar{F}, \bar{F}) < D \tag{10.3}$$

and

$$d(g_i x, x_i) < D, \quad d(g_i x', x_{i+1}) < D. \tag{10.4}$$

Proof. Set $x_0 := x$, $x_1 := x'$, and $g_0 = e \in G$. Using the coboundedness of $G \curvearrowright X$, pick $g_1 \in G$ such that $d(g_1 x_0, x_1) < D_1$, where D_1 depends only on the quasi-action $G \curvearrowright X$. Then $d_H(g_1\bar{F}, \bar{F}) < D_2$ where D_2 depends only on the constants of the coarse fibration $\bar{\mathcal{F}}$. So we can pick $x_2 \in \bar{F}$ such that $d(x_2, g_1 x_1) < D_2$. Proceeding by induction, we define g_i, x_{i+1} for $i > 0$. Define g_i, x_i for $i < 0$ inductively as follows. Assume g_k, x_k have been defined for $k > i$. Pick $g_i \in G$ such that $d(g_i x_1, x_{i+1}) < D_1$, and $x_i \in \bar{F}$ such that $d(x_i, g_i x_0) < D_2$. The sequences $\{x_i\}$, $\{g_i\}$ satisfy (10.3) and (10.4) with $D = \max(D_1, D_2)$. $\square$

Lemma 10.5. *Suppose $G \curvearrowright \mathbb{R}^n$ is a cobounded quasi-action by translations, and $\bar{\mathcal{F}}$ is a G-invariant coarse fibration by lines. Then $\bar{\mathcal{F}}$ is at finite Hausdorff distance from an affine foliation by lines: there is a line $L \subset \mathbb{R}^n$ so that every fiber $\bar{F} \in \bar{\mathcal{F}}$ is at uniformly bounded Hausdorff distance from a line parallel to L.*

Proof. Pick $\bar{F} \in \bar{\mathcal{F}}$, $p \in \bar{F}$. For each $q \in \bar{F}$, we apply Lemma 10.2 with $x = p$ and $x' = q$ to obtain sequences $\{x_i(q)\} \in \bar{F}$, $\{g_i(q)\} \in G$. Since G quasi-acts by translations, for all $q \in \bar{F}$, $i \in \mathbb{Z}$, we have $|(x_{i+1}(q) - x_i(q)) - (q - p)| < 2D$. This means that for all $i < j \in \mathbb{Z}$, $q \in \bar{F}$,

$$|(x_j(q) - x_i(q)) - (j - i)(q - p)| < 2D(j - i). \tag{10.6}$$

So when $|q - p| > 2D$, the map $i \mapsto x_i(q) \in \bar{F} \subset \mathbb{R}^n$ is biLipshitz, which clearly implies that $\{x_i(q)\}$ is a net in $\bar{F}$. Let L_q be the line $\{p + t(q - p) \mid t \in \mathbb{R}\}$. By (10.6), we get that $\{x_i(q)\}$ is contained in the cone

$$\Omega_q := \left\{ x \in \mathbb{R}^n \mid \frac{d(x, L_q)}{d(x, p)} \leq \frac{2D}{|q - p| - 2D} \right\},$$

when $|q - p| > 2D$, and hence $\bar{F}$ is in a finite neighborhood of Ω_q. This clearly implies that the family of subspaces $\mathbb{R}(q - p) \subset \mathbb{R}^n$ converges to a line $\mathbb{R}v$ as $|q| \to \infty$, for otherwise for some $q_1, q_2 \in \bar{F}$, the sets $N_r(\Omega_{q_1})$ and $N_r(\Omega_{q_2})$ would have bounded intersection for all r.

Let L be the line $\{p + tv \mid t \in \mathbb{R}\}$. We claim that $\bar{F} \subset N_{2D}(L)$. For if $q \in \bar{F} \setminus N_{2D}(L)$, then

$$d(x_i(q), L) \geq d(p + i(q-p), L) - |(p - i(q-p)) - x_i(q)|$$

$$\geq d(q, L)i - 2Di = i(d(q, L) - 2D)$$

so no subsequence of the lines $\mathbb{R}(x_i(q) - p)$ can converge to L, which is a contradiction. This proves the claim.

Since $\bar{F}$ is a quasi-line contained in $N_{2D}(L)$, we clearly have $d_H(\bar{F}, L) < \infty$. Translation invariance then implies that every fiber in $\bar{\mathcal{F}}$ is a uniformly bounded Hausdorff distance from a translate of L. □

We now consider the case of Nil geometry. Let Nil denote the Heisenberg group endowed with a left invariant Riemannian metric. We will use the notation Z for the center of the group Nil, recall that Z is isomorphic to $\mathbb{R}$. Let $\pi : \mathrm{Nil} \to \mathbb{R}^2 \simeq \mathrm{Nil}/Z$ be canonical epimorphism, and endow $\mathbb{R}^2$ with the submersion metric, so that π has Lipschitz constant 1. Let $G \curvearrowright \mathrm{Nil}$ be a cobounded quasi-action by left translations which preserves a coarse fibration $\bar{\mathcal{F}}$ of Nil by lines.

Lemma 10.7. *There is a constant C such that every fiber is a Hausdorff distance at most C from a coset of the center Z.*

Proof. Pick $\bar{F} \in \bar{\mathcal{F}}$, $p \in \bar{F}$. Let D be the constant given by Lemma 10.2 with $x = p$ and $x' = q$, and for every $q \in \bar{F}$, let $\{x_i(q)\}$, $\{g_i(q)\}$ be the sequences provided by that lemma. Set $y_i(q) := \pi(x_i(q))$.

We now suppose that $\mathrm{diam}(\pi(\bar{F})) > 2D$, and will derive a contradiction.

The quasi-action $G \overset{\Psi}{\curvearrowright} \mathrm{Nil}$ covers a quasi-action $G \overset{\Psi_{\mathbb{R}^2}}{\curvearrowright} \mathbb{R}^2$ by translations. Hence we can repeat the calculations of the previous lemma, to get that for all $i < j \in \mathbb{Z}$,

$$|(y_j(q) - y_i(q)) - (j-i)(\pi(q) - \pi(p))| < 2D(j-i),$$

and

$$|y_j(q) - y_i(p)| \geq (j-i)(|\pi(q) - \pi(p)| - 2D). \tag{10.8}$$

So when $|\pi(q) - \pi(p)| > 2D$, the map $i \mapsto y_i(q)$ is biLipschitz, which means that $i \mapsto x_i(q)$ is also biLipschitz. Hence $\{x_i(q)\}$ is a net in $\bar{F}$, and $y_i(q)$ is a net in $\pi(\bar{F})$. Repeating arguments from Lemma 10.5, we conclude that $\pi(\bar{F}) \subset N_{2D}(L)$ for some line $L \subset \mathbb{R}^2$ passing through $\pi(p)$.

Now consider the net $\{x_i(q)\} \subset \bar{F}$ for some $q \in \bar{F}$ with $|\pi(q) - \pi(p)| > 2D$. Pick $g \in G$ such that $\Psi_{\mathbb{R}^2}(g)(L) \neq L$. Since $\Psi_{\mathbb{R}^2}(g) : \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, it has displacement bounded by some $\delta \in \mathbb{R}$, and hence

$$\begin{aligned} d(g(x_i(q)), x_j(q)) &\geq d(\Psi_{\mathbb{R}^2}(g)(y_i(q)), y_j(q)) \\ &\geq |j-i||\pi(q) - \pi(p)| - 2D - \delta. \end{aligned} \tag{10.9}$$

On the other hand, the displacement function $\delta_g : \mathrm{Nil} \to \mathbb{R}$ of g is constant on fibers of π, and unbounded on $\pi^{-1}(L)$. Hence $d(g(x_i(q)), x_i(q))$ is unbounded in i. Combining this with (10.9) contradicts the fact that $d_H(g(\bar{F}), \bar{F}) < \infty$.

$\square$

We now turn to Solv geometry. We view Solv as $\mathbb{R}^3$ endowed with the metric $h = (e^z)^2 dx^2 + (e^{-z})^2 dy^2 + dz^2$. Let $\mathbb{R}^2_{xz}$ and $\mathbb{R}^2_{yz}$ denote the xz and yz planes respectively, endowed with the hyperbolic metrics $(e^z)^2 dx^2 + dz^2$ and $(e^{-z})^2 dy^2 + dz^2$, respectively. The projections $\pi_{xz} : \mathrm{Solv} = \mathbb{R}^3 \to \mathbb{R}^2_{xz}$, $\pi_{yz} : \mathrm{Solv} = \mathbb{R}^3 \to \mathbb{R}^2_{yz}$ are 1-Lipschitz.

Lemma 10.10. *Let* $G \overset{\Psi}{\curvearrowright} \mathrm{Solv}$ *be an isometric action of a lattice on* Solv *by left translations, and let* $\bar{\mathcal{F}}$ *be a* G*-invariant coarse fibration of* Solv*. Then* $\bar{\mathcal{F}}$ *is at finite Hausdorff distance from a foliation of* $\mathrm{Solv} \simeq \mathbb{R}^3$ *by lines parallel to the* x*-axis or by lines parallel to the* y*-axis.*

Proof. Pick $\bar{F} \in \bar{\mathcal{F}}$, $p \in \bar{F}$, and as before, for each $q \in \bar{F}$, we apply Lemma 10.2 to get sequences $\{x_i(q)\} \subset \bar{F}$, $\{g_i(q)\} \in G$.

Suppose for some $q \in \bar{F}$, $|z(q) - z(p)| > 2D$, where $z : \mathbb{R}^3 \to \mathbb{R}$ is the third coordinate function. Since the map $z : \mathrm{Solv} \to \mathbb{R}$ is equivariant with respect to a translation action $G \curvearrowright \mathbb{R}$, we can obtain estimates similar to (10.8) for the sequence $\{z(x_i(q))\} \in \mathbb{R}$, and so the sequences $\{\pi_{xz}(x_i(q))\}$, $\{\pi_{yz}(x_i(q))\}$ are bilipschitz embeddings of $\mathbb{Z}$ in $\mathbb{R}^2_{xz}$ and $\mathbb{R}^2_{yz}$ respectively. By the Morse Lemma, there are vertical lines $L_x \subset \mathbb{R}^2_{xz}$, $L_y \subset \mathbb{R}^2_{yz}$ such that

$$d_H(\pi_{xz}(\bar{F}), L_x) < \infty, \quad d_H(\pi_{yz}(\bar{F}), L_y) < \infty.$$

For all $g \in G$, we have $d_H(g(\bar{F}), \bar{F}) < \infty$, which forces $\Psi_{xz}(g)(L_x) = L_x$, $\Psi_{yz}(g)(L_y) = L_y$ for all $g \in G$, where $G \overset{\Psi_{xz}}{\curvearrowright} \mathbb{R}^2_{xz}$ is the action on $\mathbb{R}^2_{xz}$ covered by $G \overset{\Psi}{\curvearrowright} \mathrm{Solv}$, and likewise $G \overset{\Psi_{yz}}{\curvearrowright} \mathbb{R}^2_{yz}$ is the action on $\mathbb{R}^2_{yz}$ covered by $G \overset{\Psi}{\curvearrowright} \mathrm{Solv}$. Clearly this contradicts the cocompactness of $G \overset{\Psi}{\curvearrowright} \mathrm{Solv}$.

Hence all fibers $\bar{F} \in \bar{\mathcal{F}}$ lie in the $2D$ neighborhood of a horizontal plane. Therefore $\bar{\mathcal{F}}$ induces a coarse fibration of each horizontal plane. The stabilizer of the plane $\{z = c\}$ in G is a subgroup H which acts cocompactly on $\{z = c\}$ by translations. Applying Lemma 10.5, we conclude that the induced coarse fibration of $\{z = c\}$ is at finite Hausdorff distance from a fibration by parallel lines. Unless the lines are parallel to the x-axis or y-axis, an element $g \in G \setminus H$ will move a fiber $\bar{F}$ to a set at infinite Hausdorff distance from $\bar{F}$, contradicting the G-invariance of $\bar{\mathcal{F}}$.

$\square$

11. $PD(2)$ GROUPS ARE VIRTUALLY SURFACE GROUPS

In this section, we show how to apply the surrounding function to prove that $PD(2)$ groups over arbitrary commutative rings $\mathcal{R}$ are virtually surface groups. This was proven by Eckmann, Linnell and Müller [16, 17] in the case of $\mathcal{R} = \mathbb{Z}$ and, more recently, by Brian Bowditch for field coefficients, [9]. Our main result is

Theorem 11.1. *Suppose that G is a 2-dimensional Poincare duality group over a commutative ring $\mathcal{R}$ with a unit. Then G is virtually a surface group.*

11.1. $PD(n)$ **groups over a ring.** Poincare duality groups of dimension n over rings are generalizations of the fundamental groups of closed aspherical n-manifolds. Thus we beging our discussion with the motivating example of Poincare duality (over $\mathbb{Z}$) for manifolds. If M is an oriented triangulated (connected) n-manifold, Poincare duality induces a chain homotopy equivalence $C_c^*(M) \to C_{n-*}(M)$ between the complex of compactly supported simplicial cochains and the simplicial chain complex. If a group G acts freely simplicially on M, then Poincare duality is G-equivariant, up to twisting by the orientation module. More precisely, let $D := H_c^n(M)$ equipped with the left G-action $g \cdot \alpha := (g^{-1})^* \alpha$. Then Poincare duality defines a G-equivariant map $C_c^*(M) \to D \otimes_{\mathbb{Z}} C_{n-*}(M)$ where G acts on $D \otimes_{\mathbb{Z}} C_{n-*}(M)$ by the diagonal action. The task of this subsection is to establish a similar statement for Poincare duality groups in general.

Let $\mathcal{R}$ be a commutative ring with unit. Recall [4], that an n-dimensional Poincare duality group over $\mathcal{R}$ (for short, $PD(n)$ group over $\mathcal{R}$), is an FP-group over R such that $H^i(G, \mathcal{R}G)$ is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module when $i = n$ and is trivial otherwise. This implies that the cohomological dimension of G over $\mathcal{R}$ is equal to n [11, p. 202], and so we may choose a resolution

$$0 \to A_n \to \ldots \to A_0 \to \mathcal{R} \to 0 \tag{11.2}$$

of the trivial $\mathcal{R}G$-module $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules, [11, p. 199]. Applying the functor $\mathrm{Hom}_{\mathcal{R}G}(\cdot, \mathcal{R}G)$ to the resolution A_*, we obtain cochain complex

$$0 \leftarrow B^n \leftarrow \ldots \leftarrow B^0 \leftarrow 0$$

where $B_i := \mathrm{Hom}_{\mathcal{R}G}(A_i, \mathcal{R}G)$. The complex B_* computes $H^*(G, \mathcal{R}G)$ (recall that $H^n(G, \mathcal{R}G) \cong \mathcal{R}$ and $H^i(G, \mathcal{R}G) = 0$ if $i \neq r$), and hence we can extend B^* to a "dual" resolution

$$0 \leftarrow D \leftarrow B^n \leftarrow \ldots \leftarrow B^0 \leftarrow 0 \tag{11.3}$$

where, following [11, p. 219], we set $D := H^n(G, \mathcal{R}G)$. The $\mathcal{R}G$-bimodule structure on $\mathcal{R}G$ induces a right $\mathcal{R}G$-module structure on the resolution (11.3). We will find it more convenient to convert the right G-action to a left G-action by taking inverses of group elements; in particular, we will view D as a left G-module.

For each left $\mathcal{R}G$-module M, define a new left $\mathcal{R}G$-module $\tau(M)$, by "twisting" with the G-action on D, i.e. $\tau(M)$ is the $\mathcal{R}$-module $D \otimes_{\mathcal{R}} M$ equipped with the diagonal G action $g \cdot (d \otimes m) = gd \otimes gm$. Consider the chain complex of left $\mathcal{R}G$-modules obtained by applying the functor τ to the resolution (11.2):

$$0 \to \tau(A_n) \to \ldots \to \tau(A_0) \to \tau(\mathcal{R}) \to 0. \tag{11.4}$$

Lemma 11.5. *1. The $\mathcal{R}G$-chain complex (11.4) is a resolution of $\tau(\mathcal{R})$.*

2. The left G-modules $\tau(\mathcal{R})$ and D are isomorphic.

3. $\tau(A_i)$ is a finitely generated projective $\mathcal{R}G$-module for each i.

Proof. The module D is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module, and hence tensoring with D over $\mathcal{R}$ does nothing to $\mathcal{R}$-module structure. So clearly (11.4) is still a resolution by $\mathcal{R}$-modules, and therefore a resolution by $\mathcal{R}G$-modules. This proves 1.

For any $\mathcal{R}G$-module M, the $\mathcal{R}$-module $M \otimes_{\mathcal{R}} \mathcal{R}$ with the diagonal G-action is $\mathcal{R}G$-isomorphic to M, which proves 2.

Before proving 3, we first make a few observations about modules isomorphic to $\mathcal{R}$ and the operation τ.

Let E be an $\mathcal{R}$-module isomorphic to $\mathcal{R}$, and let $\mathcal{R}^\times$ denote the group of units in $\mathcal{R}$. Then the map $\mathcal{R}^\times \to \mathrm{Aut}_{\mathcal{R}}(E)$ which assigns to $r \in \mathcal{R}^\times$ the automorphism $E \to E$ given by $e \mapsto re$, is an isomorphism. Hence if E is an $\mathcal{R}G$-module whose underlying $\mathcal{R}$-module is isomorphic to $\mathcal{R}$, then we obtain a character $\chi : G \to \mathcal{R}^\times$ from the equation

$$g \cdot e = \chi(g)e,$$

$g \in G$, $e \in E$. When we tensor E over R with an $\mathcal{R}G$-module M and let G act diagonally, then up to $\mathcal{R}G$-isomorphism, we may think of the resulting $\mathcal{R}G$-module $M' = E \otimes_{\mathcal{R}} M$ as having the same underlying $\mathcal{R}$-module as M, where the G-action is twisted by the character χ:

$$(g, m) \mapsto \chi(g)gm,$$

for $g \in G$, $m \in M$. Keeping the above remarks in mind, it is then clear that when we apply τ to the group ring $\mathcal{R}G$, we get back an RG-module isomorphic to $\mathcal{R}G$: to get the isomorphism $\phi : \mathcal{R}G \to \tau(\mathcal{R}G)$ just set $\phi(\delta_g) := \chi(g)\delta_g$ for all $g \in G$. Direct sums are respected by τ, so τ transforms free RG-modules into free $\mathcal{R}G$-modules, and hence projective modules into projective modules. Obviously τ preserves finite generation. Combining these statements yields assertion 3. $\square$

The lemma implies that the resolution (11.3) and the resolution (11.4) both define resolutions of D by left $\mathcal{R}G$-modules. Therefore the two resolutions are $\mathcal{R}G$-chain homotopy equivalent, i.e. there are $\mathcal{R}G$-chain mappings

$$P : B^* \to \tau(A_{n-*}), \quad \bar{P} : \tau(A_*) \to B^{n-*} \tag{11.6}$$

and $\mathcal{R}G$-chain homotopy operators

$$\bar{P} \circ P \overset{H}{\sim} \mathrm{id}_{B^*}, \quad P \circ \bar{P} \overset{\bar{H}}{\sim} \mathrm{id}_{\tau(A_*)}\,. \tag{11.7}$$

11.2. $PD(2)$ **groups and quasiplanes.** The goal of this section is to show that for each $PD(2)$-group G over $\mathcal{R}$ there is a 2-dimensional metric cell complex Y on which G acts freely properly discontinuously and cocompactly so that Y satisfies coarse Poincare duality in dimensions 0 and 1.

Being FP_2 over R, the group G acts freely, cocompactly, and cellularly on a 2-dimensional cell complex Y which is 1-acyclic over $\mathcal{R}$, [11] (?). In what follows we will also need to analyze the case of $PD(2)$-groups which act cocompactly on an $(n-1)$-acyclic (over $\mathcal{R}$) n-dimensional complex Y, $n \geq 2$. Henseforth we assume that Y is such a complex. All (co)homology groups through this section will be with coefficients in $\mathcal{R}$.

It will be convenient to metrize the 1-skeleton of Y so that the edges have unit length, thus Y has a natural structure of a metric cell complex with a control map $Y \to Y^{(0)}$.

The cellular chain complex of Y determines a partial resolution of $\mathcal{R}$ by $\mathcal{R}G$-modules:

$$C_n(Y) \to ...C_1(Y) \to C_0(Y) \to \mathcal{R} \to 0.$$

The fundamental lemma of homological algebra provides chain mappings

$$(f_i : A_i \to C_i(Y))_{0\leq i\leq n}, \text{ and } (\bar{f}_i : C_i(Y) \to A_i)_{0\leq i\leq n}$$

and homotopy operators

$$(\bar{f}_i \circ f_i \overset{K_i}{\sim} \mathrm{id}_{A_i})_{0\leq i\leq n}, \quad (f_i \circ \bar{f}_i \overset{\bar{K}_i}{\sim} \mathrm{id}_{C_i(Y)})_{0\leq i<n}.$$

Using f_*, $\bar{f}_*$, K_*, and $\bar{K}_*$, we can transfer the operators P, $\bar{P}$, H, $\bar{H}$ to $C_*(Y)$ and $C_c^*(Y)$, as chain mappings

$$(P_i : C_c^i(Y) \to \tau(C_{2-i}(Y))_{0\leq i\leq n}, \quad (\bar{P}_i : \tau(C_{2-i}(Y)) \to C_c^i(Y))_{0\leq i\leq n}$$

and chain homotopies

$$(H_i : C_c^i(Y) \to C_c^{i-1}(Y))_{0<i\leq n}, \quad (\bar{H}_i : \tau(C_i(Y)) \to \tau(C_{i+1}(Y)))_{0\leq i<n}.$$

Here we have used the natural isomorphisms of $\mathcal{R}G$-modules

$$\mathrm{Hom}_{\mathcal{R}G}(C_i(Y), \mathcal{R}G) \simeq C_c^i(Y),$$

where as with (11.4), right G-actions are converted to left G-actions. The key fact for us is that $(H_i)_{0<i\leq n}$ gives a chain homotopy $\bar{P}_i \circ P_i \sim \mathrm{id}_{C_c^i(Y)}$, and $(\bar{H})_{0\leq i<n}$ gives a chain homotopy $P_i \circ \bar{P}_i \sim \mathrm{id}_{\tau(C_i(Y))}$.

As there are only finitely many G-orbits of cells in Y, it follows that if σ is an oriented i-cell in Y and $\hat{\sigma} \in C_c^i(Y)$ is the associated element, then

$$P_i(\hat{\sigma}) \in \tau(C_{2-i}(Y)) = D \otimes_{\mathcal{R}} C_{2-i}(Y)$$

can be expressed as a sum $\sum_j d_j \otimes \beta_j$ where the β_j's are oriented $(2-i)$-cells contained in a D-neighborhood of σ, where D is a universal constant. Similar statements apply to the operators $\bar{P}_*$, H_*, and $\bar{H}_*$. We therefore deduce that when T is one of these operators and $c \in \mathrm{Domain}(T)$, then $T(c)$ is supported in the D_0-neighborhood of the support of c, where D_0 is independent of c. We now forget about the G-action on Y; henceforth we will identify $\tau(C_*(Y))$ with $C_*(Y)$ and $\tau(C_c^*(Y))$ with $C_c^*(Y)$.

Lemma 11.8. *Y satisfies coarse 2-dimensional Poincare duality over $\mathcal{R}$ in dimensions $0 \leq i < n$.*

Proof. This lemma is a special case of the *Coarse Poincare Duality Theorem* 6.7 proven in [26] (for the integer coefficients). For the reader's convenience we will outline a proof.

When $K \subset Y$ is a (nonempty) subcomplex we will consider the direct system of tubular neighborhoods $\{N_R(K)\}_{R\geq 0}$ of K and the inverse system of the closures of their complements

$$\{V_R := \overline{Y - N_R(K)}\}_{R\geq 0}.$$

We get four inverse and four direct systems of (co)homology groups with coeffients in $\mathcal{R}$:

$$\{H_c^i(N_R(K))\},\ \{H_i(Y, V_R)\},\ \{H_c^i(Y, N_R(K))\},\ \{H_i(V_R)\}$$
$$\{H_c^i(V_R)\},\ \{H_1(Y, N_R(K))\},\ \{H_c^i(Y, V_R)\},\ \{H_i(N_R(K))\}$$

with the usual restriction and projection homomorphisms. Note that by excision, we have isomorphisms

$$H_i(Y, V_R) \simeq H_i(N_R(K), \partial N_R(K)),\ \text{etc.}$$

Extension by zero defines a group homomorphism $C_c^i(N_{R+D_0}(K)) \overset{ext}{\subset} C_c^i(Y)$. When we compose this with

$$C_c^i(Y) \xrightarrow{P} C_{2-i}(Y) \xrightarrow{proj} C_{2-i}(Y, V_R)$$

we get a well-defined induced homomorphism

$$P_{R+D_0} : H_c^1(N_{R+D_0}(K)) \to H_1(Y, V_R).$$

We get, in a similar fashion, homomorphisms

$$H_c^i(N_{R+D_0}(K)) \overset{P_{R+D_0}}{\longrightarrow} H_{2-i}(Y, V_R) \overset{\bar{P}_R}{\longrightarrow} H_c^i(N_{R-D_0}(K)) \tag{11.9}$$

$$H_c^i(V_R) \overset{P_R}{\longrightarrow} H_{2-i}(Y, N_{R+D_0}(K)) \overset{\bar{P}_{R+D}}{\longrightarrow} H_c^i(Y_{R+2D_0}) \tag{11.10}$$

$$(11.11) \qquad H_c^i(Y, N_{R+D_0}(K)) \xrightarrow{P_{R+D_0}} H_{2-i}(V_R) \xrightarrow{\bar{P}_R} H_c^i(Y, N_{R-D_0}(K))$$

$$(11.12) \qquad H_c^{2-i}(Y, V_R) \xrightarrow{P_R} H_{2-i}(N_{R+D_0}(K)) \xrightarrow{\bar{P}_{R+D_0}} H_c^{2-i}(Y, V_{R+2D_0})$$

Note that the homomorphisms in (11.9), (11.10), (11.11) and (11.12) inherit the bounded displacement property of P and $\bar{P}$.

We now check that the maps P_R satisfy the approximate monomorphism property stated in Definition 5.2, we leave verification of the rest of the assertions to the reader. Let

$$\xi \in Z_c^i(N_{R+2D_0}(K))$$

be a cocycle representing an element $[\xi] \in Ker(P_{R+2D_0})$, and let $\xi_1 \in C_c^i(Y)$ be the extension of ξ by zero. Then we have

$$P(\xi_1) = \partial\eta + \zeta$$

where $\eta \in C_{2-i}(Y)$ and $\zeta \in C_{2-i}(\overline{Y - N_{R+D}(K)})$. Applying $\bar{P}$ and the chain homotopy H, we get

$$\delta H(\xi_1) + H\delta(\xi_1) = \bar{P} \circ P(\xi_1) - \xi_1 = \bar{P}(\partial\eta + \zeta) - \xi_1$$

so

$$\xi_1 = \delta\bar{P}(\eta) + \bar{P}(\zeta) - \delta H(\xi_1) - H\delta(\xi_1).$$

The second and fourth terms on the right hand side vanish upon projection to $H_c^i(N_R(K))$, so $[\xi] \in Ker(H_c^i(N_{R+2D_0}(K)) \to H_c^i(N_R(K))$. □

Remark 11.13. The 2-complex Y is 1-acyclic and satisfies coarse 2-dimensional Poincare duality in dimension 1. Hence Y is 1-ended by Lemma 5.4.

11.3. **Coarse geometry of quasi-planes.** *Proof of Theorem 11.1.* We prove the theorem by analyzing the coarse geometry of the complex Y along the lines of the sections 6 and 7. The difference however is that (unlike in sections 6 and 7) we do not know that Y is simply-connected; on the other hand, in our present situation Y is quasi-isometric to the finitely-generated group G.

Case 1. G is Gromov-hyperbolic. Then G admits a free discrete cocompact action on a contractible metric complex Y' (see corollary 2.23). Although the complex Y' is not finite-dimensional we get a resultion

$$\ldots \to C_n(Y', \mathcal{R}) \to \ldots \to C_0(Y', \mathcal{R}) \to \mathcal{R} \to 0$$

by free $\mathcal{R}G$-modules. Thus we get a chain-homotopy equivalence between the above resultion and and

$$0 \to \tau(A_2) \to \tau(A_1) \to \tau(A_0) \to \tau(\mathcal{R}) \to 0.$$

Using this chain-homotopy-equivalence we can transfer the Poincare duality operators to operators

$$\mathrm{Hom}_{\mathcal{R}G}(C_i(Y'), \mathcal{R}G) \xrightarrow{P'} C_{2-i}(Y', \mathcal{R}G), C_i(Y', \mathcal{R}G) \xrightarrow{\bar{P}'} \mathrm{Hom}_{\mathcal{R}G}(C_{2-i}(Y'), \mathcal{R}G)$$

which are homotopy-inverses of each other. We note that the complex

$$\mathrm{Hom}_{\mathcal{R}G}(C_*(Y'), \mathcal{R}G)$$

is nothing but $C_c^*(Y', \mathcal{R})$ (after ignoring the $\mathcal{R}G$-module structure). The complex Y' is quasi-isometric to the finite-dimensional contractible Rips complex X of G. Hence, by choosing continuous quasi-isometries we get maps

$$Y' \xrightarrow{f} X \xrightarrow{\bar{f}} Y'$$

which are uniform proper homotopy-inverses. Applying these maps to $P', \bar{P}'$ we get duality operators

$$P : C_c^i(X, \mathcal{R}) \to C_{2-i}(X, \mathcal{R}), \bar{P} : C_i(X, \mathcal{R}) \to C_c^{2-i}(X, \mathcal{R})$$

so that the compositions $P \circ \bar{P}$ and $\bar{P} \circ P$ are chain-homotopic to the identity by homotopies with uniformly bounded tracks.

Applying Lemma 11.8 to X we conclude that X satisfies coarse Poincare duality of dimension 2. Thus we are in position to apply Corollary 7.7 and Proposition 7.9: the complex X is quasi-isometric to $\mathbb{H}^2$ and the discrete cocompact action $G \curvearrowright X$ is quasi-conjugate to a discrete cocompact and isometric action of G on $\mathbb{H}^2$. Therefore G is virtually a surface group. This concludes the proof in the Gromov-hyperbolic case.

Case 2. G is not Gromov-hyperbolic. Then, according to Lemma 3.8 in [32], there exist constants $L \geq 1, c > 0$, a sequence $R_j \in \mathbb{R}_+$ diverging to ∞, such that for each $R = R_j$, there exists an L-biLipshitz embedding

$$f = f_R : S_R \to Y.$$

Here S_R is the circle of radius R in $\mathbb{R}^2$. Let S'_R denote $f(S_R)$.

Lemma 11.14. *For each $R = R_j \gg 1$, the graph $N^{(1)}_{D_0}(S'_R)$ surounds a ball of radius $\frac{1}{2}LR$ in Y.*

Proof. Our proof is analogous to the proof of Lemma 6.7 and of the coarse Jordan separation theorem [26, Corollary 7.8]. We recall that Y satisfies coarse 2-dimensional Poincare duality in the dimension 1, let D_0 be the constant which appears in the definition of the coarse 2-dimensional Poincare duality.

Take $1 < r < \frac{1}{2}(\sqrt{3}LR - 1)$. Define a retraction $\bar{f} : N_r(S'_R) \to S_R$ as follows: for each vertex y in $N_r(S'_R)$ we let $\bar{f}(y) \in S_R$ be a point x such that $f(x) \in S'_R$ is a nearest point to y. Extend f linearly to the 1-skeleton of $N_r(S'_R)$. The inequality

$r < \frac{1}{2}(\sqrt{3}LR - 1)$ ensures that the map $\bar{f}$ extends to the 2-skeleton of $N_r(S'_R)$. The same inequality ensures that $\bar{f} \circ f : S_R \to S_R$ is homotopic to the indentity.

Let $\alpha \in C^1(S_R, \mathcal{R})$ be a cocycle whose support set has unit diameter and which represents the generator of $H^1(S_R, \mathcal{R})$. Then the homotopy $\bar{f} \circ f \simeq id$ implies that $\alpha' := \bar{f}^*(\alpha)$ is not null-cohomologous in $N_r(S'_R)$. We now apply the coarse Poincare duality to the cocycle α' and we get a dual relative 1-cycle

$$\sigma := P(\alpha') \in H_1(Y, Y \setminus N_{r-D_0}(S'_R))$$

which maps nontrivially to $H_1(Y, Y \setminus N_{D_0}(S'_R))$ provided that $r \geq 2D_0$. Since Y is 1-acyclic over $\mathcal{R}$, this means that the 0-chain $\partial\sigma$ is a linear combination $\partial\sigma = \sum a_i y_i$ with nonzero coefficients $a_i \in \mathbb{R}$, where the y_i's lie outside $N_{r-D_0}(S'_R)$, and $\sum a_i y_i$ represents a nontrivial class in $\tilde{H}_0(Y \setminus N_{D_0}(S'_R), \mathcal{R})$. Thus there is a pair of points y_k, y_l in the support of $\sum a_i y_i$, which cannot be joined by a curve in $Y \setminus N_{D_0}(S'_R)$.

It follows that the connected graph $N^{(1)}_{D_0}(S'_R)$ separates one of the balls $B(y_k, r - 2D_0)$, $B(y_l, r - 2D_0)$ from infinity. ☐

Combining the above lemma with Proposition 6.5 we get:

Corollary 11.15. *The group G is virtually nilpotent and has at most quadratic growth.*

Combining the above with the fact that G is one-ended we conclude that G contains a nilpotent subgroup Γ of finite index so that Γ fits into a short exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z} \to 1;$$

since Γ is nilpotent, this sequence splits and hence $\Gamma \cong \mathbb{Z}^2$. This concludes the proof of Theorem 11.1. ☐

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