Polygons in symmetric spaces and buildings

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1 Introduction

In this paper we study polygons in Euclidean and spherical buildings, symmetric spaces of nonpositive curvature and related geometries. We are interested in restrictions on the congruence classes of their edges other than the triangle inequality. Algebraic applications of the *generalized triangle inequalities* obtained in this paper are presented in [KLM].

We recall that for a rank 1 symmetric spaces the only isometry invariant of a geodesic segment is its metric length. In higher rank the congruence classes of oriented segments are parameterized by the Weyl chamber Δ. We call the parameter $\sigma(\gamma) \in \Delta$ corresponding to an oriented segment γ its Δ-*length*. The same notion of Δ-length can be defined in Euclidean buildings. Here Δ is the Weyl chamber for the finite Weyl group in the associated Euclidean Coxeter complex.

Question 1.1. *Let X be a symmetric space of nonpositive curvature or a Euclidean building. Describe the set $D_n(X) \subset \Delta^n$ of Δ-side lengths which occur for closed n-gons in X.*

In the case of buildings there is a finer invariant for oriented geodesic segments. Let X be a spherical or Euclidean building modelled on the Coxeter complex (A, W). We assign the *refined length* $\sigma_{ref}(x, y)$ to the geodesic oriented segment $\overline{xy}$ by projecting the pair (x, y) to $W\backslash A \times A$. (In the case of rank 0 spherical buildings, one has to work with pairs of points instead of geodesic segments.)

The following refined version of Question 1.1 will be relevant for the algebraic applications of our work presented in [KLM].

Refined Question 1.2. *Describe the set $D_n^{ref}(X) \subset (W\backslash A \times A)^n$ of refined side lengths which occur for closed n-gons in X.*

The Questions 1.1 and 1.2 for a Euclidean building are different only if the affine Weyl group does not act transitively. Although we cannot answer Question 1.2 except in special cases, we are able to relate the sets $D_n^{ref}(X)$ for different buildings:

Theorem 1.3. *If X is a thick spherical or Euclidean building then $D_n^{ref}(X)$ depends only on the associated Coxeter complex.*

Suppose that X and X' are thick spherical or Euclidean buildings with associated Coxeter complexes (A, W) and (A', W'), respectively. Then an embedding $(A, W) \hookrightarrow (A', W')$ of Coxeter complexes induces a natural embedding $D_n^{ref}(X) \hookrightarrow D_n^{ref}(X')$.

We note that there is an infinitesimal version of the question about polygons in symmetric spaces. For a symmetric space X of nonpositive curvature consider a tangent space $\mathfrak{p} = T_x X$ and the group of isometries of $\mathfrak{p}$ generated by the point stabilizer $K = G_x$ and all translations. We call this geometry an *infinitesimal symmetric space* or a *Cartan motion space* (its group of automorphism is a *Cartan motion group* $K \ltimes \mathfrak{p}$). Again there is a notion of Δ-length for segments in $\mathfrak{p}$.

Theorem 1.4. *Let X be either a thick Euclidean building, symmetric space or an infinitesimal symmetric space. Then*

1. $D_n(X)$ is a finite-sided polyhedral cone in Δ^n.

2. $D_n(X)$ depends only on the spherical Coxeter complex associated to X. More generally, suppose that X, X' are metric spaces (each of which is either an infinitesimal symmetric space or a nonpositively curved symmetric space or a Euclidean building) modelled on Euclidean Coxeter complexes $(E, W_{aff}), (E', W'_{aff})$ respectively. Then each isometric embedding[1] $f : E \to E'$ which induces an embedding ϕ of the spherical Coxeter groups $W_{sph} \to W'_{sph}$, also induces an embedding $D_n(X) \to D_n(X')$.

The proof of this theorem uses a relation between polygons in a symmetric space (or a Euclidean building) X and weighted configurations on the spherical building $B = \partial_\infty X$ at infinity.[2] Points $(\xi_i) \in \Delta^n$ can be regarded as maps

$$\tau : Z = (\mathbb{Z}/n\mathbb{Z}, \nu) \to \Delta_{sph}$$

from a finite measure space Z into the model Weyl chamber $\Delta_{sph} = \partial_\infty \Delta$ of B; we put masses $\nu(i) = \|\xi_i\|$ at $i \in Z$ and let $\tau(i)$ be the direction of ξ_i. A *weighted configuration* on B is a map $\psi : Z \to B$. We say that ψ has *type* τ if the composition with the standard projection $p : B \to \Delta_{sph}$ satisfies $p \circ \psi = \tau$.

There is a notion of (semi)stability for weighted configurations on B generalizing Mumford's concept [Mu], see [LeMi] and section 4.1 for details. At this stage we only point out that semistability is defined *intrinsically* in terms of the geometry of B. Let $\Delta_n^{ss}(B) \subset \Delta^n$ denote the set of possible types of semistable weighted configurations on B.

Theorem 1.5 (Stability inequalities [LeMi]). *Let X be a symmetric space of nonpositive curvature. Then the set $\Delta_n^{ss}(\partial_{Tits} X)$ is a polyhedral cone. It can be explicitly described by a finite set of homogeneous linear inequalities in terms of Schubert calculus on the generalized Grassmannians associated with the isometry group of X.*

See section 4.1 for a precise statement of the inequalities. A version of this result for complex Lie groups has been proven independently in [BeSj]. Note that the system of stability inequalities depends on the symmetric space X. However it follows from Theorems 1.4 and 1.6 that the set of solutions $\Delta_n^{ss}(\partial_{Tits} X)$ depends only on the associated spherical Coxeter complex.

In the present paper we show that the problems of determining the possible side lengths of closed polygons and types of semistable configurations are equivalent:

[1] An affine embedding would also suffice.

[2] If $\mathfrak{p}$ is an infinitesimal symmetric space then we assign to it the spherical building B which is the ideal boundary of the symmetric space X of nonpositive curvature such that $\mathfrak{p} = T_x X$.

Theorem 1.6. *Let X be as in Theorem 1.4. For $\xi \in \Delta^n$ the existence of a closed n-gon in X with Δ-side lengths ξ is equivalent to the existence of a semistable weighted configuration of type ξ on the associated spherical building B at infinity.*

For infinitesimal symmetric spaces this result was established in [LeMi].

Combining our results one obtains the following recipe for determining the polytope $D_n(X)$ for any of the spaces X as in Theorem 1.4: Given X, find a complex semisimple Lie group G with isomorphic spherical Coxeter complex. Using Schubert calculus for G, compute the system of stability inequalities describing Δ_n^{ss}. The polytopes Δ_n^{ss} and $D_n(X)$ are equal.

Remark 1.7. *One can show that if X is a symmetric space or an infinitesimal symmetric space then there is a natural homeomorphism between congruence classes of polygons in X and the moduli space of nice semistable configurations on B.*

Theorems 1.5 and 1.6 answer Question 1.2 if the affine Weyl group acts transitively, i.e. in particular if X is a symmetric space or an infinitesimal symmetric space. There are also non-discrete Euclidean buildings with transitive affine Weyl group, for instance, asymptotic cones of symmetric spaces as in [KlLe] or Euclidean buildings associated to algebraic groups over fields with non-archimedean valuations whose value group is $\mathbb{R}$.

For our algebraic applications it will be important to consider polygons with *integral refined side-lengths.* Pick a *special vertex* 0 in a Euclidean coxeter complex (E, W_{aff}). Let L be a lattice in E which contains the translation subgroup L_{trans} of W_{aff} and so that L contained in the normalizer N_{aff} of W_{aff} in the group of translations of E. We say that a polygon P has *refined L-integral side-lengths* if for each edge e of P, $\sigma_{ref}(e)$ is represented by a geodesic segment in E connecting points of the orbit $L \cdot 0$. We will identify 0 with the origin in the vector space underlying E and thus the orbit $L \cdot 0$ will be identified with L. Let $D_n^{ref,L}(X)$ denote the set of refined L-integral side-lengths of n-gons in X. We will be interested in the image of the natural embedding

$$\iota : D_n^{ref,L}(X) \to \Delta^n.$$

It is clear that the image of ι is contained in $D_n(X) \cap L^n$. We have equality in the following case.

Theorem 1.8. *Assume that $L = L_{trans}$ and the normalizer N_{aff} acts transitively on the vertex set $E^{(0)}$ of the Coxeter complex. Then the image of $D_n^{ref,L}(X)$ in Δ^n equals $D_n(X) \cap L^n$.*

Remark 1.9. *In the case of discrete irreducible Coxeter complexes, transitivity of the action of N_{aff} on $E^{(0)}$ is equivalent to the assumption that the Coxeter complex has type A.*

In section 5 we will show that for discrete Coxeter complexes of type C_2 and $L \neq L_{trans}$, the image of the map ι is strictly less than $D_n(X) \cap L^n$ $(n = 3)$. In [KLM] we construct analogous examples with $L = L_{trans}$.

Outline. We now outline the proofs of the main theorems of this paper.

To prove Theorem 1.3 we transfer polygons between buildings of the same type by a direct geometric construction: Coning off a closed polygon at a vertex yields a ruled surface. We then show that such ruled surfaces can be developed into buildings with isomorphic Coxeter complexes. The same argument shows that one can transfer polygons from a building X to a building X' if the Coxeter complex of X embeds into the Coxeter complex of X'.

The proof of Theorem 1.6 is more involved. Each n-gon in X with the Δ-side lengths $(\sigma_1, ..., \sigma_n)$ gives rise to a collection of *Gauss maps* to the *Tits boundary of* X:

$$\psi : \mathbb{Z}/n \to \partial_{Tits} X,$$

which we view as weighted configurations on $\partial_{Tits} X$. One then associates with each ψ a measure μ on $\partial_{Tits} X$. It is easy to see that the measure μ (and configuration ψ) is *semistable*. We then prove the converse: in the case of locally compact spaces X and Euclidean buildings which are cones over spherical buildings, nice semistable configurations on $\partial_{Tits} X$ correspond to polygons in X. More precisely, the first vertex of such polygon appears as a fixed point of a certain 1-Lipschitz map $\Phi : X \to X$ associated with a configuration on $\partial_\infty X$. To prove that Φ fixes a point in X we need μ to be semistable (nice semistable in the Riemannian case), this assumption guarantees that the dynamical system (Φ^n) has a bounded orbit. [3] Thus the existence of a polygon in X with the Δ-side lengths $(\sigma_1, ..., \sigma_n)$ is equivalent to the existence of a semistable configuration ψ on $\partial_{Tits} X$ of the *type* $(\sigma_1, ..., \sigma_n)$.

To prove Theorem 1.6 for Euclidean buildings X which are not locally compact we take a semistable configuration ψ on $\partial_{Tits} X$; this configuration is also a semistable weighted configuration on the Tits boundary of *another* Euclidean building X': The building X' is the Euclidean cone over $\partial_{Tits} X$. The weighted configuration ψ on $\partial_{Tits} X'$ then yields a polygon $P' \subset X'$ with the same Δ-side lengths as before. The transfer theorem 1.3 allows us to transfer the polygon P' to a polygon $P \subset X$ with the same Δ-side lengths.

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[3] Our proof does not work in the case of buildings which are neither locally compact and nor are cones over spherical buildings.

Contents

2 Preliminaries

2.1 Singular spaces of nonpositive curvature

In this section we review several notions from the geometry of, possibly *singular*, spaces of nonpositive curvature. General references for this material are [Ba, BrHae, KlLe] and, in the smooth case, [ChEb, BGS, Eb]. Both symmetric spaces of nonpositive curvature and Euclidean buildings fit into this geometric framework: The symmetric spaces are Hadamard manifolds, i.e. smooth Riemannian manifolds of nonpositive sectional curvature. Euclidean buildings are singular spaces of nonpositive curvature; in the discrete case, they are piecewise Euclidean complexes.

A metric space (X, d) is called *geodesic* if any two points $x, y \in X$ can be connected by a distance minimizing geodesic segment, i.e. if there exists an isometric embedding $f : [0, d(x, y) = l] \to X$ such that $f(0) = x$, $f(l) = y$. The image of such a map f is called a *geodesic segment* connecting x and y and will be denoted by $\overline{xy}$. Note that this is an abuse of notation since, in general, there may be more than one geodesic segment connecting x and y. A *convex function* on X is a function whose restriction to each geodesic segment is convex.

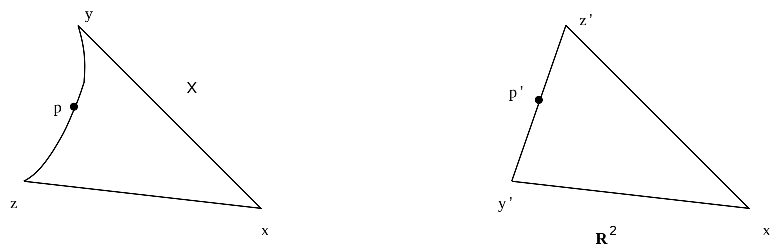

Figure 1: *Triangle comparison.*

Let X be a complete geodesic metric space. We do *not* assume that X is locally compact. One can define *curvature bounds* for such metric spaces by comparison with model spaces of constant curvature. For instance, one can compare the thickness of *geodesic triangles*. Here, by a triangle we mean a one-dimensional object: A triangle in X with the vertices x, y, z, denoted by $\Delta(x, y, z)$, is the union of three geodesic segments $\overline{xy}, \overline{yz}$ and $\overline{zx}$.

Although we will be mainly interested in the upper curvature bound 0, we give a general definition. Let M^2_κ be a complete simply-connected surface of the constant curvature $\kappa \in \mathbb{R}$. A complete geodesic metric space X is called a *CAT(κ)-space* if geodesic triangles in X are "thinner" than triangles with the same side lengths in M^2_κ. More precisely, for any geodesic triangle $\Delta(x, y, z)$ in X of perimeter $< 2\pi/\sqrt{\kappa}$ and any point p on the side $\overline{xy}$, let $\Delta(x', y', z')$ be a *comparison triangle* in $\mathbb{R}^2$, i.e. a triangle with the same side lengths, and let p' be a point on the side $\overline{x'y'}$ such that $d(p', x') = d(p, x)$. Then we require that the CAT(κ) *comparison inequality*

$$d(p, z) \leq d(p', z') \tag{1}$$

holds.

Note that the restriction on the perimeter on $\Delta(x, y, z)$ is vacuous if $\kappa \leq 0$.

A more general chord comparison is implied by (1): If q is a point on $\overline{xz}$ and q' is the corresponding point on $\overline{x'z'}$ then we have

$$d(p, q) \leq d(p', q'). \tag{2}$$

A *Hadamard space* is a *CAT(0)-space.* Other useful inequalities for Hadamard spaces can be deduced from (1), for instance the *parallelogram inequality:* Let x, y, z,

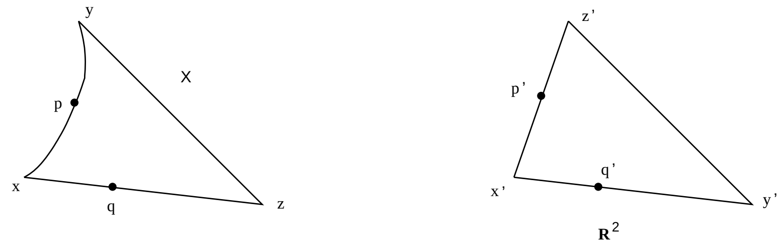

Figure 2: *Chord comparison.*

m be points in a Hadamard space X, so that m is the midpoint of $\overline{yz}$. Then:

$$4d(x,m)^2 + d(y,z)^2 \leq 2\big(d(x,y)^2 + 2d(x,z)^2\big). \tag{3}$$

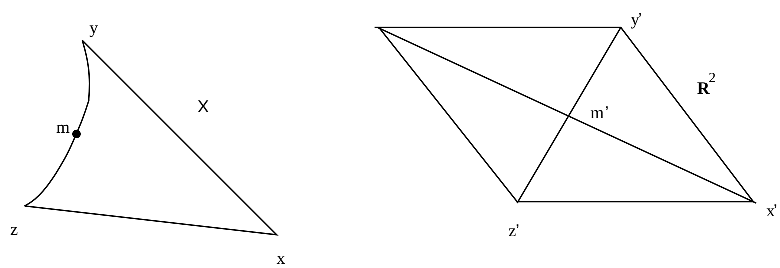

Figure 3: *Parallelogram inequality.*

An immediate consequence of the CAT(0)-property is the *convexity* of the distance function, i.e. for any two constant speed geodesic segments $\sigma_1, \sigma_2 : [a,b] \to X$, the function $t \mapsto d(\sigma_1(t), \sigma_2(t))$ is convex. It follows that any two points can be connected by a unique geodesic segment. In particular, Hadamard spaces are contractible.

Angles. The concept of *angle* generalizes as follows to the singular setting. Let X be a CAT(κ)-space. Consider two unit speed geodesic segments $\rho_1, \rho_2 : [0,\epsilon) \to X$ with the same initial point $\rho_1(0) = \rho_2(0) = p$. Their angle at p is defined by

$$\angle_p(\rho_1,\rho_2) = 2 \cdot \lim_{t \searrow 0} arcsin \frac{d(\rho_1(t), \rho_2(t))}{2t}. \tag{4}$$

The expression on the right hand side is monotone in t and the limit therefore exists. The angles define a pseudo metric on segments initiating in p, and we call the metric space $(\Sigma_p X, \angle_p)$ obtained by metric completion the *space of directions* at p. In the smooth case, $\Sigma_p X$ is the unit tangent sphere. It turns out that Σ_p is a CAT(1)-space.

If $\Delta(x,y,z)$ is a geodesic triangle in a Hadamard space and $\Delta(x',y',z')$ is a comparison triangle in Euclidean plane, one defines the *comparison angle* $\tilde{\angle}_x(y,z)$ of $\Delta(x,y,z)$ at x as $\angle_{x'}(y',z')$. (2) implies the *angle comparison* inequality

$$\angle_x(y,z) \leq \tilde{\angle}_x(y,z). \tag{5}$$

Boundary at infinity. A geodesic *ray* is an isometric embedding $\rho : [0,\infty) \to X$. By abusing notation, we will frequently identify geodesic rays with their images. We say that two rays are *asymptotic* if they have bounded Hausdorff distance from each other. This is an equivalence relation, and the set of equivalence classes of geodesic rays is called the *ideal boundary* or *boundary at infinity* $\partial_\infty X$ of a Hadamard space X. An element $\xi \in \partial_\infty X$ is an *ideal* point or a point *at infinity.* A ray representing ξ is said to be *asymptotic to* ξ. We will use the notation $\overline{x\xi}$ to denote the unique geodesic ray from $x \in X$ asymptotic to $\xi \in \partial_\infty X$.

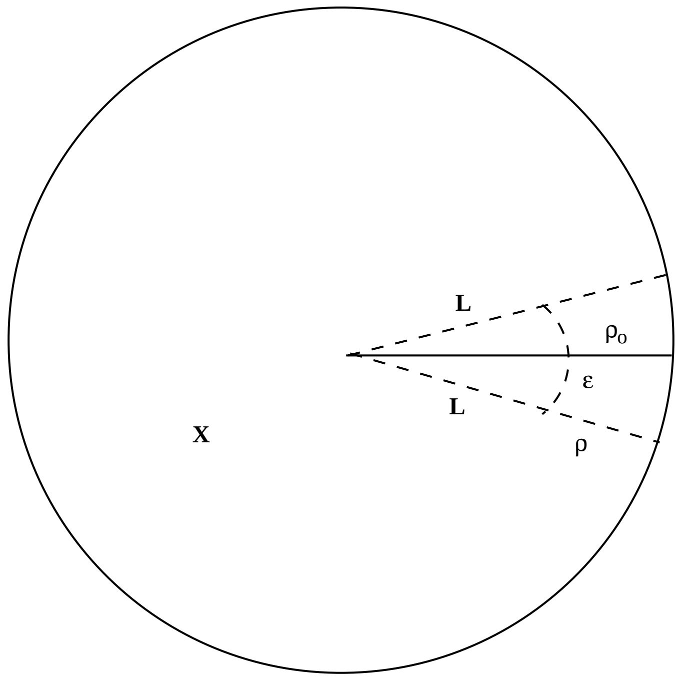

Figure 4: *Cone topology on the ideal boundary.*

The ideal boundary $\partial_\infty X$ carries several natural structures. First, there is a natural topology, called *cone topology.* For a ray $\rho_0 : [0, \infty) \to X$ and numbers $l, \epsilon > 0$ consider all rays $\rho : [0, \infty) \to X$ such that $d(\rho(t), \rho_0(t)) < \epsilon$ for $0 \le t \le l$. The ideal points represented by these rays form a subset of $\partial_\infty X$, and the family of all such subsets is a basis for the cone topology.

Second, there is a natural metric on $\partial_\infty X$. Namely, given two geodesic rays $\rho_1, \rho_2 : [0, \infty) \to X$, their *Tits angle* at infinity is defined in analogy with (4) as

$$\angle_{Tits}(\rho_1, \rho_2) = 2 \cdot \lim_{t \nearrow \infty} arcsin \frac{d(\rho_1(t), \rho_2(t))}{2t}. \tag{6}$$

The Tits angle depends only on the ideal points represented by the rays and induces the *Tits metric* $\angle_{Tits}$ on $\partial_\infty X$. The metric space $\partial_{Tits} X = (\partial_\infty X, \angle_{Tits})$ is called the *Tits boundary.* As for the spaces of directions, it turns out that the Tits boundary is a CAT(1)-space. The Tits metric is lower semicontinuous with respect to the cone topology and hence induces a topology which is finer than the cone topology. In general, it is strictly finer.

For ideal points $\xi, \eta \in \partial_{Tits} X$ and a ray ρ asymptotic to ξ holds

$$\angle_{Tits}(\xi, \eta) = \lim_{t \to \infty} \angle_{\rho(t)}(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta),$$

in particular

$$\angle_x(\xi, \eta) \le \angle_{Tits}(\xi, \eta). \tag{7}$$

Rigidity. If equality occurs in one of the comparison inequalities, we find two-dimensional flat objects isometrically embedded in a Hadamard space. The basic example for this rigidity phenomenon is:

Flat Triangle Lemma 2.1. *If equality occurs in (1) or (2) and p is an interior point of the segment $\overline{xy}$, or if equality occurs in (5), then $\Delta(x, y, z)$ bounds an isometrically embedded two-dimensional Euclidean triangle.*

This is shown by verifying that equality in (1) holds for all chords. Using uniqueness of geodesics, one then produces the embedded flat triangle.

A similar conclusion holds if equality occurs in (7). In this case, it follows that there exists an embedded flat sector bounded by the rays $\overline{x\xi}$ and $\overline{x\eta}$.

2.1.1 Asymptotic geodesics and parallel sets

The equivalence relation of asymptoticity can be refined as follows. Let us fix an ideal point $\xi \in \partial_\infty X$ and consider all rays asymptotic to ξ. We define the distance between any two such rays $\rho_1, \rho_2 : [0, \infty) \to X$ as

$$d_\xi(\rho_1, \rho_2) := \lim_{t_1, t_2 \to \infty} d(\rho_1(t_1), \rho_2(t_2)) = \inf_{t_1, t_2 \to \infty} d(\rho_1(t_1), \rho_2(t_2)) \tag{8}$$

The rays ρ_1 and ρ_2 are said to be *strongly asymptotic* if $d_\xi(\rho_1, \rho_2) = 0$. The distance (8) induces a metric on the set X_ξ^* of strong asymptote classes. In general, X_ξ^* is not complete, and we call its metric completion X_ξ the *space of strong asymptote classes* at ξ.

Example 2.2. *(i) If X is a CAT(-1) space then any two asymptotic rays are strongly asymptotic and the spaces of strong asymptote classes consist of one point.*

(ii) If X is a nonpositively curved symmetric space of rank r then the spaces of strong asymptote classes X_ξ are symmetric spaces of rank $r - 1$. Moreover, we have $X_\xi^* = X_\xi$, i.e. all points in X_ξ are represented by rays. Analogous assertions hold for Euclidean buildings.

If ρ, ρ' are asymptotic geodesic rays in a Euclidean building, then ρ, ρ' eventually bound a flat half-strip. In particular, if ρ, ρ' are strongly asymptotic then they eventually overlap.

Parallel sets. If two complete geodesics γ, γ' in a Hadamard space X have bounded Hausdorff distance, then the convex function $\mathbb{R} \to \mathbb{R}; t \mapsto d(\gamma(t), \gamma'(t))$ is bounded and therefore constant. It follows from the Flat Triangle Lemma 2.1 that γ and γ' bound an isometrically embedded flat strip. Two such geodesics are called *parallel*.

Parallelism is an equivalence relation, and one can show more generally that the union P_l of all geodesics parallel to a given geodesic l is a convex subset of X which splits as a metric product

$$P_l \cong l \times Y_l.$$

If X is a symmetric space (resp. a Euclidean building) then the parallel sets P_l are again symmetric spaces (resp. buildings) of the same rank.

A subset $Y \subset X$ is called *totally-geodesic* if it is closed and together with each pair of distinct points $y, y' \in Y$, the subset Y contains *a complete geodesic* through y, y'. Totally-geodesic subspaces $A, B \subset X$ are *parallel* if they are within finite Hausdorff distance from each other. Thus for any pair of points $a, a' \in A$ and the nearest points $b, b' \in B$, the points a, a', b', b span a totally-geodesic flat rectangle in X. It follows that there exists an isometric embedding $f : A \times [a, b] \to X$ such that $f|A \times \{a\}$ is an isometry onto A and $f|A \times \{b\}$ is an isometry onto B. The *parallel set* $P_A \subset X$ of A is the union of all totally-geodesic subspaces $B \subset X$ which are parallel to A. Then P_A is convex (since parallelism is an equivalence relation) and it splits as the metric product $A \times C$, for some convex subset $C \subset X$. The set C is the inverse image of a point $a \in A$ under the nearest-point projection $P_A \to A$. If X is a symmetric space or a Euclidean building then for each symmetric subspace (resp. subbuilding) $A \subset X$, the parallel set P_A is a symmetric subspace (resp. subbuilding) in X. See [Eb, 1.6.7] for a more detailed discussion of parallel sets in the Riemannian case.

2.1.2 Busemann functions

Busemann functions measure the relative distance from points at infinity. Namely, consider an ideal point $\xi \in \partial_\infty X$. Using a ray $\rho : [0, \infty) \to X$ asymptotic to ξ, we define the *Busemann function* b_ξ as the monotone limit:

$$b_\xi(x) := \lim_{t \to \infty} (d(x, \rho(t)) - t)$$

One checks that, up to an additive constant, b_ξ does not depend on the chosen ray ρ. As a limit of convex functions, b_ξ is convex. Note that along every ray ρ asymptotic to ξ holds

$$b_\xi \circ \rho(t) = -t + const.$$

Example 2.3. *Consider Euclidean space $\mathbb{R}^n$. The Busemann functions on $\mathbb{R}^n$ are linear functions with unit gradient vectors. If $\rho : [0, \infty) \to \mathbb{R}^n$ is a ray asymptotic to ξ then $-\nabla b_\xi$ is parallel to ρ.*

More generally, if $F \subset X$ is a flat and if $\xi \in \partial_\infty F \subset \partial_\infty X$, then the Busemann function b_ξ restricts to a linear function on F.

Lemma 2.4. *Let $x, y \in X$ and $\eta \in \partial_\infty X$. Then*

$$b_\eta(y) - b_\eta(x) \leq -\cos \angle_x(y, \eta) \cdot d(x, y). \tag{9}$$

In the case of equality there exists an isometrically embedded flat half-strip bounded by the rays $\overline{x\eta}$, $\overline{y\eta}$ and the segment $\overline{xy}$.

Proof: Let $\rho : [0, \infty) \to X$ be a parameterization for the ray $\overline{x\eta}$. Looking at the comparison triangle for $\Delta(x, y, \rho(t))$ we obtain

$$d(x, y) \cdot \cos \tilde{\angle}_x(y, \rho(t)) + d(y, \rho(t)) \geq d(x, \rho(t)).$$

Combining this with the angle comparison (5) yields

$$d(x, y) \cdot \cos \angle_x(y, \rho(t)) + d(y, \rho(t)) \geq d(x, \rho(t)).$$

The assertion follows because $d(\rho(t), x) - d(\rho(t), y) \to b_\eta(x) - b_\eta(y)$ as $t \to \infty$. This proves the inequality.

We omit the proof of rigidity in the equality case. It is not hard to derive it from the Flat Triangle Lemma 2.1. $\square$

Lemma 2.5. *(See Lemma 6.2 in Appendix.) Busemann functions on buildings are eventually linear. More precisely, if X is a Euclidean building, $\eta, \xi \in \partial_\infty X$, ρ is a ray in X asymptotic to ξ, then for certain $T \geq 0$ the restriction of b_η to $\rho([T, \infty))$ is a linear function.*

For a Busemann function b_ξ on a Hadamard space X define the *gradient vector* $\nabla(-b_\xi)(x)$ at a point $x \in X$ to be the direction τ in $\Sigma_x X$ tangent to (i.e. represented by) the geodesic ray $\overline{x\xi}$. We note that in the case when X is a Riemannian manifold, the function b_ξ is twice differentiable on X, and τ is indeed the gradient of $-b_\xi$ at x.

Let $p : [0, T] \to X$ be a geodesic segment with the image $\overline{xy}$ so that $p(0) = x$, $p(T) = y$. For $t \in [0, T)$ let $p'(t) \in \Sigma_{p(t)} X$ denote the velocity vector of p, i.e. the direction in $\Sigma_{p(t)} X$ represented by the segment $\overline{p(t)y}$.

Lemma 2.6. *We have:*

$$-b_\xi(y) + b_\xi(x) = \int_0^T \cos \angle\big(p'(t), \nabla(-b_\xi)(p(t))\big) dt.$$

Proof: Pick a geodesic ray ρ in X which is asymptotic to ξ.

Given a point $z \in \rho \subset X$ we first analyze the increment of the distance function $f(t) := f_z(t) = d(p(t), z)$ along the geodesic segment $\overline{xy}$. Recall that the function $f(t)$ is convex 1-Lipschitz, hence it is absolutely continuous and

$$f(T) - f(0) = \int_0^T f'(t) dt.$$

For $0 \leq t < T$ let $\alpha_z(t) = \angle_{p(t)}(y, z)$. We have:

$$-\cos(\alpha(t)) = f'_+(t)$$

(the first variation formula, [BrHae, page 185]). Therefore

$$f_z(T) - f_z(0) = -\int_0^T \cos(\alpha_z(t)) dt. \tag{10}$$

We now let z converge to ξ along ρ, then

$$\lim_{z \to \xi} (f_z(T) - f_z(0)) = b_\xi(y) - b_\xi(x),$$

$$\lim_{z \to \xi} \alpha_z(t) = \angle_{p(t)}(y, \eta) = \angle\big(p'(t), \nabla(-b_\xi(p(t))\big),$$

and, by applying Lebesgue convergence theorem to (10) as z converges to ξ along ρ, we get:

$$-b_\xi(y) + b_\xi(x) = \int_0^T \cos(\angle_{p(t)}(y, \xi)) dt = \int_0^T \cos\big(\angle(p'(t), \nabla(-b_\xi)(p(t))\big). \quad \square$$

Busemann functions on products.

Suppose that a Hadamard space X splits isometrically as $X = X_1 \times X_2$. Recall that the Tits boundary $\partial_{Tits}X$ of X is the metric join $\partial_{Tits}X_1 \circ \partial_{Tits}X_2$, where the Tits distance between any $\xi_1 \in \partial_{Tits}X_1$ and $\xi_2 \in \partial_{Tits}X_2$ equals $\pi/2$. We thus have natural projections

$$p_i : \partial_{Tits}X \setminus \partial_{Tits}X_{3-i} \to \partial_{Tits}X_i$$

which extend of the orthogonal projections $X \to X_i$, $i = 1, 2$. Pick base-points $o_i \in X_i$ and let $o = (o_1, o_2)$ be the base-point in X. In what follows we will normalize all Busemann functions to be zero at the points o_1, o_2 and o.

Lemma 2.7. *Let $\xi \in \partial_{Tits}X$, and assume that $\xi_i = p_i(\xi) \in \partial_{Tits}X_i$, $i = 1, 2$. Let α denote the angle $\angle_{Tits}(\xi_1, \xi) = \alpha$. Then*

$$b_\xi = \cos(\alpha) b_{\xi_1} + \sin(\alpha) b_{\xi_2}. \tag{11}$$

Proof: We first note that the equality (11) holds at o because of the normalization of the Busemann functions. Pick $z \in X$. There exists a 2-flat $E \subset X$ which contains z and so that $\xi_1, \xi_2 \in \partial_\infty E$. Then the restrictions of the Busemann functions b_ξ, b_{ξ_1} and b_{ξ_2} to E are Euclidean Busemann functions on $\mathbb{R}^2 = E$; hence they are linear functions and we have:

$$\nabla(-b_\xi)(x) = \cos(\alpha)\nabla(-b_{\xi_1})(x) + \sin(\alpha)\nabla(-b_{\xi_2})(x), \forall x \in E. \tag{12}$$

To prove (11) at the point $z \in X$ consider the geodesic segment $\overline{oz} \subset X$. Since for each $x \in \overline{oz}$ we have (12) and

$$0 = b_\xi(o) = \cos(\alpha) b_{\xi_1}(o) + \sin(\alpha) b_{\xi_2}(o),$$

Lemma 2.6 implies that

$$b_\xi(z) = \cos(\alpha) b_{\xi_1}(z) + \sin(\alpha) b_{\xi_2}(z). \quad \Box$$

Pick a geodesic $l \subset X_1$ asymptotic to $\eta \in \partial_\infty X_1$ and set $L := l \times o_2 \subset X$.

Lemma 2.8. *1. For each $x \in X$, $d(x, o) \geq -b_\eta(x)$.*

2. Fix $r < \infty$. For $t \in \mathbb{R}$ let $x_t \in X$ be such that $d(x_t, L) \leq r$ and $-b_\eta(x_t) = t$. Then

$$\lim_{t \to \infty} d(x_t, o) + b_\eta(x_t) = 0.$$

Proof: Let $x \in X$, $t := -b_\eta(x)$. Consider the 2-flat E in X which contains L and x. Then

$$d(x, o)^2 = t^2 + d(x, L)^2.$$

Hence $d(x, o) \geq t$ which implies (1). To get the second assertion note that for $x := x_t$:

$$0 \leq d(x, o) + b_\eta(x) = d(x, o) - t \leq \frac{r^2}{d(x, o) + t}.$$

Hence $d(x_t, o) + b_\eta(x_t)$ tends to zero as $t \to \infty$. $\Box$

2.1.3 A fixed point theorem for weakly contracting maps

The following result extends Cartan's theorem about the existence of fixed points for isometric actions on Hadamard manifolds to weakly contracting maps of Hadamard spaces. Our argument is a variation of the original one.

Proposition 2.9. *Let X be a Hadamard space and $\Phi : X \to X$ a 1-Lipschitz self map. If the forward orbits $(\Phi^n p)_{n\geq 0}$ are bounded then Φ has a fixed point in X.*

Proof: Consider an orbit $y_n = \Phi^n y_0$ of a point $y_0 \in X$ and define the distance from its "tail" by

$$r(x) := \limsup_{n\to\infty} d(y_n, x).$$

Note that r inherits from the distance function the convexity and the 1-Lipschitz continuity. The assumption that Φ is 1-Lipschitz implies

$$r(\Phi x) = \limsup_{n\to\infty} d(y_n, \Phi x) = \limsup_{n\to\infty} d(\Phi y_{n-1}, \Phi x) \leq \limsup_{n\to\infty} d(y_{n-1}, x) = r(x),$$

that is,

$$r \circ \Phi \leq r. \tag{13}$$

It suffices to show that r has a unique minimum since this would then be a fixed point of Φ. We denote

$$\rho := \inf_X r.$$

For $\epsilon > 0$, let x, x' be points with $\rho(x) = \rho(x') < \rho + \epsilon$. Then there exists n_0 such that for $n \geq n_0$ we have

$$d(y_n, x), d(y_n, x') < \rho + \epsilon.$$

On the other hand, let m be the midpoint of $\overline{xx'}$. Since $r(m) \geq \rho$, we have

$$d(y_n, m) > \rho - \epsilon$$

for infinitely many n. We apply the parallelogram inequality (3) to these points y_n and obtain

$$d(x, x')^2 + 4\underbrace{d(y_n, m)}_{>\rho-\epsilon}^2 \leq 2\Big(\underbrace{d(y_n, x)}_{<\rho+\epsilon}^2 + \underbrace{d(y_n, x')}_{<\rho+\epsilon}^2\Big)$$

and

$$d(x, x')^2 < 16\rho\epsilon + 8\epsilon^2. \tag{14}$$

It follows that any sequence (x_k) with $r(x_k) \searrow \rho$ must be a Cauchy sequence. The completeness of X implies that r attains its minimum. If r attains it minimum at points x, x' then for each $\epsilon > 0$ we have the inequality (14); it follows that $x = x'$ and thus the minumum of r is unique. $\square$

2.2 Coxeter complexes

In the following two sections we introduce a special class of metric spaces: *metric spaces modelled on Coxeter complexes.* This class of spaces is flexible enough to include infinitesimal symmetric spaces, nonpositively curved symmetric space and buildings. We refer to [KlLe, section 3] for a more detailed discussion.

2.2.1 Spherical Coxeter complexes

A *spherical Coxeter complex* (S, W_{sph}) consists of a unit sphere S and a finite subgroup $W_{sph} \subset \mathrm{Isom}(S)$ generated by reflections. By a reflection, we mean a reflection at a great sphere of codimension one. W_{sph} is called *Weyl group*, and the fixed point sets of the reflections in W_{sph} are called *walls.*

S carries a natural structure of a cellular (polysimplicial) complex. The top-dimensional cells, the *chambers*, are fundamental domains for the action $W_{sph} \curvearrowright S$. If convenient, we identify the *spherical model Weyl chamber* $S/W_{sph} = \Delta_{sph}$ with one of the chambers in S.

An *embedding* $(S, W_{sph}) \hookrightarrow (S', W'_{sph})$ of spherical Coxeter complexes consists of an isometric embedding $S \to S'$ and a compatible monomorphism $W_{sph} \to W'_{sph}$ which sends reflections to reflections.

We will only be interested in those spherical Coxeter complexes which are attached to root systems, see [Se, chapter V.15] for their classification. We call these spherical Coxeter complexes *algebraic.*

2.2.2 Euclidean Coxeter complexes

A *Euclidean Coxeter complex* (E, W_{aff}) consists of a Euclidean space E and a subgroup $W_{aff} \subset \mathrm{Isom}(E)$ generated by reflections at hyperplanes. We require moreover that the induced reflection group on the sphere $\partial_{Tits}E$ at infinity is finite, and that the associated spherical Coxeter complex $(\partial_{Tits}E, W_{sph})$ is algebraic. Here $W_{sph} := rot(W_{aff})$ where $rot : \mathrm{Isom}(E) \to \mathrm{Isom}(\partial_{Tits}E)$ is the natural homomorphism mapping an affine transformation to its linear part.

A *wall* in the Coxeter complex (E, W_{aff}) is a hyperplane fixed by a reflection in W_{aff}. *Singular subspaces* are defined as intersections of walls, and *vertices* are zero-dimensional singular subspaces. In the case of reducible Coxeter complexes it could happen that there are no zero-dimensional singular subspaces in E. In this case we declare *all* points of the singular subspaces of the smallest dimension to be vertices of E. We let $E^{(0)}$ denote the vertex set of E.

We denote the kernel of $rot : W_{aff} \to W_{sph}$ by L_{trans} and we refer to it as the *translation subgroup.* The exact sequence $0 \to L_{trans} \to W_{aff} \to W_{sph} \to 1$ splits, i.e. the affine Weyl group decomposes as the semidirect product $W_{aff} \cong W_{sph} \ltimes L_{trans}$. A vertex of E is called *special* if its stabilizer maps onto W_{sph} via rot.

An *embedding* $(E, W_{aff}) \hookrightarrow (E', W'_{aff})$ of Coxeter complexes consists of an affine embedding $E \to E'$ and a compatible monomorphism $W_{aff} \to W'_{aff}$ which sends reflections to reflections. If E is irreducible then $E \to E'$ must be a homothetic embedding.

We will be interested in Euclidean Coxeter complexes attached to symmetric spaces of nonpositive curvature and Euclidean buildings. In the case of symmetric spaces of noncompact type, the affine Weyl group contains the full group of translations of E.

A Euclidean Coxeter complex (E, W_{aff}) is called *discrete* if W_{aff} is a discrete subgroup of $\mathrm{Isom}(E)$. This includes the degenerate case when $L_{trans} = 0$ and $W_{aff} \cong W_{sph}$. We assume now that L_{trans} acts cocompactly on E. There is a polyhedral fundamental domain Δ_{aff} for the action $W_{aff} \curvearrowright E$, called a *Weyl alcove.* The group W_{aff} is generated by the reflections at the faces of Δ_{aff}. There is a tessellation of E by the W_{aff}-images of the Weyl alcove. This defines a natural structure of polysimplicial cell complex on E.

Lemma 2.10. *The stabilizer $Stab_{W_{aff}}(x)$ of a point $x \in E$ is generated by the reflections at the walls through x.*

Proof: The tessellation of E induces a tessellation of the unit tangent sphere $\Sigma_x E$ at x. It is the tessellation by fundamental domains for the group generated by the reflections at walls through x. □

2.3 Geometries modelled on Coxeter complexes

Fix a spherical or Euclidean Coxeter complex (A, W). Let Z be a metric space. A *geometric structure* on Z *modelled on* (A, W) consists of an atlas of isometric embeddings $\phi : A \hookrightarrow Z$ satisfying the following compatibility condition: For any two charts ϕ_1 and ϕ_2, the transition map $\phi_2^{-1} \circ \phi_1$ is the restriction of an isometry in W. The images of these charts are called *apartments.* We will require that there are *plenty of apartments* in the sense that any two points in Z lie in a common apartment. All W-invariant notions introduced for the Coxeter complex (A, W), such as walls, singular subspaces, chambers etc., carry over to geometries modelled on (A, W). The *rank* of X is the dimension of A.

Examples of such geometries are provided by symmetric spaces of nonpositive curvature and *infinitesimal symmetric spaces.* These are modelled on Euclidean Coxeter complexes with transitive affine Weyl group (provided that the symmetric spaces in question have noncompact type, i.e. no flat deRham factors). In the case of a symmetric space X, the apartments are the maximal flats. The associated Coxeter complex has the form (E, W_{aff}) where E is an apartment and W_{aff} is the group generated by reflections at singular hyperplanes.

A Cartan motion space is a Euclidean space Y which is given a structure of a space modelled on a Euclidean Coxeter complex (A, W) such that the group of automorphisms of Y acts transitively on the chambers. In the present paper we consider only those Cartan motion spaces which are isomorphic to *infinitesimal symmetric spaces.* Namely, consider a nonpositively curved symmetric space X. The (E, W_{aff})-structure on X induces a (E, W_{aff})-structure on $Y := \mathfrak{p} = T_p X$ such that the apartments in $\mathfrak{p}$ are the translates of the Cartan subspaces (i.e. the maximal abelian subalgebras) in $\mathfrak{p}$. The apartments through 0 in $\mathfrak{p}$ are the tangent spaces to the apartments through p in X. The chamber-transitive group of automorphisms of Y is the semidirect product $K \ltimes \mathfrak{p}$ where K is the stabilizer of p in $\mathrm{Isom}(X)$ and $\mathfrak{p}$ acts on Y by translations.

Then Y with this structure of a Cartan motion space will be called an *infinitesimal symmetric space.*

Remark 2.11. *Each Euclidean space E can be given structure of a space modelled on* $(\mathbb{R}, \mathbb{Z}/2 \ltimes \mathbb{R})$. *The group of automorphisms of such a space is* $\mathrm{Isom}(E)$, *which is chamber-transitive.*

The third kind of geometry considered in this paper consists of spherical and Euclidean buildings.

Definition 2.12. *A* spherical building *is a CAT(1)-space modelled on a spherical Coxeter complex.*

Spherical buildings have a natural structure as polysimplicial piecewise spherical complexes. We prefer the geometric to the combinatorial view point because it appears to be more flexible.

Definition 2.13. *A discrete Euclidean building is CAT(0)-space modelled on a discrete Euclidean Coxeter complex.*

In the non-discrete case the definition of Euclidean buildings is more subtle (it requires one more axiom), see [KlLe, section 4.1.2]. We refer to [KlLe] for a thorough discussion of buildings from the geometric viewpoint.

A building is called *thick* if every wall is an intersection of apartments. A non-thick building can always be equipped with a natural structure of a thick building by reducing the Weyl group.

Trivial Example 2.14. *A 0-dimensional spherical building modelled on* $(\mathrm{S}^0, \mathbb{Z}_2)$ *is a discrete metric space where any two distinct points have distance* π.

A 1-dimensional Euclidean building modelled on $(\mathbb{R}, D_\infty)$ *is a simplicial tree. Here* D_∞ *denotes the infinite dihedral group* $\mathbb{Z}/2 \ltimes \mathbb{Z}$.

Example 2.15. *If X is a symmetric space of nonpositive curvature or a thick Euclidean building modelled on the Coxeter complex (E, W_{aff}), then its ideal boundary $\partial_{Tits}X$ is a thick spherical building modelled on $(\partial_{Tits}E, W_{sph})$. In the case that X is a building, the spaces of directions $\Sigma_x X$ are spherical buildings modelled on $(\partial_{Tits}E, W_{sph})$. The building $\Sigma_x X$ is thick if and only if x is a special vertex of X.*

Let B be a spherical building modelled on (S, W_{sph}). The quotient map $S \to S/W_{sph} \cong \Delta_{sph}$ induces a canonical projection $\theta : B \to \Delta_{sph}$ folding the building onto its model Weyl chamber. The θ-image of a point in B is called its *type*.

Two points ξ and η in B are called *antipodal* if they have maximal distance π. If rank of B is ≥ 1 then the union of all geodesic segments between ξ and η forms a subbuilding $B(\xi, \eta)$ which is a spherical suspension.

2.4 Polygons and side lengths

An *n-gon* $z_1 \ldots z_n$ in a metric space Z is a map $\mathbb{Z}/n\mathbb{Z} \to Z$ carrying i to the *vertex* z_i. If Z is a geodesic metric space, we define a *geodesic polygon* as a polygon $z_1 \ldots z_n$ together with a choice of *sides* $\overline{z_i z_{i+1}}$.

Let (A, W) be a spherical or Euclidean Coxeter complex. The complete invariant of a pair of points (x, y) is its image $\sigma_{ref}(x, y)$ under the canonical projection to $A \times A/W$. We define the *refined length* of a geodesic segment $\overline{xy}$ as $\sigma_{ref}(x, y)$. This notion carries over to geometries modelled on the Coxeter complex (A, W): For a pair of points (x, y) pick an apartment a containing x, y and, after identifying a with A, let $\sigma_{ref}(x, y)$ be the projection to $A \times A/W$.

For a *Euclidean* Coxeter complex (E, W_{aff}) there is a coarser notion of Δ-*length* obtained from composing σ_{ref} with the natural forgetful map

$$E \times E/W_{aff} \to E/W_{sph} \cong \Delta.$$

Here we regard E as a vector space, and the origin 0 is chosen as a special vertex, i.e. $Stab_{W_{aff}}(0) \cong W_{sph}$. To compute the Δ-length $\sigma(x, y)$ we regard the oriented geodesic segment $\overline{xy}$ as a vector in E and project it to Δ.

Again, the concept of Δ-length carries over to the geometries modelled on (E, W_{aff}). Note that Δ-length and refined length coincide for symmetric spaces and infinitesimal symmetric spaces of noncompact type because the affine Weyl group acts transitively.

Suppose now that the Euclidean Coxeter complex (E, W_{aff}) is *discrete*, and we have a lattice L in E such that

$$L_{trans} \subset L \subset N_{aff}$$

where N_{aff} is the normalizer of W_{aff} in the full group of translations in E. We identify the orbit $L \cdot 0$ with L. Then we define the sets of L-*integral* Δ-lengths and L-*integral* refined lengths as the subsets

$$\Delta \cap L \subset \Delta$$

and $(L \times L)/W_{aff} \subset (E \times E)/W_{aff}$, respectively. Note that a segment with L-integral Δ-length has L-integral refined length iff its vertices lie in the distinguished orbit $W_{aff} \cdot 0$. For segments with endpoints of type $W_{aff} \cdot 0$ the notions of Δ-length and refined length are equivalent.

3 Transfer of polygons between buildings

3.1 Proof of Theorem 1.3

In this section we prove that in thick (spherical or Euclidean) buildings X and X' with isomorphic Coxeter complexes, the same refined side lengths occur for polygons, cf. Theorem 1.3. To prove this, we will transfer a polygon P in X to a polygon P' in X' preserving the refined lengths of their edges. The idea is to consider a "ruled surface" spanned by the polygon P. It is obtained by choosing a point, say a vertex of P, at which we cone P off. This ruled surface is a finite complex. It decomposes into finitely many triangles each of which is contained in an apartment, cf. Lemma 3.1. We develop the ruled surface into X' by an inductive process and obtain a ruled surface in X' spanned by a polygon with the same refined side lengths as P.

Lemma 3.1. *Let X be a Euclidean or a spherical building, and let $\Delta(x,y,z) \subset X$ be a geodesic triangle which is not contained in an apartment. Then there is a finite subdivision of the edge $\overline{xy}$ by points $x_0 = x, x_1, ..., x_{k-1}, x_k = y$ such that each geodesic triangle $\Delta(z, x_i, x_{i+1})$ is contained in an apartment.*

Proof: We first consider the case when X is a spherical building. Notice that the edge $\overline{xy}$ contains no points antipodal to z because otherwise $\Delta(x,y,z)$ would be contained in a great circle. We choose the subdivision of $\overline{xy}$ so that each subsegment $\overline{x_i x_{i+1}}$ is contained in a chamber Δ_i. This is possible because $\overline{xy}$ is contained in an apartment. Each chamber Δ_i is contained in an apartment through z. This proves the lemma in the case of spherical buildings.

In the case when X is a Euclidean building the lemma was proven in [KlLe, Corollary 4.6.8]. $\square$

Proof of Theorem 1.3. We start with the case of spherical buildings. It will suffice to treat the case of triangles. The general case follows by induction on the number of edges.

We consider thick spherical buildings B and B' with associated spherical Coxeter complexes (A, W) and (A', W'), respectively. We are given an embedding $\iota : (A, W) \hookrightarrow (A', W')$ of Coxeter complexes. The transfer of triangles from B to B' proceeds by induction on the rank of the building B.

The case of $rank(B) = 0$ is trivial. We explain the inductive step from rank $r-1$ to rank r. Suppose $\Delta(x,y,z)$ is a geodesic triangle in B. We subdivide the edge $\overline{xy}$ as in Lemma 3.1. We need to find triangles $\Delta(z', x'_i, x'_{i+1})$ in B' such that for all i, the refined side lengths of $\Delta(z', x'_i, x'_{i+1})$ are the ι-images of the refined side lengths of $\Delta(z, x_i, x_{i+1})$, and $\angle_{x'_i}(x'_{i-1}, x'_{i+1}) = \pi$. This will be done inductively. Suppose that $\Delta(z', x'_{i-1}, x'_i)$ has been found. In order to find the direction $\overrightarrow{x'_i x'_{i+1}}$ at x'_i, we look at the points x_i and x'_i. The links $\Sigma_{x_i} B$ and $\Sigma_{x'_i} B'$ are thick spherical buildings, and ι induces an embedding of their Coxeter complexes. $\Sigma_{x_i} B$ has rank $r-1$ and we may apply the induction hypothesis to transfer the triangle $\Delta(\overrightarrow{x_i x_{i-1}}, \overrightarrow{x_i z}, \overrightarrow{x_i x_{i+1}})$ in $\Sigma_{x_i} B$ to a triangle $\Delta(\overrightarrow{x'_i x'_{i-1}}, \overrightarrow{x'_i z'}, \overrightarrow{x'_i x'_{i+1}})$ in $\Sigma_{x'_i} B'$. We choose an apartment in B' containing $\overline{z' x'_i}$ and tangent to the direction $\overrightarrow{x'_i x'_{i+1}}$. Inside this apartment there is a unique choice for x'_{i+1} with the desired properties. After transferring all triangles, the concatenation of the segments $\overline{x_i x_{i+1}}$ forms a geodesic segment $\overline{x'y'}$ whose refined length is the ι-image of the refined length of $\overline{xy}$. This concludes the proof in the spherical case.

The same argument works in the Euclidean case, applying the result for spherical buildings of one rank lower. $\square$

Note that one can apply Theorem 1.3 to automorphisms or, more generally, to self embeddings $(A, W) \hookrightarrow (A, W)$.

Question 3.2. *Suppose that X, X' are spherical or Euclidean buildings with isomorphic Coxeter complexes. Is it true that for each finite configuration $F = \{x_1, ..., x_m\} \subset X$ there exists an embedding $h : F \to X'$ which preserves the refined lengths of the segments $\overline{x_i x_j}$, $1 \le i, j \le m$? Note that it is not even clear if one can construct a map h which preserves the ordinary distances.*

3.2 A variation: Folding polygons into apartments

We describe a modification of the transfer construction which produces from triangles in a building broken triangles in an apartment.

Suppose that $\Delta(x,y,z)$ is a triangle in a (Euclidean or spherical) building B. In general it is not contained in an apartment. Consider the subdivision of the side $\overline{yz}$ as in Lemma 3.1. For each i, let a_i be an apartment containing $\Delta(z,x_i,x_{i+1})$. We will produce points x''_i in the first apartment $a=a_0$ such that the triangles $\Delta(z,x''_i,x''_{i+1})$ are congruent to the triangles $\Delta(z,x_i,x_{i+1})$ via apartment isomorphisms $\alpha_i : a_i \to a$. This is done inductively as follows. We start with $x''_0 = x_0$ and $x''_1 = x_1$. Suppose that x''_i has been constructed. To find x''_{i+1}, choose $\alpha_i : a_i \to a$ so that it carries $\overline{zx_i}$ to $\overline{zx''_i}$. Put $x''_{i+1} = \alpha_i(x_{i+1})$. The procedure yields a *broken triangle* in the apartment a consisting of two geodesic sides $\overline{zx''_0}$ and $\overline{zx''_k}$ with the same refined lengths as the corresponding sides of the original triangle $\Delta(x,y,z)$, and one piecewise geodesic path $x''_0 x''_1 \dots x''_k$. The refined lengths of $\overline{x_i x_{i+1}}$ and $\overline{x''_i x''_{i+1}}$ are equal. Hence, at every break point x''_i, the directions $\overrightarrow{x''_i x''_{i-1}}$ and $\overrightarrow{x''_i x''_{i+1}}$ in the spherical Coxeter complex $\Sigma_{x''_i} a$ are antipodal modulo the action of the stabilizer of x''_i in the Weyl group of a.

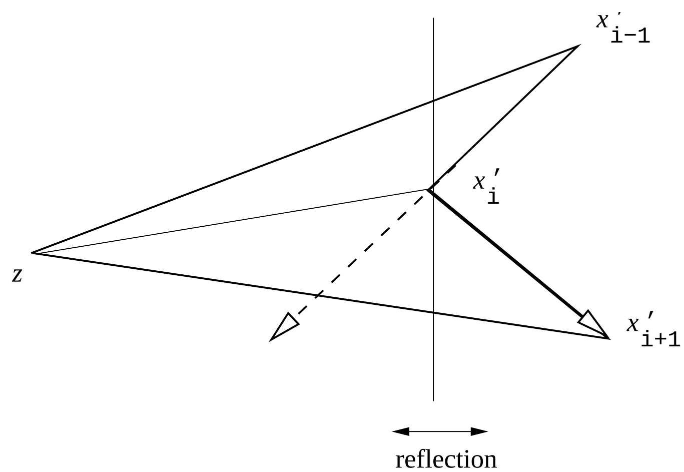

Figure 5: *A broken triangle.*

A broken triangle in the apartment can be unfolded to a geodesic triangle in the building if and only if, for each break point x''_i, $0 < i < k$, the following holds. Let $\xi''_i, \eta''_i, \zeta''_i \in \Sigma_{x''_i} a$ be the directions towards x''_{i-1}, x''_{i+1}, z. The necessary and sufficient condition is that there exists a triangle $\Delta(\xi_i, \eta_i, \zeta_i)$ in the spherical building $\Sigma_{x''_i} B$ so that $\angle(\xi_i, \eta_i) = \pi$ and the refined lengths of $\overline{\xi_i \zeta_i}$ and $\overline{\eta_i \zeta_i}$ are the same as for $\overline{\xi''_i \zeta''_i}$ and $\overline{\eta''_i \zeta''_i}$. Notice that in general the buildings $\Sigma_{x''_i} B$ may have smaller Weyl groups than W_{sph} with respect to their natural structure as *thick* spherical buildings. Namely, their Weyl groups are the stabilizers of x''_i in the Weyl group of a.

4 Polygons and weighted configurations at infinity

The goal of this section is to establish a correspondence between geodesic polygons in a space X (which is either a symmetric space, a Cartan motion space or a Euclidean building) and *weighted configurations* on the Tits boundary of X. After defining semistable (as well as stable and nice semistable) weighted configurations we discuss their properties and describe a certain convex polyhedral cone Δ^n_{ss} in Δ^n (here Δ is a model Euclidean Weyl chamber of X). The cone Δ^n_{ss} consists of *types* of semistable weighted configurations on the Tits boundary of X, where X is an (infinitesimal) symmetric space corresponding to a complex reductive Lie group. Then to geodesic polygons P in X we associate *semistable* weighted configurations (Gauss maps) on $\partial_{Tits}X$ via a construction analogous to the Gauss map for smooth surfaces in $\mathbb{R}^3$. The most difficult part of this section is to establish a correspondence in the opposite direction: given a nice semistable configuration ψ on the Tits boundary of X (in case of Euclidean buildings a semistable configuration suffices) we prove existence of a geodesic polygon in X for which ψ is a Gauss map. This is done under the assumption that X is either locally compact or is a Euclidean building which is a cone over a spherical building. These results in conjunction with the Transfer Theorem 1.3 allow us to prove Theorems 1.4 and 1.6 stated in the Introduction.

4.1 Weighted configurations at infinity and stability

In this section we review some material from [LeMi] concerning measures and weighted configurations on the Tits boundaries of $CAT(0)$-spaces. Proofs of most of these results are presented in the Appendix.

Let X be a nonpositively curved symmetric space or a Euclidean building. A collection of weights $m_1, ..., m_n \geq 0$ and ideal points $\xi_1, ..., \xi_n \in \partial_\infty X$ determines a *weighted configuration*

$$\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_\infty X$$

at infinity. Here the measure ν on $\mathbb{Z}/n\mathbb{Z}$ is defined by $\nu(i) = m_i$. The *type* $\tau(\psi) = (\tau_1, \ldots, \tau_n) \in \Delta^n$ of the weighted configuration ψ is given by $\tau_i = m_i \cdot \theta(\xi_i)$ where we identify Δ_{sph} with the unit vectors in Δ. The canonical projection $\theta : \partial_{Tits}X \to \Delta_{sph}$ has been defined in section 2.3.

The *weighted Busemann function* on X associated with the pushed forward measure $\mu = \psi_* \nu = \sum_{i=1}^n m_i \delta_{\xi_i}$ is given by

$$b_\mu = \sum_{i=1}^n m_i b_{\xi_i}.$$

It is well defined up to an additive constant. Weighted Busemann functions are convex. We define the *asymptotic slope* of b_μ in the direction of an ideal point η as

$$\text{slope}_\mu(\eta) = \lim_{t \to \infty} \frac{b_\mu(\rho(t))}{t}$$

where $\rho(t)$ is a geodesic ray in X asymptotic to η. The asymptotic slope is given by the formula (see the Appendix):

$$\text{slope}_\mu(\eta) = -\sum_{i=1}^n m_i \cos \angle_{Tits}(\xi_i, \eta). \tag{15}$$

Let B be a spherical building with the metric denoted by $\angle_{Tits}$. Motivated by (15) we define the *slope* of a measure μ on B with finite total mass $|\mu|$ as

$$\mathrm{slope}_\mu(\eta) = -\int_B \cos\angle_{Tits}(\xi,\eta)\, d\mu(\xi).$$

In this paper we will only be interested in finitely supported measures.

Definition 4.1 (Stability). *The measure μ on B is called* semistable *if* $\mathrm{slope}_\mu(\eta) \geq 0$ *and* stable *if* $\mathrm{slope}_\mu(\eta) > 0$ *for all $\eta \in B$.*

We recall some properties of the slope function established in [LeMi, Section 2.2] (see also the Appendix): The function slope_μ is Lipschitz continuous with Lipschitz constant $|\mu|$. The set $\{\mathrm{slope}_\mu \leq 0\}$ is convex in the sense that, with any two points ξ_1 and ξ_2 satisfying $\angle_{Tits}(\xi_1,\xi_2) < \pi$, it contains the segment $\overline{\xi_1\xi_2}$. The function slope_μ is convex on $\{\mathrm{slope}_\mu \leq 0\}$ and strictly convex on $\{\mathrm{slope}_\mu < 0\}$. If μ is a semistable measure then $\{\mathrm{slope}_\mu = 0\}$ is a subcomplex with respect to the canonical cell structure on B.

There is a refinement of the notion of semistability motivated by a corresponding concept in geometric invariant theory.

Definition 4.2 (Nice semistability). *A measure μ on B is called* nice semistable *if μ is semistable and $\{\mathrm{slope}_\mu = 0\}$ is a subbuilding or empty. In particular, stable measures are nice semistable.*

For the purposes of this paper, i.e. the study of polygons, nice semistability plays only a role in the case of symmetric spaces and infinitesimal symmetric spaces, and not in the case of Euclidean buildings. Therefore we will limit our discussion of nice semistability to measures on the Tits boundaries of symmetric spaces. There, nice semistability of a measure is equivalent to the existence of a minimum for the associated weighted Busemann function, see Lemma 6.11 in the Appendix. In Proposition 4.5 below, we will describe structure of nice semistable measures.

A weighted configuration ψ on B is called *stable, semistable* or *nice semistable*, respectively, if the corresponding measure $\psi_*\nu$ has this property.

Example 4.3. *(i) Let B be a spherical building of rank 0. Then a measure μ on B is stable iff it contains no atoms of mass $\geq \frac{1}{2}|\mu|$, semistable iff it contains no atoms of mass $> \frac{1}{2}|\mu|$, and nice semistable iff it is either stable or consists of two atoms of the same mass.*

(ii) Suppose that B is a unit sphere and regard it as the ideal boundary of a Euclidean space E, $B = \partial_{Tits}E$. Then all weighted Busemann functions are (affine) linear. Thus a semistable measure μ has slope zero everywhere.

(iii) Let B be an arbitrary spherical building, and let $\xi,\eta \in B$ be antipodal. Suppose that the measure μ is supported in the suspension $B(\xi,\eta)$. Then $\mathrm{slope}_\mu(\xi) + \mathrm{slope}_\mu(\eta) = 0$ *and hence μ cannot be stable. If μ is semistable then* $\mathrm{slope}_\mu(\xi) = \mathrm{slope}_\mu(\eta) = 0$.

Projecting measures on spherical joins. Consider a spherical building which decomposes as a spherical join $B = B_1 \circ B_2$ of spherical buildings. Let X_i be symmetric spaces or Euclidean buildings such that $\partial_{Tits}X_i = B_i$. Then $\partial_{Tits}(X_1 \times X_2) = B$. Given a measure μ on B, we construct measures μ_i on B_i so that the decomposition

$$b_\mu = b_{\mu_1} + b_{\mu_2} \tag{16}$$

holds for the Busemann functions on X. Namely, for $\eta_i \in B_i$ and $\xi \in \overline{\eta_1\eta_2}$, we project the atomic measure δ_ξ on B to the measures $\cos\angle_{Tits}(\xi, \eta_i) \cdot \delta_{\eta_i}$ on B_i. Extending this linearly yields well-defined maps of measures $\mu \mapsto \mu_i$ such that (16) holds, see Lemma 2.7. The Busemann functions b_{μ_i} of course descend to X_i.

Similarly, we can project a weighted configuration $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to B$ to weighted configurations $\psi_i : (\mathbb{Z}/n\mathbb{Z}, \nu_i) \to B_i$. We choose the maps ψ_i so that $\psi(j)$ lies on the segment joining the points $\psi_i(j)$, and define the ν_i by

$$\nu_i(j) = \nu(j)\cos\angle_{Tits}(\psi(j), \psi_i(j)).$$

Clearly $\mu_i = (\psi_i)_*\nu_i$. Notice that $\nu_{3-i}(\psi^{-1}(B_i)) = 0$.

Lemma 4.4 ([LeMi]). *μ is semistable if and only if both μ_1 and μ_2 are semistable.*

Proof: We have: $b_\mu = b_{\mu_1} + b_{\mu_2}$ and hence

$$\mathrm{slope}_\mu = \mathrm{slope}_{\mu_1} + \mathrm{slope}_{\mu_2}.$$

Thus semistability of μ_1, μ_2 implies semistability of μ. Conversely, since for each $\eta_i \in B_i$, $\mathrm{slope}_{\mu_{3-i}}(\eta_i) = 0$, $i = 1, 2$; it follows that

$$\mathrm{slope}_\mu\Big|_{B_i} = \mathrm{slope}_{\mu_i}.$$

Hence semistability of μ implies semistability of each μ_i. ☐

We have the following structure result for nice semistable measures.

Proposition 4.5 (Cf. [LeMi, Thm. 2.13]). *Let X be a symmetric space and μ be a measure on $\partial_{Tits}X$. Suppose that $F \times Z \subseteq X$ is a smallest symmetric subspace whose ideal boundary contains* Supp(μ)*, and let F be its Euclidean deRham factor. Then:*

1. μ is nice semistable if and only if the projection of μ to $\partial_\infty F$ is semistable, and the projection to $\partial_\infty Z$ is stable.

2. In the case when μ is nice semistable, $F \times Z$ is unique up to parallel translation, and $\{\mathrm{slope}_\mu = 0\} = \partial_{Tits}(F \times Y)$ where Y is the symmetric subspace such that $F \times Z \times Y$ is the parallel set of $F \times Z$.

We refer the reader to the Appendix of this paper for a proof of this and other properties of nice semistable measures.

Folding in buildings and symmetric spaces. Cf. [LeMi, Section 2.2.5]. Suppose that B is a spherical building and ξ is a point in B. For an antipode η of ξ there is a canonical map

$$Fold = Fold_{\xi,\eta} : B \longrightarrow B(\xi, \eta) \tag{17}$$

called *folding at ξ*. It is a piecewise isometric 1-Lipschitz map characterized by the properties that $\angle_{Tits}(Fold(\cdot), \xi) = \angle_{Tits}(\cdot, \xi)$ and that $Fold$ restricts to the identity on the neighborhood $Star(\xi)$ of ξ. $Fold$ maps every apartment a through ξ isometrically to the unique apartment through ξ and η coinciding with a near ξ. Notice that for another antipode η' of ξ, there is a natural isometry $\iota : B(\xi,\eta) \to B(\xi,\eta')$ satisfying $\iota \circ Fold_{\xi,\eta} = Fold_{\xi,\eta'}$.

If B is the ideal boundary of a symmetric space or Euclidean building X, $B = \partial_{Tits} X$, then the folding in B can be extended to X.

Folding in X is defined as follows. Recall that $B(\xi,\eta)$ is the ideal boundary of the parallel set $P_l \subseteq X$ of a geodesic l asymptotic to ξ and η. For a point $x \in X$, let $\rho : [0,\infty) \to X$ be the unit speed geodesic ray starting in x and asymptotic to ξ. There is a unique ray ρ' in P_l which is strongly asymptotic to ρ, i.e. it satisfies $\lim_{t\to\infty} d(\rho'(t),\rho(t)) = 0$. We set $Fold(x) = \rho'(0)$. Notice that for any geodesic l' asymptotic to l, the restriction of folding to $P_{l'}$ is an isometry onto P_l.

Remark 4.6. *In the case when X is a symmetric space, folding can be described as a projection along horocycles as follows. Let G be the identity component of* $\mathrm{Isom}(X)$. *Let U denote the unipotent radical of the stabilizer of ξ in G. The orbits of U in X are called* horocycles. *Each horocycle $U \cdot x$ intersects P_l in a unique point which equals $Fold(x)$.*

Folding of measures and weighted configurations. Consider now a measure μ on B. We define the *folded measure* on the suspension $B(\xi,\eta)$ as

$$\mu' = (Fold_{\xi,\eta})_* \mu.$$

Then $\mathrm{slope}_{\mu'} = \mathrm{slope}_{\mu}$ on the neighborhood $Star(\xi)$ of ξ.

In general, folding does not preserve the (semi)stability of measures. However we have the following result.

Lemma 4.7. *If the measure μ is semistable and if* $\mathrm{slope}_\mu(\xi) = 0$ *then the folded measure μ' is again semistable.*

Proof: Since $\mathrm{slope}_{\mu'} = \mathrm{slope}_{\mu}$ on $Star(\xi)$, we have $\mathrm{slope}_{\mu'}(\xi) = 0$ and $\mathrm{slope}_{\mu'} \geq 0$ on a neighborhood of ξ. The convexity of the slope function on its zero sublevel set implies that $\mathrm{slope}_{\mu'} \geq 0$ everywhere. ☐

Remark 4.8. *By choosing ξ appropriately, the folded measure can be made nice semistable, cf. [LeMi, Section 2.2.5].*

The folding construction applies to weighted configurations as well. Notice that folding preserves the type of a configuration. According to the previous remark, for any semistable weighted configuration there exists a nice semistable configuration of the same type.

4.2 The stability inequalities for symmetric spaces

Let X be a symmetric space of noncompact type (thus X has no Euclidean deRham factors) and let G be the identity component of its isometry group. The ideal boundary $\partial_{Tits} X$ is a spherical building modelled on a spherical Coxeter complex (S, W)

with model spherical Weyl chamber $\Delta_{sph} \subset S$. We regard S as an apartment in $\partial_{Tits} X$. Let Δ denote the Euclidean Weyl chamber of X. We thus identify Δ_{sph} with $\partial_{Tits}\Delta$.

Let B be the stabilizer of Δ_{sph} in G. For each vertex ζ of $\partial_{Tits} X$ one defines the generalized Grassmannian $Grass_\zeta = G\zeta$. It is a compact homogeneous space stratified into B-orbits called *Schubert cells.* Every Schubert cell is of the form $C_\eta = B\eta$ for a unique vertex $\eta \in W\zeta \subset S^{(0)}$. The closures $\overline{C_\eta}$ are called *Schubert cycles.* They are unions of Schubert cells and represent well defined elements in the homology $H_*(Grass_\zeta, \mathbb{Z}_2)$.

For each vertex ζ of Δ_{sph} and each n-tuple $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ of vertices in $W\zeta$ consider the following homogeneous linear inequality for $\xi \in \Delta^n$:

$$\sum_i \xi_i \cdot \eta_i \leq 0 \qquad (*_{\zeta;\vec{\eta}})$$

Here we identify the η_i's with unit vectors in Δ. Let $I_{\mathbb{Z}_2}(G)$ be the subset consisting of all data $(\zeta, \vec{\eta})$ such that the intersection of the Schubert classes $[\overline{C}_{\eta_1}], \ldots, [\overline{C}_{\eta_n}]$ in $H_*(Grass_\zeta, \mathbb{Z}_2)$ equals $[pt]$.

Theorem 4.9 ([LeMi]). *$\Delta^n_{ss}(X) \subset \Delta^n$ consists of all solutions ξ to the system of inequalities $(*_{\zeta;\vec{\eta}})$ where $(\zeta, \vec{\eta})$ runs through $I_{\mathbb{Z}_2}(G)$.*

Remark 4.10. *This system of inequalities depends on the Schubert calculus for the Lie group G. It is one of the results of this paper that the set of solutions only depends on the spherical Coxeter complex.*

Typically, the system of inequalities in Theorem 4.9 is redundant. If G is a *complex* Lie group one can use the complex structure to obtain a smaller system of inequalities. In this case, the homogeneous spaces $Grass_\zeta$ are complex manifolds and the Schubert cycles are complex subvarieties and hence represent classes in *integral* homology. Let $I_{\mathbb{Z}}(G) \subset I_{\mathbb{Z}_2}(G)$ be the subset consisting of all data $(\zeta, \vec{\eta})$ such that the intersection of the Schubert classes $[\overline{C}_{\eta_1}], \ldots, [\overline{C}_{\eta_n}]$ in $H_*(Grass_\zeta, \mathbb{Z})$ equals $[pt]$. We have the analogous result which was proven independently and with different methods by Berenstein and Sjamaar:

Theorem 4.11 (Stability inequalities [BeSj],[LeMi]). *$\Delta^n_{ss}(X) \subset \Delta^n$ consists of all solutions ξ to the system of inequalities $(*_{\zeta;\vec{\eta}})$ where $(\zeta, \vec{\eta})$ runs through $I_{\mathbb{Z}}(G)$.*

Question 4.12. *Can one describe the stability inequalities directly in terms of the Coxeter complex?*

Suppose now that X is a nonpositively curved symmetric space which does not necessarily have noncompact type, i.e. $X = F \times X'$, where F is the flat deRham factor of X, and X' has noncompact type. Then Lemma 4.4 implies that $\Delta^n_{ss}(X) = \Delta^n_{ss}(F) \times \Delta^n_{ss}(X')$. Note that $\Delta^n_{ss}(F)$ equals the set of semistable configurations on $\partial_{Tits} F$, i.e. configurations $\psi = ((m_1, \xi_1), ..., (m_n, \xi_n))$ for which

$$\sum_{i=1}^{n} m_i \xi_i = 0.$$

Thus we get a complete description of $\Delta^n_{ss}(X)$ in terms of stability inequalities for $\Delta^n_{ss}(X')$.

4.3 Gauss maps

We now relate polygons in X and weighted configurations on the boundary $\partial_{Tits}X$ at infinity.

Consider a closed polygon $P = x_1x_2\dots x_n$ in X, i.e. a map $\mathbb{Z}/n\mathbb{Z} \to X$. The distances $m_i = d(x_i, x_{i+1})$ determine a finite measure ν on $\mathbb{Z}/n\mathbb{Z}$ by $\nu(i) = m_i$. The polygon P gives rise to a collection $Gauss(P)$ of *Gauss maps*

$$\psi : \mathbb{Z}/n\mathbb{Z} \longrightarrow \partial_\infty X \tag{18}$$

by assigning to i an ideal point $\xi_i \in \partial_\infty X$ so that the ray $\overline{x_i\xi_i}$ passes through x_{i+1}. This construction, in the case of the hyperbolic plane, already appears in the letter of Gauss to Bolyai, [Ga]. Taking into account the measure ν, we view the maps $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_\infty X$ as *weighted configurations* of points on $\partial_\infty X$. Note that if X is a Riemannian symmetric space and all m_i's are non-zero, there is a unique Gauss map due to the unique extendibility of geodesics. On the other hand, if X is a Euclidean building then, due to the branching of geodesics, there are in general infinitely many Gauss maps. However, the corresponding weighted configurations are of the same type, i.e. they project to the same weighted configuration on Δ_{sph}.

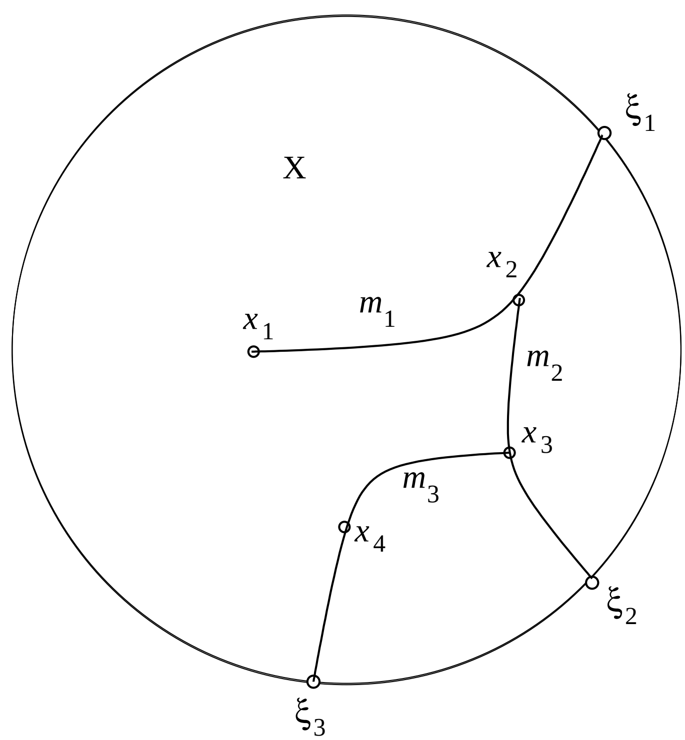

Figure 6: *The Gauss map.*

The following crucial observation explains why the notion of semistability is important for studying closed polygons.

Lemma 4.13. *The pushed forward measures $\mu = \psi_*\nu$ are semistable.*

Proof: Consider an arbitrary ideal point η and note that (Lemma 2.4)

$$b_\eta(x_{i+1}) - b_\eta(x_i) \le -m_i \cos \angle_{Tits}(\xi_i, \eta). \tag{19}$$

Summing up yields

$$0 \leq -\sum m_i \cdot \cos \angle_{Tits}(\xi_i, \eta) = \text{slope}_\mu(\eta),$$

the asymptotic slope at η of the weighted Busemann function for μ. □

In the Riemannian case there is a more precise result.

Lemma 4.14. *If X is a symmetric space of nonpositive curvature then the measure μ is nice semistable.*

Proof: We only have to treat the case when μ is not stable. Then there is an ideal point η with $\text{slope}_\mu(\eta) = 0$ and the inequalities (19) become equalities. For each i there is a unique geodesic l_i through x_i asymptotic to η. Lemma 2.4 implies that all geodesics l_i are parallel to each other. It follows that the polygon P lies in the parallel set P_l of a geodesic l asymptotic to η. Moreover, if $m_i \neq 0$ then the geodesic through x_i and x_{i+1} lies in P_l. Thus μ is supported on $\partial_\infty P_l$.

The parallel set P_l splits as a product $P_l \cong l \times Y$. By the construction explained in section 4.1, the measure μ project to measures μ_1 on $\partial_\infty l$ and μ_2 on $\partial_\infty Y$. These measures correspond to the Gauss maps of the orthogonal projections of the polygon P to l and Y, therefore they are semistable.

If μ_2 is not stable, we repeat the construction. After finitely many steps we obtain a product subspace $F \times Z \subseteq X$ containing P so that F is a flat and Z is a symmetric subspace (possibly a point). Moreover, μ is supported on $\partial_{Tits}(F \times Z)$, and its projection to Z is stable and the projection to F is semistable. According to Proposition 4.5, the measure μ is nice semistable on $\partial_{Tits}X$. □

4.4 A fixed point theorem

We are now interested in finding polygons with prescribed Gauss map. Such polygons will correspond to the fixed points of a certain dynamical system.

For $\xi \in \partial_\infty X$ and $t \geq 0$, we define the map $\phi := \phi_{\xi,t} : X \to X$ by sending x to the point at distance t from x on the geodesic ray $\overline{x\xi}$.

Lemma 4.15. *(i) ϕ is 1-Lipschitz, i.e. $d(\phi x, \phi y) \leq d(x, y)$.*

(ii) For $t > 0$ we have equality in (i) iff there is an isometrically embedded flat half strip in X bounded by $\overline{xy}$ and the rays $\overline{x\xi}$, $\overline{y\xi}$.

Proof: By nonpositivity of the curvature, the function $\delta : t \mapsto d(\phi_{\xi,t}(x), \phi_{\xi,t}(y))$ is convex. Since it is also bounded, δ is non-increasing. This implies (i).

In the case of equality in (i) for some $t > 0$, the function δ must be constant. The existence of the flat half strip follows from a basic rigidity result [BGS, Lemma 2.3]. □

Fix now a weighted configuration $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_\infty X$ with non-zero total mass. We define the 1-Lipschitz self-map

$$\Phi = \Phi_\psi : X \longrightarrow X$$

as the composition $\Phi_n \circ \cdots \circ \Phi_1$ of the maps $\Phi_i = \phi_{\xi_i, m_i}$.

Remark 4.16. *If X is a symmetric space or a building then the map Φ is not a strict contraction. To see this pick a regular point $\eta \in \partial_\infty X$ and consider the top-dimensional flats $F_i \subset X$ which are asymptotic to the points η and ξ_i, $i = 1, ..., n$. The flats F_i are strongly asymptotic at the point η and hence the map $\Phi|_{F_1}(x), x \in F_1$, converges to an isometry, as $x \in F_1$ converges to η in the cone topology.*

The fixed points of Φ are the first vertices of closed polygons $P = x_1 \dots x_n$ so that ψ is a Gauss map for P.

Example 4.17. *Suppose that $X \cong \mathbb{R}^k$. Then for each weighted configuration ψ on $\partial_\infty X$, the map $\Phi = \Phi_\psi$ is a translation. For each $\eta \in \partial_{Tits} X$ the Busemann function b_η is linear, and we have $b_\eta(\Phi(x)) - b_\eta(x) = \mathrm{slope}_\mu(\eta)$. Therefore Φ_ψ is the identity if and only if ψ is semistable.*

Regarding existence, we will prove in the subsequent sections the following result.

Theorem 4.18. *Suppose that X is either a symmetric space or a locally compact Euclidean building or a (thick) Euclidean building with one vertex. Suppose that ψ is a nice semistable weighted configuration on $\partial_{Tits} X$ (in the symmetric space case) or a semistable configuration (in the building case). Then Φ_ψ has a fixed point.*

4.4.1 Existence of a fixed point in the locally compact case

Suppose that X is a symmetric space or a locally compact Euclidean building. We start by proving Theorem 4.18 in the case when the measure μ is stable.

In view of Lemma 2.9 it suffices to show that the dynamical system $\Phi : X \to X$ has a bounded orbit $(\Phi^n(p))_{n \geq 0}$. Suppose that this is false.

Step 1. Our assumption that Φ does not have a bounded orbit implies that Φ does not map any bounded subset of X into itself. Pick a base point $o \in X$. Since no metric ball centered at o is mapped into itself there is a sequence of points x_n with $d(x_n, o) \to \infty$ which is "pulled away from" o in the sense that

$$d(\Phi x_n, o) > d(x_n, o).$$

Suppose that for some point $x \in \overline{ox_n}$ the inequality $d(\Phi x, o) > d(x, o)$ does not hold. Then, since $d(\Phi(x), \Phi(x_n)) \leq d(x, x_n)$ (recall that Φ is 1-Lipschitz), we have:

$$d(o, \Phi(x_n)) \leq d(o, \Phi(x)) + d(\Phi(x), \Phi(x_n)) \leq d(o, x) + d(x, x_n) = d(o, x_n).$$

Contradiction. Hence for each $x \in \overline{ox_n}$ we have $d(\Phi x, o) > d(x, o)$. The space X is locally compact, hence (after passing to a subsequence) the geodesic segments $\overline{ox_n}$ have a Hausdorff limit in X which is a geodesic ray $\rho([0, \infty)) = \overline{o\eta_+}$. Thus for each $t \geq 0$ we have

$$d(\Phi(\rho(t)), o) \geq d(\rho(t), o). \tag{20}$$

Step 2. Now we will use that X is a symmetric space or a Euclidean building. Choose a complete geodesic l containing the ray ρ. This geodesic is asymptotic to ideal points η_+ and η_- in $\partial_\infty X$. We apply the folding procedure described in section 4.1 and fold the weighted configuration ψ to a configuration $\psi' = Fold_{\eta_+, \eta_-} \psi$ on the ideal boundary of the parallel set P_l. We denote $\xi_i' = Fold(\xi_i) = \psi'(i)$, $\mu' = Fold_* \mu$, $\Phi_i' := \phi_{\xi_i', m_i}$ and $\Phi' := \Phi_{\psi'}$. Note that $\mathrm{slope}_{\mu'}(\eta_+) = \mathrm{slope}_\mu(\eta_+) > 0$.

Lemma 4.19. $\lim_{t\to\infty} d(\Phi'\rho(t), \Phi\rho(t)) = 0$. *If X is a Euclidean building then there exists t_0 such that $\Phi'\rho(t) = \Phi\rho(t)$ for $t \geq t_0$.*

Proof: Note that the maps Φ_i are 1-Lipschitz and Φ'_i moves geodesic rays asymptotic to η_+ in P_l to parallel rays. It therefore suffices to prove that

$$\lim_{t\to\infty} d(\Phi_1\rho(t), \Phi'_1\rho(t)) = 0. \tag{21}$$

The first assertion of lemma will then follow by induction over i. The second assertion will follow because strongly asymptotic rays in buildings eventually coincide.

We may pick a geodesic l_1 in X asymptotic to η_+ so that $\xi_1 \in \partial_\infty P_{l_1}$. We consider the canonical isometry $u : P_l \to P_{l_1}$ between parallel sets, which sends every ray in P_l to the unique strongly asymptotic ray in P_{l_1}. Then

$$u \circ \Phi_1 = \Phi'_1 \circ u \tag{22}$$

holds on P_l. The map Φ'_1 moves the ray ρ to a parallel ray within P_l, and Φ_1 moves the ray $u \circ \rho$ to a parallel ray within P_{l_1}. According to (22), the rays $\Phi_1 \circ u \circ \rho$ and $\Phi'_1 \circ \rho$ are strongly asymptotic. By the triangle inequality:

$$d(\Phi_1\rho(t), \Phi'_1\rho(t)) \leq d(\Phi_1\rho(t), \Phi_1 \circ u \circ \rho(t)) + d(\Phi_1 \circ u \circ \rho(t), \Phi'_1\rho(t))$$

Taking into account that Φ_1 is 1-Lipschitz, we see that both summands on the right tend to zero as $t \to \infty$. $\square$

Step 3. The lemma allows us to replace Φ with Φ' and to work on the parallel set P_l. $\Phi' : P_l \to P_l$ preserves the product splitting $P_l \cong l \times Y$. The level sets of $b_{\eta+}$ on P_l are the cross sections $\{t\} \times Y, t \in l$. The increment $b_{\eta+} \circ \Phi' - b_{\eta+}$ is constant with value equal to $\text{slope}_{\mu'}(\eta_+) > 0$, i.e. $\Phi'|_{P_l}$ has a translational component moving points away from η_+. Since Φ' has uniformly bounded displacement it follows (cf. Lemma 2.8) that

$$\limsup_{t\to\infty} \big(d(\Phi'\rho(t), o) - d(\rho(t), o)\big) < 0.$$

Combining this with Lemma 4.19 we obtain a contradiction with (20). This finishes the proof of Theorem 4.18 in the case when X is locally compact and the measure μ is stable.

We next extend this existence result to the semistable case. Suppose first that X is a symmetric space and ψ is nice semistable. We use the notation of Proposition 4.5.

Lemma 4.20. *Φ_ψ has a fixed point in the symmetric subspace Z.*

Proof: The measure μ is supported on $\partial_{Tits}(F \times Z)$ and thus Φ preserves $F \times Z$. We project ψ to weighted configurations ψ_1 on $\partial_{Tits}F$ and ψ_2 on $\partial_{Tits}Z$. Then, according to Proposition 4.5, ψ_1 is semistable and ψ_2 is stable. The restriction of Φ to $F \times Z$ splits as the product $(\Phi_{\psi_1}, \Phi_{\psi_2})$. Since ψ_1 is semistable, we have that $\Phi_{\psi_1} = id_F$. Furthermore, ψ_2 is stable and, by the stable case discussed above, Φ_{ψ_2} has a fixed point in Z. $\square$

Let now X be a locally compact Euclidean building and suppose that ψ is semistable. We argue by induction on the rank of X.

The assertion is clear if X has rank 0, i.e. is a point. For the induction step suppose that X has rank $r \geq 1$ and that the assertion is true for buildings of rank $< r$. We may assume that ψ is not stable. Let η be an ideal point with $\text{slope}_\mu(\eta) = 0$. Pick a geodesic l asymptotic to η and consider the parallel set P_l. We fold the configuration ψ to a semistable configuration ψ' on $\partial_{Tits} P_l$, cf. Lemma 4.4. By Lemma 4.19, Φ_ψ has a fixed point on P_l if and only if $\Phi_{\psi'}$ has one.

As in the proof of Lemma 4.20, we reduce to an analogous fixed point problem on the cross section Y of P_l. The rank of Y is $r-1$, and the claim therefore follows from the induction hypothesis.

This concludes the proof of Theorem 4.18 in the locally compact case.

4.4.2 Description of the fixed point set in the Riemannian case

Suppose that X is a symmetric space and that ψ is a weighted configuration on $\partial_{Tits} X$.

Lemma 4.21. *If ψ is stable then Φ_ψ has a unique fixed point.*

Proof: Suppose that Φ has two distinct fixed points x_1 and x'_1, and denote by $x_1 \dots x_n$ and $x'_1 \dots x'_n$ the corresponding polygons. It follows from Lemma 4.15 that $d(x_1, x'_1) = \dots = d(x_n, x'_n)$. Assuming $d(x_1, x_2) > 0$ without loss of generality, we see that the configuration lies on the ideal boundary of the parallel set of the geodesic through x_1 and x_2. In particular, the configuration ψ not stable. $\square$

The above lemma fails in the case of Euclidean buildings. For instance, it is easy to give counterexamples of measures with four atoms of equal weight on the ideal boundary of a tree.

We generalize Lemma 4.21 to nice semistable measures using the structure result described in Proposition 4.5.

Lemma 4.22. *If ψ is nice semistable, then the fixed point set of Φ_ψ is a symmetric subspace of the form $F \times \{z\} \times Y$ with $z \in Z$. The pointwise stabilizer of* $\text{Supp}(\mu)$ *in* $\text{Isom}(F \times \{z\} \times Y)$ *acts transitively on the fixed point set.*

Proof: We identify Z with the subspace $\{f_0\} \times Z$ for some point $f_0 \in F$. Φ_ψ preserves Z and has a unique fixed point z there, cf. Lemma 4.20. We may assume that Y passes through z.

Let x be another fixed point of Φ, and let l be the geodesic through z and x. As in the proof of Lemma 4.21 above we see that μ is supported on $\partial_{Tits} P_l$. Hence $(F \times Z) \cap P_l$ is a non-empty symmetric subspace containing $\text{Supp}(\mu)$ in its ideal boundary. Since $F \times Z$ is minimal with respect to this property, it follows that $F \times Z \subseteq P_l$. Parallel translation along l sends z to x and moves $F \times Z$ to a parallel symmetric subspace. Thus x lies in $F \times Z \times Y$ since the latter is the parallel set for $F \times Z$. We conclude that $Fix(\Phi) \subseteq F \times Z \times Y$, and it is then clear that $Fix(\Phi) = F \times \{z\} \times Y$. $\square$

4.4.3 Existence of a fixed point for buildings with unique vertex

We now prove Theorem 4.18 in the remaining case when X is a Euclidean building modelled on a Coxter complex with trivial translation group. Such buildings are Euclidean cones over spherical buildings; in fact X is canonically isometric to the Euclidean cone $Cone(\partial_{Tits}X)$ over its Tits boundary. We denote the tip of this cone by o.

We first treat the case when the configuration ψ on $\partial_{Tits}X$ is stable. Then X has the unique vertex o. As in the locally compact case, it suffices to prove that $\Phi = \Phi_\psi$ preserves a bounded set.

Let $m = |\mu|$ denote the total mass μ. For a chamber Δ of X, let $\Delta_m \subset \Delta$ be the subset of all points with distance $> m$ from the boundary of Δ. Since Φ has displacement $\leq m$, we have $\Phi(\Delta_m) \subset \Delta$. We observe moreover that the restriction $\Phi|_{\Delta_m}$ is a translation satisfying

$$b_\xi(\Phi x) - b_\xi(x) = \text{slope}_\mu(\xi) \tag{23}$$

for all $\xi \in \partial_\infty \Delta$. Using the stability of μ, i.e. $\text{slope}_\mu \geq \delta > 0$, we deduce that

$$d(o, \Phi(x)) \leq d(o, x) - \delta$$

holds for all $x \in \Delta_m$.

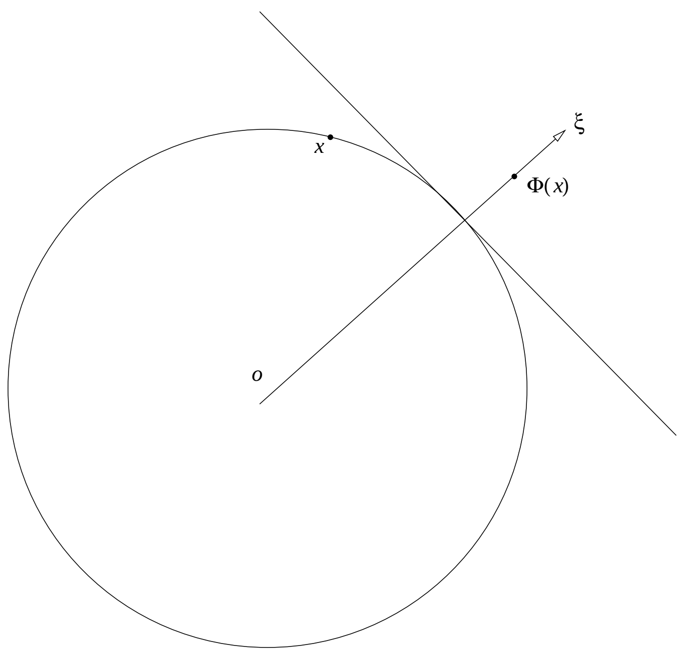

Figure 7: *$d(\Phi(x), o) > d(x, o)$ implies that $b_\xi(\Phi x) < b_\xi(x)$.*

We will now carry out similar estimates near lower-dimensional faces V of X. Let $star(V)$ denote the union of all chambers in X containing V, and let $star(V)_m \subset star(V)$ be the subset of all points at distance $> m$ from the frontier of $star(V)$. Again $\Phi(star(V)_m) \subset star(V)$. Moreover, (23) holds on $star(V)_m$ for all $\xi \in \partial_\infty V$. Due to the local product structure of $star(V)$ near V, the component of Φ in the direction of V is a translation τ. We have that $\tau(p(x)) = p(\Phi(x))$ on $star(V)_m$. The translation τ decreases the distance to the vertex o. As above, the stability of the measure μ implies that

$$d(o, p(\Phi(\cdot))) \leq d(o, p(\cdot)) - \delta$$

holds on $star(V)_m$. We use furthermore that the distance spheres around o intersect V orthogonally. More precisely, given $\epsilon, r > 0$ there exists $R = R(\epsilon, r) > 0$ such that

$$d(o, x) - \epsilon \leq d(o, p(x)) \leq d(o, x)$$

holds for all $x \in star(V)$ with $d(x, V) \leq r$ and $d(o, x) \geq R$. It follows that

$$d(o, \Phi(x)) \leq d(o, x) \tag{24}$$

for all $x \in star(V)_m$ with $d(x, V) \leq r$ and $d(o, x) \geq R(\frac{\delta}{2}, r)$. Choosing the constants r and R appropriately, depending on the dimensions of the faces V, we can cover the complement of a bounded subset of X with subsets on which (24) holds. It follows that Φ preserves sufficiently large balls around o, concluding the proof of Theorem 4.18 in the stable case.

We are left with the case when the measure μ is semistable and not stable. There exists $\eta \in B$ with $\text{slope}_\mu(\eta) = 0$. We argue as in the case of locally compact Euclidean buildings and semistable configurations by induction over the rank. Here we use the fact that every parallel set, and more generally any subbuilding of X is again a Euclidean cone over its ideal boundary and hence splits as the product of a Euclidean space and a building with one vertex.

This concludes the proof of Theorem 4.18. $\square$

4.5 Proofs of Theorems 1.4 and 1.6

Proof of Theorem 1.6: In the case that X is an infinitesimal symmetric space, Theorem 1.6 was proven in [LeMi] (see also Appendix, Theorem 6.16). The inclusion $D_n(X) \subseteq \Delta^n_{ss}(\partial_\infty X)$ has been proven in Lemma 4.13 in general. The converse inclusion follows from Theorem 4.18 in the cases when X is a symmetric space, or a thick locally compact Euclidean building or a thick Euclidean building with one vertex.

We now consider the case of a general thick Euclidean building X. Let (E, W_{aff}) denote the Euclidean Coxeter complex of X and $(\partial_\infty E, W_{sph})$ the associated spherical Coxeter complex. Let $Cone(\partial_{Tits} X)$ denote the Euclidean cone over the spherical building $\partial_{Tits} X$ at infinity. It is a building with one vertex. We know that $D_n(Cone(\partial_{Tits} X)) = \Delta^n_{ss}(\partial_{Tits} X)$. Note that (E, W_{sph}) is the *Euclidean* Coxeter complex of $Cone(\partial_{Tits} X)$, and (E, W_{sph}) naturally embeds into (E, W_{aff}). The Transfer Theorem 1.3 implies the inclusion $D_n(Cone(\partial_{Tits} X)) \subseteq D_n(X)$. Thus

$$D_n(X) \subseteq \Delta^n_{ss}(\partial_{Tits} X) \subseteq D_n(Cone(\partial_{Tits} X)) \subseteq D_n(X)$$

and we are done. $\square$

Proof of Theorem 1.4: Consider two spaces X and X' which are modelled on Euclidean Coxeter complexes $(E, W_{aff}), (E', W'_{aff})$ respectively. Let B, B' be the corresponding spherical buildings at infinity. Consider an isometric embedding $f : E \to E'$, which induces an embedding ϕ of the spherical Coxeter groups $W_{sph} \to W'_{sph}$. The Transfer Theorem 1.3 implies that we have the induced embedding

$$D_n(Cone(B)) \to D_n(Cone(B')).$$

By Theorem 1.6 we have

$$D_n(X) = \Delta^n_{ss}(B) = D_n(Cone(B)) \text{ and } D_n(Cone(B')) = \Delta^n_{ss}(B') = D_n(X').$$

Thus f induces an embedding

$$D_n(X) \to D_n(X').$$

In particular, if f and ϕ are surjective, then the map $D_n(X) \to D_n(X')$ is a bijection. □

5 Polygons with prescribed refined side lengths in discrete Euclidean buildings

Let X be a thick Euclidean building without Euclidean factor. For applications to algebra it is particularly interesting to study polygons with L-integral side lengths, cf. [KLM]. Let $D^L_n(X) \subset D_n(X)$ and $D^{ref,L}_n(X) \subset D^{ref}_n(X)$ be the subsets of possible L-integral side lengths for polygons, see section 2.4. We have a description for

$$D^L_n(X) = D_n(X) \cap L^n$$

in terms of the stability inequalities 4.9, and we are now interested in studying $D^{ref,L}_n(X)$. I.e., given $\xi \in D^L_n(X)$, to what extent can one prescribe the location of the vertices for polygons with Δ-side lengths ξ?

Lemma 5.1. *Suppose that $L = L_{trans}$. Then for any given L-integral Δ-lengths $\xi \in D^L_n(X)$ there exists a polygon with Δ-side lengths ξ and vertices in the 0-skeleton $X^{(0)}$ of X.*

Proof: We recall that the first vertex of a polygon with Δ-side lengths ξ is a fixed point of the map $\Phi_\psi : X \to X$ introduced in section 4.4, where ψ is a Gauss map of the polygon. Since ξ is L_{trans}-integral, the map Φ is simplicial and moreover preserves the type of points, i.e. commutes with the natural projection of X to its Weyl alcove. Hence $Fix(\Phi)$ is a subcomplex and it follows that it contains a vertex. □

In general, one cannot find a polygon with vertices of prescribed type, see Example 5.5 below.

Problem 5.2. *Given L, describe the image of the natural inclusion*

$$D^{ref,L}_n(X) \hookrightarrow D^L_n(X). \tag{25}$$

There is an easy positive result:

Proposition 5.3. *The inclusion (5.2) is surjective if $L = L_{trans}$ and every vertex of E is special, i.e. its stabilizer is isomorphic to W_{sph}.*

Proof: Every vertex being special is equivalent to the property that the normalizer N_{aff} of W_{aff} in the full group of translations of E acts transitively on the vertices of E.

Let P be a polygon in X with L-integral side lengths ξ. According to Lemma 5.1, we can assume that its first vertex x_1 is a vertex of the building, i.e. it lies in $X^{(0)}$.

Let $x_1' \in X^{(0)}$ be another vertex of the building. We choose an apartment a containing x_1 and x_1'. By our assumption, there exists a translation normalizing the Weyl group of a which carries x_1 to x_1'. Using transfer, cf. Theorem 1.3, we can obtain from P another polygon P' in X with the same Δ-side lengths and with first vertex x_1'. □

Remark 5.4. *In the case of irreducible buildings, on easily checks by looking at the list of possible Dynkin diagrams that the hypothesis in Proposition 5.3 is equivalent to the condition that the spherical Coxeter complex $(\partial_\infty E, W_{sph})$ is of type A_ℓ. Thus we have proven Theorem 1.8 from the Introduction.*

The next example shows that one cannot always prescribe the type of vertices.

Example 5.5. *Let X be a thick Euclidean building with associated Euclidean Coxeter complex (E, W_{aff}) of type C_2 and $L = N_{aff}$. Then the map (25) is not surjective for $n = 3$.*

Proof: Recall that the Coxeter complex (E, W_{aff}) is two-dimensional, and the Weyl alcove is an isosceles right-angled triangle. The vertex stabilizers are isomorphic to either D_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The vertices of the former type are special. There are two W_{aff}-orbits of special vertices and one orbit of non-special ones.

Consider the broken triangle shown in Figure 8. It has geodesic sides $\overline{zx}$, $\overline{zy}$ and a broken side $\overline{xuy}$. Its Δ-side lengths are L-integral and it has non-special vertices. (Note that the vector $\overrightarrow{zy}$ does not belong to L_{trans}.)

We first check that this triangle can be unfolded to a geodesic triangle $\Delta(z, x, y'')$ in X in the sense of section 3.2. This can be done as follows. We divide the apartment into two half-apartments $h_1 \ni y$ and $h_2 \ni z$ by the geodesic l through u. We take a third half-apartment h_3 with boundary l. Denote by $\psi : h_1 \to h_3$ the isometry fixing l. The point $y'' = \psi(y)$ has the desired properties.

Now we prove that there exists no geodesic triangle $\Delta(x', y', z')$ in X with special vertices and the same Δ-side lengths as $\Delta(x, y'', z)$. Suppose that such a triangle exists. As described in section 3.2, we subdivide the side $\overline{x'y'}$ and fold $\Delta(x', y', z')$ to a broken triangle in an apartment a. Notice that the side $\overline{x'y'}$ can be folded only at its midpoint u'. Since u' is a non-special vertex, y' is folded onto x', or the folded triangle remains geodesic. The former case is impossible because $\Delta(x', y', z')$ is not isosceles. The latter case is ruled out by inspection: We leave it to the reader to verify that there are no geodesic triangles contained in an apartment which have the same Δ-side lengths as $\Delta(x', y', z')$. □

We refer the reader to our paper [KLM] for more results concerning the image of the map (25) and applications to algebraic problems.

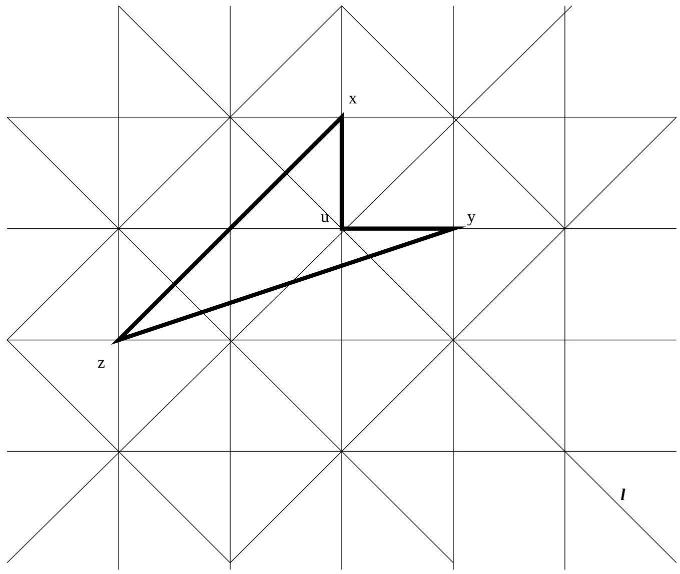

Figure 8: *Folded triangle.*

6 Appendix

In this section we prove some of the results of [LeMi] which were used in this paper.

The next lemma follows from [Eb, 1.10.10].

Lemma 6.1. *Let X be a symmetric space. Take a pair of non-zero tangent vectors $u, v \in T_xX$, let l be the geodesic with initial condition v, and suppose that u points towards the ideal point ξ. Then $D^2_{v,v} b_\xi \geq 0$ and the following are equivalent:*

1. $D^2_{v,v} b_\xi = 0$.
2. *b_ξ is affine linear on l.*
3. *The 2-plane in T_xX spanned by u and v has sectional curvature zero.*
4. *u, v are tangent to a 2-flat.*

Asymptotic slopes.

Consider a Lipschitz continuous convex function $f : X \longrightarrow \mathbb{R}$ on a Hadamard space X. It is asymptotically linear along any ray, and we define the *asymptotic slope* of f at $\xi \in \partial_\infty X$ as follows: Pick any geodesic ray ρ asymptotic to ξ and set

$$\text{slope}_f(\xi) := \lim_{t \to \infty} \frac{f(\rho(t))}{t}.$$

It is clear that slope does not depend on the choice of a ray ρ asymptotic to ξ.

Lemma 6.2. *1. Let $\xi \in \partial_\infty X$ and $\rho : [0, \infty) \to X$ be a geodesic ray (parameterized by unit speed) asymptotic to $\eta \in \partial_\infty X$. Then*

$$b_\xi \circ \rho(t) + t \cos \angle_{Tits}(\xi, \eta) \tag{26}$$

is convex and converges to a finite limit as $t \to \infty$.

2. If X *is a Euclidean building then* b_ξ *is eventually linear, i.e., for certain* $T \geq 0$ *the restriction of* b_ξ *to* $\rho([T, \infty))$ *is a linear function.*

Proof: 1. There is a ray ϕ asymptotic to ρ on which b_ξ is linear, namely a ray in a flat which contains ξ and η in its ideal boundary. The slope of b_ξ along ϕ is $-\cos \angle_{Tits}(\xi, \eta)$. This shows that (26) remains bounded as $t \to \infty$. It is clearly convex, because the Busemann function is convex.

2. Since X is a building, the ray ρ is eventually parallel to the ray ϕ. Thus there is $T \in \mathbb{R}_+$ such that the rays $\rho([T, \infty))$, $\phi([T, \infty))$ bound a flat strip. Hence linearity of b_ξ on $\phi([T, \infty))$ implies linearity of b_ξ on $\rho([T, \infty))$. $\square$

Corollary 6.3. *Let* $b_\mu = \sum_i m_i b_{\xi_i}$ *be a weighted Busemann function on a Hadamard space* X*. Then the asymptotic slope* $\text{slope}_\mu(\eta)$ *of* b_μ *in the direction of* $\eta \in \partial_{Tits} X$ *equals*

$$-\sum_{i=1}^{n} m_i \cos \angle_{Tits}(\xi_i, \eta).$$

Proof: The above lemma implies that for each i we have

$$\lim_{t \to \infty} \frac{b_\xi(\rho(t))}{t} = -\cos \angle_{Tits}(\xi_i, \eta).$$

Hence

$$\lim_{t \to \infty} \frac{1}{t} \sum_i m_i b_{\xi_i}(\rho(t)) = -\sum_i m_i \cos \angle_{Tits}(\xi_i, \eta). \quad \square$$

We next consider the properties of the asymptotic slope function.

Lemma 6.4. *Let* $f : X \to \mathbb{R}$ *be a Lipschitz convex function. Then:*

1. slope_f *is Lipschitz continuous with the same Lipschitz constant as* f.

2. The set $\{\text{slope}_f \leq 0\} \subset \partial_\infty X$ *is convex with respect to the Tits metric. The function* slope_f *is convex on* $\{\text{slope}_f \leq 0\}$ *and strictly convex on* $\{\text{slope}_f < 0\}$.

3. If X *is locally compact, then* f *is proper iff* $\text{slope}_f > 0$ *everywhere on* $\partial_\infty X$.

Proof: (1) Pick a pair of points $\xi_1, \xi_2 \in \partial_{Tits} X$. Consider geodesic rays ρ_i emanating from a base-point $o \in X$ and asymptotic to ξ_i, $i = 1, 2$. For each $t > 0$ consider the triangles $\Delta(o, \rho_1(t), \rho_2(t))$. Let $\alpha(t)$ denote the angle at $\bar{o}$ of the comparison triangle $\Delta(\overline{o}, \overline{\rho_1(t)}, \overline{\rho_2(t)}) \subset \mathbb{R}^2$. Then

$$\lim_{t \to \infty} \alpha(t) = \angle_{Tits}(\xi_1, \xi_2).$$

We have:

$$d(\rho_1(t), \rho_2(t)) \leq t\alpha(t),$$

thus, if f is L-Lipschitz, we get:

$$|f(\rho_1(t)) - f(\rho_2(t))| \leq Ld((\rho_1(t), \rho_2(t)) \leq Lt\alpha(t).$$

It follows that

$$\lim_{t\to\infty}\left|\frac{f(\rho_1(t))}{t}-\frac{f(\rho_2(t))}{t}\right|\le\lim_{t\to\infty}L\alpha(t)=L\angle_{Tits}(\xi_1,\xi_2).$$

This proves 1.

2. Take a pair of points $\xi_1,\xi_2\in\{\text{slope}_f\le 0\}$ so that $\angle_{Tits}(\xi_1,\xi_2)<\pi$ and, as before, consider rays ρ_i which are asymptotic to ξ_i. Then the midpoints $m(t)$ of the segments $\overline{\rho_1(t)\rho_2(t)}$ converge to the midpoint ξ_3 of $\overline{\xi_1\xi_2}$ in $\partial_{Tits}X$. By convexity of f we have:

$$f(m(t))\le\frac{f(\rho_1(t))+f(\rho_2(t))}{2}.$$

Since

$$d(o,m(t))\ge t\cos(\angle_{Tits}(\xi_1,\xi_2)/2),$$

we get

$$\frac{f(m(t))}{d(o,m(t))}\le\frac{f(\rho_1(t))+f(\rho_2(t))}{2t\cos(\angle_{Tits}(\xi_1,\xi_2)/2)}$$

and thus

$$\text{slope}_f(\xi_3)\le\frac{\text{slope}_f(\xi_1)+\text{slope}_f(\xi_2)}{2\cos(\angle_{Tits}(\xi_1,\xi_2)/2)}. \tag{27}$$

If $\text{slope}_f(\xi_i)\le 0$, the right-hand side is $\le(\text{slope}_f(\xi_1)+\text{slope}_f(\xi_2))/2$; this implies convexity of both $\{\text{slope}_f\le 0\}$ and of the restriction of slope_f to this set. The same inequality (27) implies strict convexity on $\{\text{slope}_f<0\}$, since for $\xi_1\ne\xi_2$ we have

$$1>\cos(\angle_{Tits}(\xi_1,\xi_2)/2)>0.$$

3. If f were not proper, sublevel sets would be noncompact and hence contain rays. It follows that there is an ideal point with asymptotic slope ≤ 0. Conversely, if f is proper then clearly $\text{slope}_f>0$. □

Notions of stability for convex functions.

Definition 6.5 (Stability of convex functions). *A Lipschitz convex function f on X is called*

- **semistable** *if f is bounded below,*
- **unstable** *otherwise,*
- **nice semistable** *if f has a minimum, and*
- **stable** *if f is proper and bounded below.*

Our next goal is to verify that in the case when X is a symmetric space, a measure $\mu=\sum_{i=1}^n m_i\delta_{\xi_i}$ is stable (resp. nice semistable) iff the weighted Busemann function $f=b_\mu$ is.

Remark 6.6. *In fact μ is semistable iff b_μ is semistable (see [LeMi]) but we do not need this fact here.*

Lemma 6.7. *If X is a locally compact Hadamard space then μ is stable iff $f = b_\mu$ is stable.*

Proof: If f is proper and bounded from below, it follows that for each geodesic ray $\rho = \overline{x\eta}$ in X

$$\lim_{t\to\infty} f(\rho(t)) = \infty,$$

thus $\text{slope}_f(\eta) = \text{slope}_\mu(\eta) > 0$. This implies stability of μ.

Conversely, suppose that f has an unbounded sublevel set $S = \{x \in X : f(x) < const\}$. Then convexity of f and local compactness of X imply that there exists a geodesic ray $\rho = \overline{x\eta} \subset S$. Thus

$$\lim_{t\to\infty} \frac{f(\rho(t))}{t} \leq 0,$$

which implies that μ is not a stable measure. $\square$

Nice semistable measures. The case of nice semistable measures is more complicated, we begin by analyzing the sets of minima of weighted Busemann functions.

Lemma 6.8. *Suppose that X is a Hadamard space, μ is a measure on $\partial_\infty X$ and b_μ is the associated weighted Busemann function on X. Assume that $X' \subset X$ is a closed convex subspace such that* $\text{Supp}(\mu) \subset \partial_\infty X'$. *Then the minimum set of $b_\mu\big|_{X'}$ is contained in the minumum set of b_μ on the entire X, in paricular, b_μ is nice semistable iff $b_\mu|_{X'}$ is nice semistable.*

Proof: The proof is based upon the following elementary observation: if $\xi \in \partial_\infty X'$ then for each $x \in X$ and its nearest point $x' \in X'$ we have

$$b_\xi(x') \leq b_\xi(x).$$

By linearity, the same inequality applies to the weighted Busemann function b_μ. Thus, if $b_\mu|_{X'}$ attains its minimum, it also attain minimum on X. $\square$

We assume from now on that X is a symmetric space.

Lemma 6.9. *Let μ be an atomic measure on $\partial_\infty X$, let v be a non-zero tangent vector in $T_x X$ and let l be the geodesic through x with initial velocity v. Then the following are equivalent:*

1. $D^2_{v,v} b_\mu = 0$.
2. *b_μ is affine linear on l.*
3. *μ is supported on the boundary of the parallel set $\partial_\infty P_l$.*

Proof: Integration yields

$$D^2_{v,v} b_\mu = \int_{\partial_\infty X} D^2_{v,v} b_\xi \; d\mu(\xi).$$

Since $D^2_{v,v}b_\xi \geq 0$, we have $D^2_{v,v}b_\mu = 0$ iff $D^2_{v,v}b_\xi = 0$ for μ-almost all ξ. Thus (according to Lemma 6.1) for each $\xi \in \mathrm{Supp}(\mu)$ the geodesic l and geodesic ray $\overline{x\xi}$ are contained in a 2-flat $F \subset X$. It follows that $\xi \in \partial_\infty P_l$ and hence (1) $\Longrightarrow$ (3).

Suppose that $\mathrm{Supp}(\mu) \subset \partial_\infty P_l$. Then for each $\xi \in \mathrm{Supp}(\mu)$ there is a 2-flat $F \subset P_l$ containing l such that $\xi \in \partial_\infty F$. Since b_ξ is affine linear on F it is also affine linear on l. Thus (3) $\Longrightarrow$ (2).

The implication (2) $\Longrightarrow$ (1) is clear. □

We define $MIN(\mu)$ to be the minimum set of the function b_μ.

Corollary 6.10. *1. $MIN(\mu)$ is a complete totally geodesic subspace of X.*

2. If μ is stable then b_μ is strictly convex and has a unique minimum.

Proof: 1. It is clear that $MIN(\mu)$ is closed and convex (since b_μ is a continuous convex function). Suppose that $x, y \in MIN(\mu)$ be distinct points and l be the complete geodesic through x, y. Then for the unit vector v tangent to the segment $\overline{xy}$ at x we have $D^2_{v,v}b_\mu = 0$. Thus Lemma 6.9 implies that b_μ is affine linear along l and therefore is constant. It follows that $l \subset MIN(\mu)$ and thus $MIN(\mu)$ is totally geodesic. (Note that instead of Lemma 6.9 one could use the fact that b_μ is real-analytic on X to prove 1.)

2. Suppose that μ is stable. Then b_μ is proper and bounded from below, which implies that $MIN(\mu) \neq \emptyset$. Suppose that for some $x \in MIN(\mu)$ and a unit vector $v \in T_xX$ we have $D^2_{v,v}b_\mu = 0$. Since $\nabla b_\mu(x) = 0$, Lemma 6.9 implies that b_μ is constant along the geodesic l through x with the initial velocity v. Thus b_μ is not proper. Contradiction. □

Lemma 6.11. *b_μ is nice semistable iff μ is nice semistable.*

$$\partial_\infty MIN(\mu) = \{\mathrm{slope}_\mu = 0\}.$$

Proof: Corollary 6.10 implies that b_μ is nice semistable iff $MIN(\mu)$ is a symmetric subspace X_0 in X. It is clear that in the case b_μ is nice semistable, $\partial_\infty X_0 \subset \{\mathrm{slope}_\mu = 0\} = C$. Let's prove the opposite inclusion. Pick a point $x \in X_0$ and for $\eta \in C$ consider the geodesic ray $\rho = \overline{x\eta}$. Since x is a point of minimum of b_μ, and the function b_μ is convex on ρ and has zero asymptotic slope, it follows that b_μ is constant along ρ. Thus $\eta \in \partial_\infty X_0$ and we have proven the equality $\partial_\infty MIN(\mu) = \{\mathrm{slope}_\mu = 0\}$. Since $\partial_\infty X_0$ is a subbuilding, we conclude that nice semistability of b_μ implies nice semistability of μ.

Consider the opposite implication. Suppose that μ is nice semistable. If μ is stable then $MIN(\mu)$ is a single point and there is nothing to prove. Thus we assume that $C = \{\mathrm{slope}_\mu = 0\} \neq \emptyset$ is a subbuilding. Pick a spherical apartment s in the building C, then $\dim(s) = \dim(C)$. For each flat $F \subset X$ with $s = \partial_\infty F$ the function b_μ is constant along F. Consider the parallel set $P_F = Y \times F$ of the flat F. If the function b_μ is not proper along Y then there is a geodesic ray $\rho = \overline{x\eta} \subset Y$ such that b_μ has zero asymptotic slope along ρ. By convexity of the function slope_μ it follows that the metric join $J = \{\eta\} \circ s$ is contained in C. However $\dim(J) > \dim(s)$.

Contradiction. Hence b_μ is proper and convex on Y, which means that b_μ attains minimum on $P_F = Y \times F$.

Lemma 6.8 implies that b_μ attains its minimum on X as well which means that b_μ is nice semistable. □

Structure of minimum sets for nice semistable measures.

Theorem 6.12 (Structure of minimum sets). *For each nice semistable measure μ on $\partial_\infty X$ there is a product subspace $Y_0 \times F_0 \times Z_0 \subset X$ such that the following holds:*

1. *Y_0 and Z_0 have no Euclidean factor. F_0 is Euclidean.*
2. *$MIN(\mu) = Y_0 \times F_0$.*
3. *$\mathrm{Supp}(\mu) \subseteq \partial_\infty(Z_0 \times F_0)$.*
4. *$b_\mu|_{F_0}$ is constant and $b_\mu|_{Z_0}$ is proper and bounded below.*
5. *The projection of μ to $\partial_\infty Z_0$ is stable, the projection of μ to $\partial_\infty F_0$ is semistable.*

Proof: It suffices to consider the case when the measure μ is not stable. The totally geodesic subspace $MIN(\mu)$ splits as $MIN(\mu) = Y_0 \times F_0$ where F_0 is the Euclidean deRham factor of $MIN(\mu)$. The parallel set of $MIN(\mu)$ splits as

$$P_{MIN(\mu)} = Y_0 \times F' \times Z_0$$

where $F' \supseteq F_0$ is a flat and Z_0 has no Euclidean factor. According to Lemma 6.9, μ is supported on the ideal boundary of $\cap_{l \subseteq MIN(\mu)} P_l$. Since

$$\bigcap_{l \subseteq MIN(\mu)} P_l \subset P_{MIN(\mu)} = Y_0 \times F' \times Z_0,$$

it follows that

$$\bigcap_{l \subseteq MIN(\mu)} P_l = F' \times Z_0.$$

Since $\mathrm{Supp}(\mu) \subset \partial_\infty(F' \times Z_0)$, each translation along a geodesic in F' fixes $\mathrm{Supp}(\mu)$ pointwise; thus the Busemann function b_μ must be constant on $F' \times \{z\}$ for each $z \in Z_0$. Since $F_0 \subset F'$, the flat F' is also contained in $MIN(\mu)$. Therefore $F_0 = F'$. Since $\partial_\infty(Y_0 \times F_0) = \{\mathrm{slope}(b_\mu) = 0\}$, the function b_μ must be proper on $x \times Z_0$ for each $x \in Y_0 \times F_0$. This proves assertions 1–4.

Let μ', μ'' denote the projections of μ to $\partial_\infty F_0$ and $\partial_\infty Z_0$ respectively (see section 4.1). Then both μ', μ'' are semistable. Recall that (see equation (16))

$$b_\mu \Big|_{F_0 \times Z_0} = b_{\mu'} + b_{\mu''};$$

thus semistability of μ' implies that $b_{\mu'} = 0$. Since $b_\mu = b_{\mu''}$ is proper on Z_0 we conclude that the measure μ'' on $\partial_\infty Z_0$ is stable. This proves 5. □

We now can prove Proposition 4.5:

Proposition 6.13. *Let X be a symmetric space and μ be a measure on $\partial_{Tits}X$. Suppose that $F \times Z \subseteq X$ is a smallest symmetric subspace whose ideal boundary contains* $\mathrm{Supp}(\mu)$*, and let F be its Euclidean deRham factor. Then:*

1. μ is nice semistable if and only if the projection of μ to $\partial_\infty F$ is semistable, and the projection to $\partial_\infty Z$ is stable.

2. In the case when μ is nice semistable, $F \times Z$ is unique up to parallel translation, and $\{\mathrm{slope}_\mu = 0\} = \partial_{Tits}(F \times Y)$ where Y is the symmetric subspace such that $F \times Z \times Y$ is the parallel set of $F \times Z$.

Proof: 1. For a measure μ on $\partial_\infty X$, let $X' \subset X$ be a smallest symmetric subspace whose ideal boundary contains $\mathrm{Supp}(\mu)$, and let $F \subset X'$ be the Euclidean deRham factor of $X' = F \times Z'$. If the projection μ' of μ to $\partial_\infty F$ is semistable, and the projection μ'' to $\partial_\infty Z$ is stable then $b_{\mu'} \equiv 0$ and hence

$$b_\mu \Big|_{X'} = b_{\mu''}.$$

Thus b_μ attains minimum on X', and therefore, by Lemma 6.8, on the entire X. It follows that μ is nice semistable.

Conversely, suppose that μ is nice semistable. Let $X' \subset X$ be a smallest symmetric subspace such that $\mathrm{Supp}(\mu) \subset \partial_\infty X'$. Let $X' = F \times Z$ where F is the flat deRham factor of X'; clearly μ is nice semistable as a measure on $\partial_\infty X'$. Thus, by applying Theorem 6.12 to X' and using minimality of X', we get: $F \subset MIN(\mu)$, the projection of μ to $\partial_\infty F$ is semistable, the projection to $\partial_\infty Z$ is stable. This proves 1.

2. Consider the product subspace $Y_0 \times F_0 \times Z_0 \subset X$ given by Theorem 6.12. By minimality of X', $\partial_\infty X' \subset \partial_\infty(F_0 \times Z_0)$. Since $Y_0 \times F_0 \times Z_0$ is the parallel set of $F_0 \times Z_0$, it follows that $X' \subset \{y\} \times F_0 \times Z_0$ for some $y \in Y_0$. The inclusion $F \subset MIN(\mu) = Y_0 \times F_0$ implies that $F \subset F_0$. Stability of the projection of μ to $\partial_\infty Z$ implies that $Z \subset Z_0$. Splitting F_0 as $F \times F'$ we get $Y := F' \times Y_0$, and therefore $\{\mathrm{slope}_\mu = 0\} = \partial_\infty(F \times Y)$.

It remains to show that $Y \times F \times Z$ is the parallel set of $F \times Z$. The inclusion $Y \times F \times Z \subset P_{F \times Z}$ follows from the equality $Y_0 \times F_0 \times Z_0 = P_{F_0 \times Z_0}$. The opposite inclusion follows from the fact that $Y \times F$ is the minimum set of b_μ on X. $\square$

Polygons in infinitesimal symmetric spaces and weighted configurations.

Let X be a nonpositively curved symmetric space. We will identify X with the quotient G/K where G is the connected component of $\mathrm{Isom}(X)$ and K is a maximal compact subgroup of G, the stabilizer of a base-point $o \in X$. Then we have the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ of the Lie algebra of G so that T_oX is identified with $\mathfrak{p}$.

Suppose that $\psi : (\mathbb{Z}/n, \nu) \to \partial_\infty X$ is a weighted configuration on $\partial_\infty X$, and let $\mu := \psi_*(\nu)$. Pick a point $x \in X$. The configuration $\psi = ((m_1, \xi_1), ..., (m_n, \xi_n))$ corresponds to a collection of tangent vectors $v_1, ..., v_n \in T_xX$ so that $\|v_i\| = m_i$ and the geodesic ray γ_i emanating from x and having initial velocity v_i, is asymptotic to ξ_i, $i = 1, ..., n$. Note that v_i is the gradient (at x) of the function $-m_i b_{\xi_i}$. Thus

$$\sum_{i=1}^n v_i = 0 \iff \nabla b_\mu(x) = 0 \iff x \in MIN(\mu).$$

Therefore existence of a point $x \in X$ with $\sum_{i=1}^{n} v_i = 0$ is equivalent to nice semistability of ψ. Note that the vectors $v_1, ..., v_n \in T_x X$ satisfying $v_1 + ... + v_n$ form a (closed) n-gon P in the infinitesimal symmetric space $T_x X$; we regard the weighted configuration $\psi : \mathbb{Z}/n \to \partial_\infty X$ as an (infinitesimal) Gauss map of the polygon P. Hence we get:

Proposition 6.14 ([LeMi]). *A configuration ψ is nice semistable iff there exists a point $x \in X$ and an n-gon P in the infinitesimal symmetric space $T_x X$ such that ψ is an (infinitesimal) Gauss map of P.*

To get a polygon in the distinguished tangent space $\mathfrak{p}$ at the base-point $o \in X$ we apply an element $g \in G$ such that $g(x) = o$:

Corollary 6.15 ([LeMi]). *A configuration ψ is nice semistable iff there exists an element $g \in G$ and an n-gon P in the infinitesimal symmetric space $\mathfrak{p}$ such that $g \circ \psi$ is an (infinitesimal) Gauss map of P.*

Let $\partial_\infty \mathfrak{p}$ be given structure of a spherical building B. Then we get

Theorem 6.16 ([LeMi]). $\tau = (\tau_1, ..., \tau_n) \in \Delta_{ss}^n(B)$ *iff there exists an n-gon P in the infinitesimal symmetric space $\mathfrak{p}$ with the Δ-side lengths $\tau_1, ..., \tau_n$.*

Question 6.17. *Does the above theorem hold for Cartan motion spaces which are not isomorphic to infinitesimal symmetric spaces?*

Douady-Earle barycenter.

If μ is a stable measure then the function b_μ has unique minimum $x \in X$ which is also the unique (nondegenerate) critical point of b_μ. This point x is called the *Douady-Earle barycenter $DE(\mu)$ of the measure μ.* This definition makes sense even if $\mathrm{Supp}(\mu)$ is not a finite set; the Douady-Earle barycenter was first introduced by Douady and Earle in [DE] for hyperbolic spaces. We note that for each $t > 0$ the measure $t\mu = \psi_*(t\nu)$ is also stable. Let Φ_{ψ_t} denote the mapping $X \to X$ associated with the weighted configuration ψ_t:

$$\psi : (\mathbb{Z}/n, t\nu) \to \partial_\infty X.$$

Each mapping Φ_{ψ_t} has a unique fixed point $x_t \in X$. Repeating the arguments from [KMT] one can show that $x_t, t \in (0, \infty)$, is a smooth curve in X and that

$$\lim_{t \to 0} x_t = DE(\mu).$$

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