

Convex projective Gromov–Thurston examples

Michael Kapovich

April 12, 2006

Abstract

We consider Gromov–Thurston examples of negatively curved n -manifolds which do not admit metrics of constant sectional curvature. We show that for each $n \geq 4$ some of the Gromov–Thurston manifolds admit strictly convex real–projective structures.

1 Introduction

Gromov and Thurston in [5] constructed, for each $n \geq 4$, examples of compact n -manifolds which admit metrics of negative curvature, with arbitrarily small pinching constants, but do not admit metrics of constant curvature. We review these examples in section 2.

The main goal of this paper is to put *convex projective structures* on Gromov–Thurston examples. Suppose that $\Omega \subset \mathbb{R}P^n$ is an open subset and $\Gamma \subset PGL(n+1, \mathbb{R})$ is a subgroup acting properly discontinuously on Ω . The quotient orbifold $Q = \Omega/\Gamma$ has natural projective structure c . The structure c is said to be *(strictly) convex* iff Ω is a (strictly) convex proper subset of $\mathbb{R}P^n$. In this case we refer to Q as (strictly) convex projective orbifold.

Our main result then is:

Theorem 1.1. *Gromov–Thurston examples admit strictly convex projective structures.*

We refer the reader to section 6 for the more precise statement. Our theorem will be proven in section 6 via “bending” of the original hyperbolic structure on a certain hyperbolic manifold M (used to construct Gromov–Thurston examples) in the manner similar to [5], where flat-conformal structures were constructed on certain negatively curved manifolds.

There are two parts in this proof: (1) Producing a projective structure, (2) proving that the structure is convex. Then strict convexity of the structure follows from Benoist’s theorem below (theorem 1.2), since Gromov-Thurston examples have Gromov-hyperbolic fundamental groups.

Part (1) is dealt with by solving a certain *product of matrices* problem, which is a special case of a Lie-theoretic problem interesting on its own right, see section 4. The projective manifolds M' are then built by gluing convex subsets of the hyperbolic manifolds M . By passing to the universal cover we obtain a tessellation of \tilde{M}' by convex polyhedra in \mathbb{H}^n , each of which has infinitely many facets.

Dealing with (2) is especially interesting, since, at present, there is only one general method for proving convexity of projective structures, namely via Vinberg–Tits fundamental domain theorem [9]. Unfortunately, this theorem applies only to reflection groups, which cannot be used in higher dimensions. Our approach to proving convexity is to adapt Vinberg’s arguments in a more general context of manifolds obtained by gluing convex cones with *infinitely many faces*. In this setting, Vinberg’s arguments (requiring polyhedrality of the cones) do not directly apply and we modify them by appealing to the *small cancellation theory*, see section 3.

The main motivation for this paper comes from the following beautiful

Theorem 1.2. (*Y. Benoist, [3]*) *Suppose that a convex projective orbifold M is compact. Then M is strictly convex iff $\Gamma = \pi_1(M)$ is Gromov-hyperbolic.*

Examples of convex-projective structures on compact orbifolds are provided by the quotients of round balls in $\mathbb{R}P^n$ by discrete cocompact groups of automorphisms. The Hilbert metric on such examples is a Riemannian metric of constant negative sectional curvature. Thus such orbifolds are *hyperbolic*. By deforming the above examples in $\mathbb{R}P^n$ one obtains other examples of strictly convex projective manifolds/orbifolds.

In 2002 I was asked by Bruce Kleiner and Francois Labourie if one can construct examples of compact strictly convex projective manifolds which are not obtained by deforming hyperbolic examples. The main goal of this paper is to prove that such examples indeed exist in all dimensions ≥ 4 . Independently, such examples were constructed by Yves Benoist in dimension 4 using reflection groups, see [2]. The paper [2] also produces “exotic” strictly convex subsets Ω in $\mathbb{R}P^n$ for all $n \geq 3$: The metric space (Ω, d_H) is Gromov-hyperbolic but is not quasi-isometric to \mathbb{H}^n , where d_H is the Hilbert metric

on Ω . However these examples do not appear to admit discrete cocompact groups of automorphisms.

Acknowledgments. During the work on this paper I was partially supported by the NSF grant DMS-04-05180. I am grateful to John Millson for explaining to me the construction of arithmetic examples in section 2 and to Bruce Kleiner for useful conversations.

2 Gromov-Thurston examples

In this section we review Gromov-Thurston examples [5] of compact n -manifolds/orbifolds which admit metrics of negative curvature but do not admit metrics of constant curvature. (Note that Gromov and Thurston [5] have other examples as well: These examples will not be discussed here.)

Consider the quadratic form

$$\varphi(x) = x_1^2 + \dots + x_n^2 - \sqrt{p}x_{n+1}^2$$

where p is a (positive) prime number, $n \geq 2$. Let $\tilde{\Gamma} = \text{Aut}(\varphi) \cap GL(n+1, \mathbb{Z})$; then $\tilde{\Gamma}$ is a cocompact arithmetic subgroup in $\text{Aut}(\varphi) \cong O(n, 1)$.

We let H denote the Lorentzian model of the hyperbolic space \mathbb{H}^n :

$$\{x : \varphi(x) = -1, x_{n+1} > 0\}.$$

Consider the linear subspace

$$V = \{x \in \mathbb{R}^{n+1} : x_1 = x_2 = 0\}.$$

The intersection $V \cap H$ is a totally-geodesic codimension 2 hyperbolic subspace. The stabilizer of V in $\tilde{\Gamma}$ acts cocompactly on $V \cap H$ since it is isomorphic to the set of integer points in the algebraic group

$$\text{Aut}(x_3^2 + \dots + x_n^2 - \sqrt{p}x_{n+1}^2).$$

Suppose that $W \subset \mathbb{R}^{n+1}$ is a *rational* codimension 1 linear subspace containing V . Then the Lorentzian (with respect to φ) involution τ_W fixing W pointwise belongs to $GL(n+1, \mathbb{Q})$. Observe that the groups $\tilde{\Gamma}$ and $\tau_W \tilde{\Gamma} \tau_W$ are commensurable. Therefore, there exists a finite index subgroup $\Gamma_W \subset \tilde{\Gamma}$ which is normalized by τ_W . By applying this procedure to two appropriately chosen rational hyperplanes passing through V we obtain

Lemma 2.1. *Given a number $m \geq 1$ there exists a subgroup $\hat{\Gamma} \subset \text{Aut}(\varphi)$ commensurable to $\tilde{\Gamma}$, which contains a dihedral subgroup D_m fixing V point-wise. The generating involutions in D_m acts as reflections.*

By passing to an appropriate torsion-free normal subgroup $\Gamma \subset \hat{\Gamma}$ we get a compact hyperbolic manifold $M = H/\Gamma$. Let $\bar{\Gamma}$ denote the subgroup of $\hat{\Gamma}$ generated by Γ and the dihedral subgroup D_m .

The group D_m acts on M isometrically with a fundamental domain O (that can be identified with the orbifold $M/D_m = \mathbb{H}^n/\bar{\Gamma}$), which is a manifold with corners so that the corner (possibly disconnected) corresponds to the hyperbolic subspace $V \cap \mathbb{H}^n$. The dihedral angle at this corner is π/m . By abusing notation we will keep the notation V for this codimension 2 totally-geodesic submanifold of M .

The boundary of $O \setminus V$ is the union of two codimension 1 totally-geodesic (possibly disconnected) submanifolds. We denote the closures of these submanifolds W_1, W_2 : these are submanifolds with boundary (which is equal to V) in M . Then we can think of the manifold M as obtained by gluing $2m$ copies of O .

Assumption 2.2. *We assume from now on that the manifold M admits an isometric action $D_{2m} \curvearrowright M$ of a dihedral group D_{2m} which contains D_m as an index 2 subgroup.*

Then there is an isometric involution $\iota : O \rightarrow O$ which interchanges W_1 and W_2 .

We now construct new manifolds (without boundary) M' by gluing $2m - 2$ copies of O .

Remark 2.3. *Another class of Gromov–Thurston examples M'' is obtained by gluing $2m + 2$ copies of O . Construction of projective structures on such manifolds is a bit more complicated than the one explained in this paper, therefore the manifolds M'' will not be discussed here.*

We will think of M, M' as *doubles* of the manifolds N, N' which are obtained by gluing $m, m - 1$ copies of O respectively. Thus M' is obtained by “subtracting” two copies of O to M .

In section 6 we will use an alternative description of M' . Assumption 2.2 implies that the manifold N' admits a reflection symmetry θ' fixing the submanifold V . Then M' is diffeomorphic to the manifold obtained by gluing two copies of N' via the involution $\theta'|\partial N'$ of the boundary.

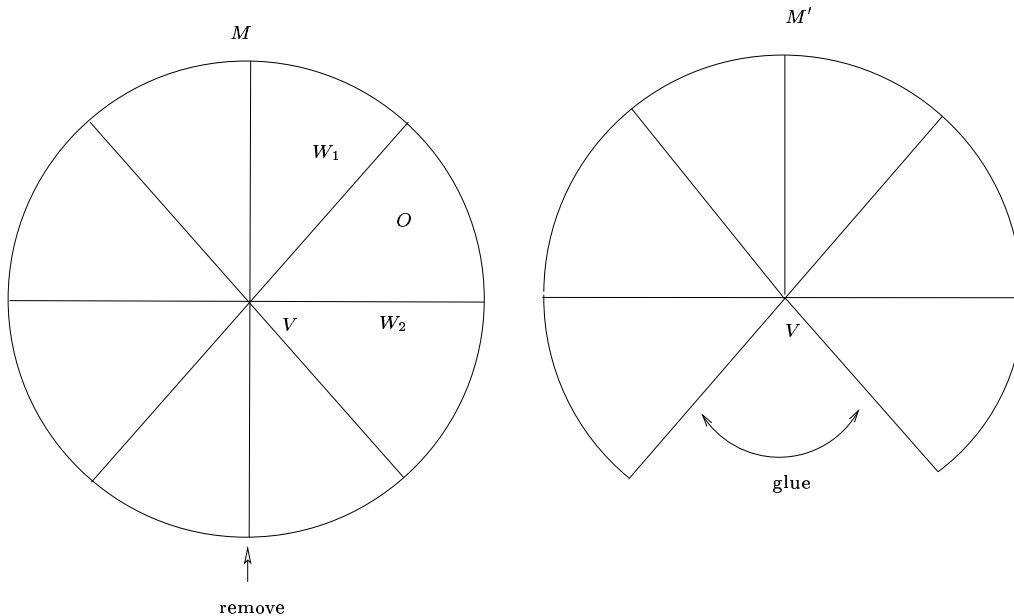


Figure 1: *Constructing the manifold M' .*

Proposition 2.4 (Gromov, Thurston, [5]). *For sufficiently large m , the manifold M' admits a metric of negative sectional curvature varying in the interval $[-1 + \epsilon_m, -1]$. Moreover, $\lim_{m \rightarrow \infty} \epsilon_m = 0$.*

Remark 2.5. *Note that M' admits a canonical singular Riemannian metric which is smooth and hyperbolic away from V . The negatively curved Riemannian metric on M' is obtained by modifying the above singular metric on a regular R -neighborhood of V . This modification works provided that R is sufficiently large, which is achieved by taking large m . Alternatively, one can fix m and pass to an appropriate finite-index subgroup of Γ .*

Thus $\pi_1(M')$ is Gromov-hyperbolic provided that m is sufficiently large.

Proposition 2.6 (Gromov, Thurston, [5]). *If $n \geq 4$ then M' does not admit a metric of constant (negative) curvature.*

Proof. Our argument is a variation on the argument given in [5]. The idea of the proof is to apply Mostow Rigidity Theorem several times both in dimension n and $n - 1$.

Suppose that M' admits a hyperbolic metric g . Observe that the group $F = D_{m-1}$ acts (via homeomorphisms) on M' . Therefore, by Mostow Rigidity Theorem, $F \curvearrowright M'$ is homotopic to an isometric action $F \curvearrowright (M', g)$; the fixed-point set of this action is a submanifold V' homotopic to V .

The fundamental domain for the latter action is a submanifold with boundary $O' \subset M'$ homotopic to O . Thus the dihedral angle of O' along V' equals $\frac{\pi}{m-1}$.

The group $F \curvearrowright M'$ contains a topological reflection σ fixing $S := \partial N'$ pointwise. Therefore, σ is homotopic to an isometric reflection $\sigma' \in F \curvearrowright M'$, whose fixed-point set is a hypersurface S' homotopic to S . Then S' is a hyperbolic $n - 1$ -dimensional manifold homotopy-equivalent to ∂N . Since ∂N is also a hyperbolic manifold, it follows from Mostow Rigidity Theorem that ∂N and S' are isometric. Let $L' \subset M'$ denote the submanifold bounded by S' and homotopic to N' .

Then we can glue N and L' along their totally-geodesic boundaries via the isometry $\partial N \rightarrow S'$. The result is a compact hyperbolic manifold K which is obtained by gluing $2m - 1$ submanifolds O_j , each of which is homotopy-equivalent to O . Since O admits a reflection symmetry ι , it follows (from Mostow Rigidity Theorem) that K admits an isometric dihedral group action

$$D_{2m-1} \curvearrowright K,$$

whose fundamental domain is a submanifold O'' with the dihedral angle $\frac{2\pi}{2m-1}$ along the fixed-point set V'' of D_{2m-1} .

Note that the hyperbolic manifolds with boundary O , O' and O'' are homotopy-equivalent to each other, where the homotopy-equivalences restrict to isometries between their boundaries. The boundary of each manifold is totally-geodesic away from an $n - 2$ -dimensional submanifold V, V', V'' ; the dihedral angles equal $\frac{\pi}{m}$, $\frac{\pi}{m-1}$ and $\frac{2\pi}{2m-1}$ respectively.

We now take m copies of O , $m - 1$ copies of O' and glue them together (using isometries of the components of $\partial O \setminus V, \partial O' \setminus V'$) to form a manifold Q . In the manifold Q the submanifolds V, V' are identified with a codimension 2 submanifold U (isometric to $V \cong V'$). The total dihedral angle along U equals

$$m \frac{\pi}{m} + (m - 1) \frac{\pi}{m - 1} = 2\pi.$$

Thus the manifold Q is hyperbolic.

On the other hand, we can glue together

$$2m - 1 = m + (m - 1)$$

copies of the manifold O'' to form a hyperbolic manifold Q' . Thus there exists a homotopy-equivalence $h : Q \rightarrow Q'$ which carries copies of O, O' to copies of O'' . However, if there is an isometry $h' : Q \rightarrow Q'$ homotopic to h , then h' would have to carry a copy of O to a copy of O'' . The latter is impossible since these orbifolds have different dihedral angles along V and V'' . Contradiction. \square

Definition 2.7. *When $n \geq 4$ (and m is sufficiently large), we will refer to the manifolds M' as Gromov-Thurston examples.*

3 A convexity theorem

Given a point $x \in \mathbb{R}^n$ and a set $B \subset \mathbb{R}^n$ let $\Sigma = Cone_x(B)$ denote the union of all segments $\overline{xb}, b \in B$. We will refer to x as the *tip* and B as the *base* of this cone.

Lemma 3.1. *If B is convex then Σ is also convex.*

Proof. Let $p, q \in \Sigma$. Then there exist $a, b \in B$ such that $p \in \overline{ax}, q \in \overline{bx}$. Thus the segment \overline{pq} is contained in the planar triangle $\Delta(a, b, x)$ with the vertices a, b, x . Since $\overline{ab} \subset B$, it follows that

$$\overline{pq} \subset \Delta(a, b, x) \subset \Sigma. \quad \square$$

Throughout this section we will assume that C is an open convex homogeneous cone in \mathbb{R}^n which is different from \mathbb{R}^n itself.

Definition 3.2. *1. An (open) facet of C is an open convex homogeneous $(n - 1)$ -dimensional cone contained in the boundary of C .*

2. A codimension k face of C is an open convex $(n - k)$ -dimensional cone contained in the boundary of C . (We will mostly need this for $k = 2$.)

We will use the notations \bar{F}, \bar{C} , etc. to denote the closures of faces, cones, etc. Accordingly, we will refer to closed facets, closed codimension k faces of C , etc.

For each face F of C let $H = Span(F)$ denote the hyperplane in \mathbb{R}^n spanned by F .

Let D be a convex subset of \mathbb{R}^n , so that $C \subset D \subset \bar{C}$, and which is obtained by adding to C some of the faces.

Lemma 3.3. *Let $x, y \in \bar{C}$ be such that $y \in D$. Then the half-open interval $(x, y] := \overline{xy} \setminus \{x\}$ is contained in D .*

Proof. The point y belongs to a certain open convex cone E which is either C , or a face of C . Let $B \subset E$ be an open round ball centered at y . Let $\Sigma = \text{Cone}_x(B)$ denote the cone with the tip at x and base B , which is the union of segments connecting x to the points of B . Then Σ has nonempty interior $\Sigma^0 \subset \text{Span}(E)$ containing the open segment $(x, y]$. By convexity, $\Sigma \subset \bar{E}$, hence

$$(x, y] \subset \Sigma^0 \cup \{y\} \subset E \subset C. \quad \square$$

Let \mathcal{F} be a certain collection of faces of F , so that $C \in \mathcal{F}$. We define a new convex cone C' as

$$C' = \bigcup_{F \in \mathcal{F}} F.$$

Then $C \subset C' \subset \bar{C}$. For a face $F \in \mathcal{F}$ we let F' denote the closure of F in C' . By abusing the notation we will continue to refer to the F' 's as *faces* of C' .

Assumption 3.4. *We assume that C and \mathcal{F} are such that:*

1. *Each face $F \in \mathcal{F}$ of codimension ≥ 2 is the intersection of the higher-dimensional faces G'_i .*
2. *For each pair of distinct facets $F_1, F_2 \in \mathcal{F}$, with the $n - 2$ -dimensional intersection $\bar{F}_1 \cap \bar{F}_2$, we require this intersection to contain a codimension 2 face $F \in \mathcal{F}$.*

Suppose now that X is a simply-connected affine n -manifold obtained by gluing infinitely many copies $C'_j, j \in J$, of the cone C' via linear isomorphisms of faces F' of C' . We will refer to the cones C_j as *cells* in X . We will assume that each face of X is contained in only finitely many other faces.

In a similar fashion we define a cell complex \bar{X} by extending the above gluing maps to the closed cells \bar{C}_j . The *origin* $0 \in \bar{X}$ is the point corresponding to 0 in the closed cone C . Since X contains infinitely many cells, the space \bar{X} is not locally compact (the origin is incident to infinitely many cells).

Remark 3.5. *More generally, one can allow spaces X built out of non-isomorphic convex cones C_j . However we do not need this for the purposes of this paper.*

We then have a *developing map* $dev : X \rightarrow \mathbb{R}^n$, which is a linear isomorphism on each cell C_j . The developing map extends naturally to a map $dev : \bar{X} \rightarrow \mathbb{R}^n$. This map determines the notion of a *segment* $[xy]$ in \bar{X} , which is defined as a path which is mapped by dev homeomorphically to a straight-line segment in \mathbb{R}^n .

Definition 3.6. *A subset S of \bar{X} is called convex if every two points in S can be connected by a segment which is contained in S .*

Assumption 3.7. *1. We assume that X is such that for each point in every codimension 2 face E of $C_j \subset X$, the tessellation of X by the adjacent cells is locally isomorphic to a tessellation of \mathbb{R}^n by cones cut off by a family of $t \geq 4$ hyperplanes passing through a codimension 2 subspace. (Thus the number of adjacent cones is $2t \geq 8$.) See Figure 2.*

2. In addition, we assume that for every pair of cells C'_i, C'_j sharing a facet F , the union $C'_i \cup C'_j$ is convex.

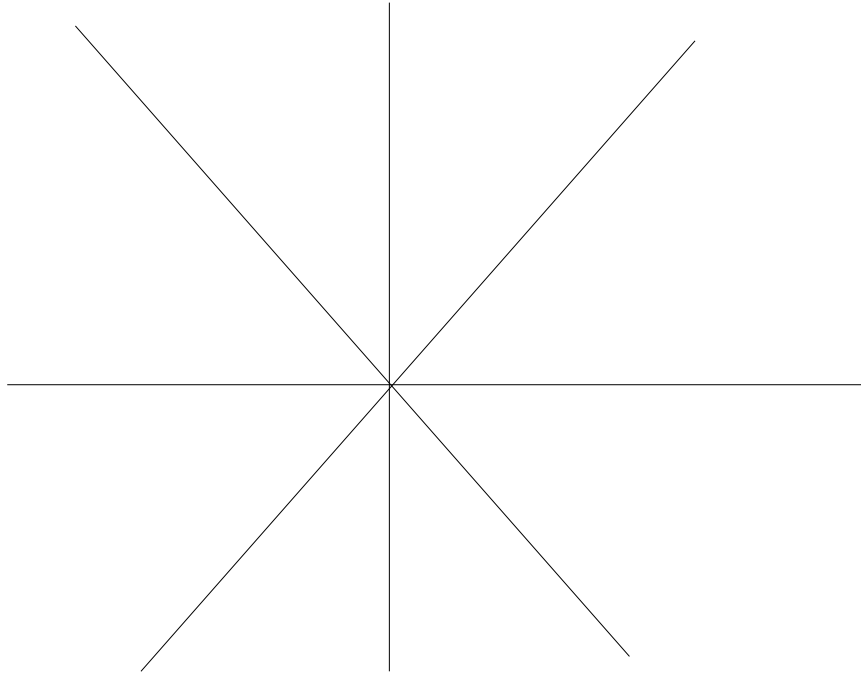


Figure 2: *In this example $t = 4$.*

Note that this assumption is satisfied in a number of important cases, e.g. for tessellations corresponding to linear reflection groups.

A wall H in X is a maximal connected subset of X which is the union of facets F'_j so that each point $x \in H$ has a neighborhood $U \subset H$ which is mapped by dev homeomorphically to an open disk (or a half-disk) of a hyperplane in \mathbb{R}^n . Thus the developing map sends each wall to a subset of a hyperplane in \mathbb{R}^n . Therefore, each segment $\sigma \subset X$ can intersect a wall H transversally in at most one point.

Remark 3.8. *One can show that each wall is a manifold without boundary.*

The main result of this section is the following

Theorem 3.9. *Suppose that X is as above and $n = \dim(X) \geq 3$. Then:*

1. \bar{X} is convex and $dev : \bar{X} \rightarrow \mathbb{R}^n$ is an isomorphism onto a convex homogeneous cone in \mathbb{R}^n .
2. X is convex and the developing map dev is an isomorphism of X onto a proper open convex homogeneous cone in \mathbb{R}^n .

Proof. Our proof is modelled on the standard arguments appearing in the proofs of Poincaré's fundamental domain theorem (cf. [9]).

Let Z denote the nerve of the collection of codimension ≥ 2 faces in X . Then $\dim(Z) \leq 2$; our assumptions on X imply that Z is a simply-connected regular cell complex. Nonempty intersections between cells in Z are again cells. Each 2-face c of Z is a $2t$ -gon for a certain $t \geq 4$ (depending on c). Since C and its facets are convex it follows that the links of vertices of Z contain no bigons. Therefore Z satisfies the *small cancellation condition* $C'(1/7)$, see [4, Appendix].

Remark 3.10. *Since Z is simply-connected and satisfies $C'(1/7)$ condition, it follows that Z is contractible. Therefore \mathcal{F} contains only faces of codimension ≥ 2 .*

However in general Z is not locally compact, since C can have infinitely many facets. This is where we deviate from Vinberg's argument [9].

Definition 3.11. *A path p in the 1-skeleton $T := Z^{(1)}$ of Z is a local geodesic (or Dehn-reduced) if it contains no backtracks and no subpaths of length $> t$ contained in a single t -gonal 2-cell of Z .*

A degenerate bigon in T has two vertices x, y and two equal edges α, β , which are local geodesics in T .

Given two vertices $x, y \in T$ which belong to a common t -gonal 2-cell c of Z and which are distance t apart, there are exactly two geodesics α, β (of

length t) connecting x to y . The union of these geodesics is the boundary of c . We then obtain an *elementary bigon* in T with the vertices x, y and edges α, β .

More generally, define a *corridor* D in Z as a union of 2-cells F_1, \dots, F_l ($l \geq 2$) of the complex Z so that for each i :

1. F_i, F_{i+1} share an edge e_i , called an *interior edge* of the corridor D .
2. The edges e_{i-1}, e_i are “antipodal” on the boundary of F_i .

Thus each corridor D yields a wall in X .

Consider the topological circle λ which is the boundary of the corridor $D = F_1 \cup \dots \cup F_l$. Pick two vertices $x, y \in \lambda$. They are connected by two curves $\beta, \gamma \subset \lambda$ whose union is λ .

Definition 3.12. *In case both β, γ are local geodesics, we refer to $\beta \cup \gamma$ as a simple bigon with the vertices x, y .*

One can actually show that in case when $\beta \cup \gamma$ is a simple bigon, at least one of the arcs β, γ is a geodesic in T . Moreover, given a corridor D , there are exactly 4 simple bigons contained in the boundary of the disk D .

Lemma 3.13. *Let $x, y \in T$, be vertices within distance d from each other. Then:*

1. *There are only finitely many local geodesics in T connecting x to y .*
2. *The union $\text{geo}(x, y)$ of geodesics connecting x to y is convex in T . The distance between any two vertices in $\text{geo}(x, y)$ is $\leq d$. Thus $\text{diam}(\text{geo}(x, y)) = d$.*

Proof. Since Z satisfies the condition $C'(1/7)$, according to [4, Proposition 39, Part (i)], all (local) geodesic bigons in T are concatenations of:

1. Degenerate bigons.
2. Elementary bigons.
3. Simple bigons.

See Figure 3.

Remark 3.14. *The proofs in [4, Appendix] are given under the assumption that the cell complex is the Cayley complex of a finitely-generated group. However the proofs needed for [4, Proposition 39, Part (i)] do not require this assumption.*

Consider a local geodesic segment γ in T with the end-points x, y . Then each subpath α of length 2 in γ is contained in at most one 2-cell c_α of Z .

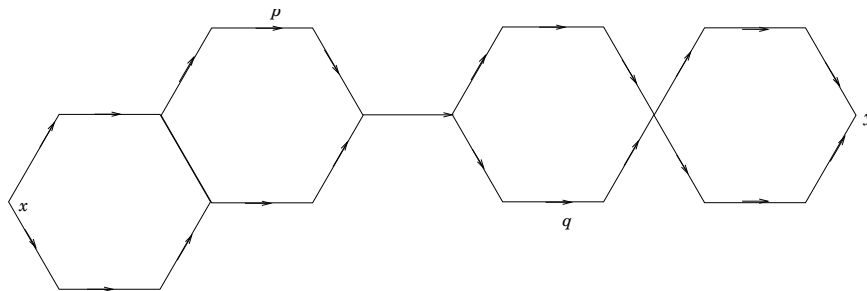


Figure 3: A geodesic bigon with the vertices x, y .

Thus the union

$$U := \bigcup_{\alpha \subset \gamma} c_\alpha$$

is compact. According to the above description of geodesic bigons, each geodesic $\beta \subset T$ connecting x to y is contained in U . Thus the set of such geodesics is finite. This proves the first assertion.

Each geodesic bigon $\beta \cup \gamma$ in T bounds a disk $D \subset Z$ of the least combinatorial area; we will refer to this area as the *area of the bigon*. Then for the given geodesic β there exists a unique geodesic γ , connecting the end-points x, y of β , so that the resulting bigon has maximal area. Namely, consider subpaths $\alpha \subset \gamma$ each of which are contained in the boundary of a 2-cell $c_\alpha \subset Z$ and the length of α is half of the perimeter of c_α . Then the distinct subpaths α, α' either do not overlap or overlap along an edge (in the case they belong to a common corridor). Let D be the union of γ with all the 2-cells c_α . Then D is the minimal disk bounding the maximal bigon $\beta \cup \gamma$ with the vertices x, y .

It follows from the description of D that all geodesics in T connecting x to y are contained in the 1-skeleton of D . If $z, w \in D^{(0)}$ are not in the same 2-cell contained in D , then each geodesic connecting p to q extends to a geodesic connecting x to y . If z, w belong to the same 2-cell c then all geodesics connecting z to w are contained in ∂c . Thus D is convex. \square

We call a segment $[x, y] \subset \bar{X}$ *generic* if the open segment (x, y) is entirely contained in the union of open cells and open facets and is not contained in a single facet.

Each generic segment $\sigma = [x, y]$ determines a path $p(\sigma)$ in T : The vertices z_0, z_1, \dots, z_k and edges of this path correspond to the open cells C_0, \dots, C_k and open facets in X which cover the open segment (x, y) .

Definition 3.15. *The union*

$$C'_0 \cup \dots \cup C'_k$$

is the gallery $gal(p(\sigma))$ corresponding to the path $p(\sigma)$. The number k is the length of the gallery, it equals the length of the path $p(\sigma)$ in T . By abusing notation we will refer to k as the distance between C_0 and C_k .

The following lemma shows that the distance between C_0 and C_k is the distance between the end-points of the path $p(\sigma)$ in the graph T , thereby justifying the above definition.

Lemma 3.16. *The path $p(\sigma)$ is a geodesic in T .*

Proof. It follows immediately from convexity of the cells in X and the Assumption 3.7, that $p(\sigma)$ is a local geodesic. Suppose that $p(\sigma)$ is not a global geodesic. Then there exists a shorter geodesic $q \subset T$ connecting the end-points of $p(\sigma)$. Thus $p(\sigma) \cup q$ is a (locally) geodesic bigon in T bounding a minimal disk $D \subset Z$. Since $p(\sigma)$ is not a geodesic, the disk D contains a corridor. Therefore it suffices to obtain a contradiction in the case when D is a corridor itself. In this case there is a wall H in X whose intersection with the gallery $gal(p(\sigma))$ is not connected. (The wall H passes through the facets in X corresponding to the interior edges of the corridor D .) However the segment σ can intersect the wall H transversally in at most one point. Contradiction. \square

Remark 3.17. *The above lemma is analogous to the familiar description of geodesics in Cayley graphs of Coxeter groups.*

We now extend the definition 3.15 to allow galleries associated with arbitrary geodesic paths $q \subset T$.

Given two cells A, B in X , let $Gal(A, B)$ denote the union of all galleries

$$A' = C'_0 \cup C'_1 \cup \dots \cup C'_l = B',$$

connecting A to B . We define the union of *closed galleries*

$$Gal(\bar{A}, \bar{B})$$

as the closure of $Gal(A, B)$ in \bar{X} .

Observe that, according to Lemmata 3.13 and 3.16, $Gal(A, B)$ is a finite union of cells.

We will prove Theorem 3.9 on existence of a segment connecting points x, y by induction on the distance between the cells.

In case when we have points $x, y \in \bar{X}$ (resp. X) which belong to the same cell, there is nothing to prove (since each cell is convex).

Suppose that for every pair of points in \bar{X} which belong to cells within distance $\leq k - 1$, there exists a segment $[x, y] \subset \bar{X}$. Our goal is to prove the same assertion for k .

Pick two cells A, B which are distance k apart and which correspond to vertices $a, b \in T$. Our goal is to show that there exists a segment $[x, y] \subset \bar{X}$ for all $x \in \bar{A}, y \in \bar{B}$.

Recall that the union $geo(a, b)$ of geodesics in T connecting the vertices a, b is convex and has diameter equal to $d(a, b) = k$. Therefore, each cell $D \subset Gal(A, B)$ which is adjacent to B , the gallery $Gal(A, D)$ is contained in $Gal(A, B)$ and its projection to T has diameter $k - 1$. Thus, by the induction hypothesis, for every pair of points $x', y' \in Gal(\bar{A}, \bar{D})$ the segment $[x', y'] \subset \bar{X}$ exists; moreover, according to Lemma 3.16, this segment is contained in $Gal(\bar{A}, \bar{D})$. Thus $Gal(\bar{A}, \bar{D})$ is convex.

Let $\Phi := \{F_1, \dots, F_l\}$ denote the set of facets of B which are contained in the interior of $Gal(A, B)$. Thus for each facet $F \in \Phi$ there exists a gallery

$$A' = C'_0 \cup C'_1 \cup \dots \cup C'_{k-1} \cup C'_k = B',$$

so that $F = C'_{k-1} \cap C'_k$.

Fix a point x in the open cell A and let $y \in \bar{B}$ vary. Let $Y \subset \bar{C}_k$ denote the set of points $y \in \bar{C}_k$ such that there exists a segment $[x, y] \subset \bar{X}$. Each facet $F \in \Phi$, is contained in a cell $D_F = \bar{C}_{k-1}$ which is distance $k - 1$ away from \bar{C}_0 . Therefore

$$F \subset Gal(\bar{A}, \bar{D}_F),$$

and the latter is convex by the induction assumption. Hence $F \subset Y$, which implies that Y is nonempty.

Let $B_{sing} \subset B$ denote the (possibly empty) set of points y such that the segment $[x, y] \subset \bar{X}$ passes through the origin or through the boundary of a codimension 2 face. Since B has dimension ≥ 3 it follows (from the dimension count) that L does not locally separate B . Set $B_{reg} := B \setminus B_{sing}$.

Lemma 3.18. 1. $cl(Y_{gen}) \subset Y$.

2. $cl_{B_{reg}}(Y_{gen}) \subset int(Y) \cap B_{reg}$.

3. For each $F \in \Phi$, F is contained in the interior of Y .

4. $int(Y) \cap B_{reg} \subset cl_{B_{reg}}(Y_{gen})$.

Proof. 1. Consider a sequence $y_j \in Y_{gen}$ which converges to some $y \in \bar{B}$. Then

$$[x, y_j] \subset Gal(\bar{A}, \bar{B}).$$

Since the above union of galleries is compact, it follows that the sequence of segments $[x, y_j]$ subconverges to a segment $[x, y]$ in \bar{X} . Thus $y \in Y$.

2. For each $y \in Y \setminus B_{sing}$ consider the segment $\sigma = [x, y]$ and define the point $z = z_\sigma \in [x, y]$ so that

$$[z, y] \subset \bar{B}, \quad [x, z] \cap \bar{B} = \emptyset.$$

Consider a sequence $y_j \in Y_{gen}$ which converges to a point $y \in B$. Then (similarly to the proof of 1), we can assume that the segments $\sigma_j = [x, y_j]$ converge to a segment $[x, y]$. For each j take the point $z_j := z_{\sigma_j}$; the limit of these points is some $z \in \partial B \cap [x, y]$. Without loss of generality we can assume that all z_j 's belong to a common facet F of B , so that $\bar{F} = \bar{D} \cap \bar{B}$, where $D \subset Gal(A, B)$. Thus $z \in \bar{F} \subset \bar{D}$.

Since $y \in B_{reg}$, there are only three possibilities:

(a) $z \in F$.

(b) z belongs to an (open) codimension 2 face contained in ∂F .

(c) z belongs to the intersection $\bar{D} \cap \bar{B}$ and is not contained in any other closed cell.

In all three cases, by Assumption 3.7, there exists a convex neighborhood U of the point z in

$$Gal(\bar{A}, \bar{D}) \cup \bar{B}$$

so that $U \cap X$ is also convex. (In case (a) it follows since X is an affine manifold; in case (b) it follows from Part 1 of Assumption 3.7; in case (c) it follows from Part 2 of this assumption.)

Recall that $x \in A \subset \text{int}(Gal(\bar{A}, \bar{D}))$, the latter is convex. Thus, by Lemma 3.3,

$$[x, z] \subset \text{int}(Gal(\bar{A}, \bar{D})) \subset X.$$

By convexity of \bar{B} (and Lemma 3.3), $(z, y] \subset B \subset X$. Pick points $z' \in \text{int}(U) \cap [x, z]$, $z'' \in \text{int}(U) \cap (z, y]$. Convexity of $X \cap U$ then implies that $z \in X$; hence the entire segment $[x, y]$ is contained in X . Therefore, since X is an affine manifold, there exists a neighborhood V of y in B such that for each $y' \in V$, there exists a segment $[x, y'] \subset X$. Thus $y \in \text{int}(Y)$.

3. The proof of this assertion is analogous to the last part of the proof of 2: For each $y \in F$ the segment $[x, y]$ is contained in X and thus we can use the neighborhood V of y as above.

4. For $y \in \text{int}(Y)$, let $V \subset Y$ be an open ball containing y . Since the union of facets of the cell C is dense in $C' \setminus C$, we see that $V \setminus Y_{gen}$ is nowhere dense in V . Hence $Y_{gen} \cap V$ is dense in V and therefore $y \in \text{cl}(Y_{gen})$. The assertion 4 follows. \square

Corollary 3.19. $\bar{B} = Y = \text{cl}(Y_{gen})$.

Proof. First, by combining 2 and 4 we see that

$$\text{cl}_{B_{reg}}(\text{int}Y \cap B_{reg}) = \text{cl}_{B_{reg}}(Y_{gen}).$$

This of course implies that $\text{cl}(\text{int}Y \setminus B_{sing}) = \text{cl}(Y_{gen})$.

Thus $\text{int}(Y \cap B_{reg})$ is both closed and open in B_{reg} . By 3, the set $\text{int}(Y \cap B_{reg})$ is nonempty. Since B_{reg} is connected (recall that B_{sing} does not locally separate), we conclude that $\text{int}(Y \cap B_{reg}) = B_{reg}$. Thus $B_{reg} \subset Y$. Since B_{reg} is dense in \bar{B} , by applying 1 we see that

$$\bar{B} \subset \text{cl}(B_{reg}) = \text{cl}(\text{int}(Y \cap B_{reg})) = \text{cl}(Y_{gen}) \subset Y.$$

Thus $\bar{B} = Y = \text{cl}(Y_{gen})$. \square

Therefore, for each point $y \in \bar{B}$ there exists a segment $[x, y] \subset \bar{X}$. Recall that we assumed that $x \in A$. For a point $y \in \bar{B}$ and $x \in A$, pick a sequence $x_j \in A$. Then there exist a sequence $y_j \in Y_{gen}$ which converges to y ; the segments $[x_j, y_j]$ are all contained in $\text{Gal}(A, B)$. Therefore, by compactness of $\text{Gal}(\bar{A}, \bar{B})$, we conclude that the segments $[x_j, y_j]$ converge to a segment $[x, y] \subset \bar{X}$.

Thus we proved that for each pair of cells A and B within distance k , and each pair of points $x \in \bar{A}, y \in \bar{B}$, there exists a segment $[x, y] \subset \bar{X}$.

Remark 3.20. Moreover, $[x, y] \subset \text{Gal}(\bar{A}, \bar{B})$.

Hence, by induction, we conclude that \bar{X} is convex. It follows that $\text{dev} : \bar{X} \rightarrow \mathbb{R}^n$ is a continuous bijection onto a convex cone $K \subset \mathbb{R}^n$. This proves the first assertion of Theorem 3.9.

Our next goal is to prove convexity of X . Let $x, y \in X$. Pick a relatively compact neighborhood U of x in X . Then U is covered by finitely many cells $C'_j \subset X$. Choose a cell A' containing y . For each $x' \in U$ the segment $[x', y] \subset \bar{X}$ is covered by a finite union of faces contained in

$$\bigcup_j \text{Gal}(\bar{C}_j, \bar{A}).$$

Therefore the cone $Cone_y(U)$ with the tip y and the base U is covered by finitely many closed cells. Thus the developing map dev sends $Cone_y(U)$ homeomorphically onto a convex subset

$$\Sigma := dev(Cone_y(U)) = Cone_{dev(y)}(dev(U)) \subset \mathbb{R}^n.$$

Clearly, the open segment $(dev(x), dev(y))$ is contained in the interior of Σ . It follows that the open segment (x, y) is also contained in the interior of

$$\bigcup_j Gal(\bar{C}_j, \bar{A}).$$

Hence the open segment (x, y) is contained in X . Thus dev is a homeomorphism of X onto a convex homogeneous cone in \mathbb{R}^n . Properness of this cone follows from infiniteness of the number of cells in X . \square

4 Products of matrices

In this section we will consider the following problem:

Problem 4.1. *Let G be a Lie group with a collection of 1-parameter subgroups $G_1, \dots, G_k \subset G$. Analyze the image of the map*

$$Prod : \prod_{i=1}^k G_i \rightarrow G$$

given by $Prod(g_1, \dots, g_k) = g_1 \cdot \dots \cdot g_k$.

In the case when $G = SO(3)$, this problem is ultimately related to the variety of geodesic k -gons in S^3 with the fixed side-lengths, [7]. (See [6], [8], for the relation of this product problem to bending deformations of flat conformal structures.)

Here we consider the case of $G = GL(2, \mathbb{R})$; the subgroups G_i are orthogonal conjugates of the group of diagonal matrices $\{Diag(1, e^t), t \in \mathbb{R}\}$. More specific problem then is:

Problem 4.2. *Show that under appropriate conditions on the subgroups G_i , the image of the map $Prod$ contains the subgroup $SO(2) \subset GL(2, \mathbb{R})$.*

Let $\mathfrak{gl}(2, \mathbb{R}) = \mathfrak{p} \oplus \mathfrak{o}(2)$ denote the Cartan decomposition of the Lie algebra of $GL(2, \mathbb{R})$. The Lie algebras \mathfrak{p}_i of G_i 's are contained in \mathfrak{p} . Let $e := (1, \dots, 1) \in \prod_i G_i$. Then derivative

$$dProd_e : \oplus_i \mathfrak{p}_i \rightarrow \mathfrak{gl}(2, \mathbb{R})$$

is the map

$$(\xi_1, \dots, \xi_k) \mapsto \sum_i \xi_i.$$

Therefore its image is contained in \mathfrak{p} and hence is orthogonal to $\mathfrak{o}(2)$. Thus one cannot approach Problem 4.2 by making infinitesimal calculations.

There is probably a purely algebraic or analytic solution to Problem 4.2; we will use hyperbolic geometry instead. Given a basis (v, w) of \mathbb{R}^2 and $t \in \mathbb{R}$ we define the matrix

$$A = A_{v,w,t}$$

to be the linear transformation which fixes v and sends w to $e^t w$.

Consider the projective action of $GL(2, \mathbb{R})$ on the circle $\mathbb{R}P^1$ (which we identify with the boundary of the hyperbolic plane \mathbb{H}^2). We will use the notation $[A] \in PGL(2, \mathbb{R})$ for the projection of the matrix $A \in GL(2, \mathbb{R})$.

The vectors v, w project to fixed points $[v], [w]$ of the projective transformation $[A]$. We identify the hyperbolic plane \mathbb{H}^2 with the unit disk in \mathbb{R}^2 in such a way that the group $O(2) \subset GL(2, \mathbb{R})$ fixes the origin 0 in \mathbb{H}^2 . Then the hyperbolic geodesic $L_A = \overline{[v][w]} \subset \mathbb{H}^2$ invariant under $[A]$ passes through the origin 0 (and hence is a Euclidean straight line). We parameterize the geodesic L with the unit speed and orient L in the direction from $[v]$ to $[w]$, thereby identifying it with the real line. The origin in \mathbb{H}^2 corresponds to zero in \mathbb{R} . We let L^\pm denote the positive and negative rays (starting at 0) in L corresponding to this orientation.

In these coordinates, the isometry $[A]$ acts on L by $r \mapsto r + t$. The isometry $[A_{w,v,t}]$ acts on \mathbb{H}^2 by the translation $r \mapsto r - t$ along the geodesic L . By considering the action of $SO(2)$ by conjugation we see the following:

Let $R = R_\phi \in SO(2)$ be the rotation by the angle ϕ . Then the matrix

$$R_\phi A_{v,w,t} R_\phi^{-1} = A_{R(v), R(w), t}$$

acts on \mathbb{H}^2 by translation $r \mapsto r + t$ along the geodesic

$$\overline{R_{\phi/2}([v]) R_{\phi/2}([w])}.$$

We now assume that we are given 1-parameter groups

$$G_1 = \{A_{e_1, e_2, t} : t \in \mathbb{R}\}, \quad G_2 = R_{\pi/4} G_1 R_{-\pi/4},$$

$$G_3 = R_{\pi/2} G_1 R_{-\pi/2} = \{A_{e_2, e_1, t} : t \in \mathbb{R}\}, \quad G_4 = R_{-\pi/4} G_1 R_{\pi/4}.$$

Geometrically, these are groups of translations along two orthogonal hyperbolic geodesics L_1 and L_2 in \mathbb{H}^2 (G_i and G_{i+2} translate along L_i in the opposite directions, $i = 1, 2$). Given a matrix $A_i \in G_i$ we let $\ell_i := \ell(A_i)$ denote the translation length of $[A_i]$ along its invariant geodesic; here we are ignoring the orientation so that $\ell_i \geq 0$.

Thus, in order for $A_i \in G_i, i = 1, \dots, 4$ to have the product equal to R_ϕ it is necessary and sufficient to have:

1. The product of the eigenvalues of A_i 's is equal to 1 (i.e. the product of four matrices is in $SL(2, \mathbb{R})$). Equivalently,

$$t_1 + t_2 + t_3 + t_4 = 0.$$

2. The product of the hyperbolic translations

$$[A_4] \circ [A_3] \circ [A_2] \circ [A_1]$$

is the rotation $R_{\phi/2}$ around the origin in \mathbb{H}^2 . In particular, the above product of hyperbolic isometries has to fix the intersection $L_1 \cap L_2$.

Remark 4.3. *Similar description, of course, will be valid for more general choices of 1-parameter groups G_i which are conjugate to G_1 by rotations $R_{\theta_i}, i = 1, \dots, k$.*

With this geometric interpretation it is clear, for instance, that the product $A_3 A_2 A_1$ is never a nontrivial rotation. The reason is that unless $A_2 = 1, [A_1] = [A_3]^{-1}$, the product of the hyperbolic isometries does not fix the origin.

We now make the situation a bit more symmetric and require that

$$\ell_1 = \ell_4, \quad \ell_2 = \ell_3.$$

In particular, the Condition 1 will be satisfied provided that

$$t_1, t_3 > 0, \quad t_2, t_4 < 0.$$

We then consider the images of the origin under the compositions of the isometries $[A_1], [A_2], [A_3], [A_4]$. We let $x_0 := 0; x_i := [A_i](x_{i-1}), i = 1, \dots, 4$. Given a number $\theta \in (-\frac{\pi}{2}, 0)$ set

$$\alpha = \alpha(\theta) := \frac{\pi}{2} + \theta.$$

Lemma 4.4. *For every $\theta \in (-\frac{\pi}{2}, 0]$ there exists a pair of continuous functions $\ell_i = \ell_i(\theta), i = 1, 2$, so that:*

1.

$$\begin{cases} \cosh(\ell_2) = \cosh(\ell_1) \sin(\alpha) \\ \sinh^2(\ell_1) = \cos(\alpha) \end{cases}$$

In particular, $\ell_i(0) = 0, i = 1, 2$.

2. *For $\ell_i = \ell_i(\theta)$ the composition*

$$[A_4] \circ [A_3] \circ [A_2] \circ [A_1]$$

is the (counter-clockwise) rotation R_θ around the origin $0 \in \mathbb{H}^2$ by the angle θ .

Proof. Let L_1, L_2 be the pair of oriented geodesics in \mathbb{H}^2 (invariant under the subgroups $[G_1] = [G_3], [G_2] = [G_4]$ respectively) which intersect orthogonally at the origin.

We orient the geodesics L_1, L_2 away from the points $[e_1], [R_{\pi/4}(e_1)]$ fixed by $[A_1] \in [G_1], [A_2] \in [G_2]$. Let L_i^+ denote the positive half-rays in these geodesics.

Recall that $\alpha \in (0, \frac{\pi}{2}]$ and that we will be using the translation parameters so that

$$t_1, t_3 > 0, \quad t_2, t_4 < 0.$$

The key observation is that there exists a unique geodesic quadrilateral (a *Lambert quadrilateral*) $Q_\alpha = [0, y_1, x_2, y_2]$ in \mathbb{H}^2 with the three right angles (at the vertices $0, y_1 \in L_1^+, y_2 \in L_2^+$) and the angle α at the vertex x_2 . See Figure 4. The orientation on Q_α given by the ordering of its vertices is clockwise, which corresponds to the assumption that $\theta \leq 0$.

Set

$$\ell_2 = \ell_3 := d(0, y_1) = d(0, y_2)$$

and

$$\ell_1 = \ell_4 := d(y_2, x_2) = d(y_1, x_2).$$

It is clear that ℓ_1, ℓ_2 are continuous functions of θ so that $\ell_i(0) = 0$. The equations relating α, ℓ_1, ℓ_2 follow immediately from the hyperbolic trigonometry, see [1, Theorem 7.17.1].

Choose points $x_1 \in L_1^+, x_3 \in L_2^+$ so that

$$d(0, x_1) = \ell_1 = d(0, x_3) = \ell_4.$$

Now take the hyperbolic translations g_1, g_3 along L_1 sending 0 to x_1 and y_1 to 0 respectively. Define the hyperbolic translations g_2, g_4 along L_2 sending 0 to y_2 and x_3 to 0 respectively. Thus the isometries g_1, g_4 have the translation lengths $\ell_1 = \ell_4$; the isometries g_2, g_3 have the translation lengths $\ell_2 = \ell_3$.

It is clear from the Figure 4 that

$$g_2(x_1) = x_2, g_3(x_2) = x_3$$

and therefore

$$g_4 \circ g_3 \circ g_2 \circ g_1(0) = 0.$$

Hence the above composition of translations is a certain rotation R_ϕ around the origin. In order to compute the angle ϕ of rotation take two vectors $\xi_1, \xi_2 \in T_0\mathbb{H}^2$ tangent to the geodesic rays L_i^- , $i = 1, 2$. Then the images of ξ_1, ξ_2 under

$$d(g_2 \circ g_1), \quad d(g_3^{-1} \circ g_4^{-1})$$

are tangent to the geodesic segments $\overline{x_2y_2}, \overline{x_2y_1}$ respectively. Therefore the angle ϕ equals $\alpha - \frac{\pi}{2}$ (the rotation is in the clockwise direction). Thus

$$\phi = \theta. \quad \square$$

Let $A_i \in G_i$ denote the matrices corresponding to the hyperbolic translations g_i . Then

Corollary 4.5.

$$A_4 \cdot A_3 \cdot A_2 \cdot A_1 = R_{\frac{\alpha}{2} - \frac{\pi}{4}}.$$

Therefore we get the following:

Theorem 4.6. *For each $\tau \in (-\frac{\pi}{4}, 0]$ there is a unique pair of numbers $t_1 \geq 0, t_2 \leq 0$ so that for the set of parameters*

$$\vec{t} = (t_1, t_2, -t_2, -t_1)$$

the product of the corresponding matrices equals the rotation R_τ . Moreover, the function \vec{t} depends continuously on τ .

Projective generalization.

Let $P \subset \mathbb{R}P^n$ be a projective hyperplane, $p \in \mathbb{R}P^n \setminus P$ and $t \in \mathbb{R}$. Then there exists a unique map $A = A_{P,p,t} \in PGL(n+1, \mathbb{R})$ satisfying:

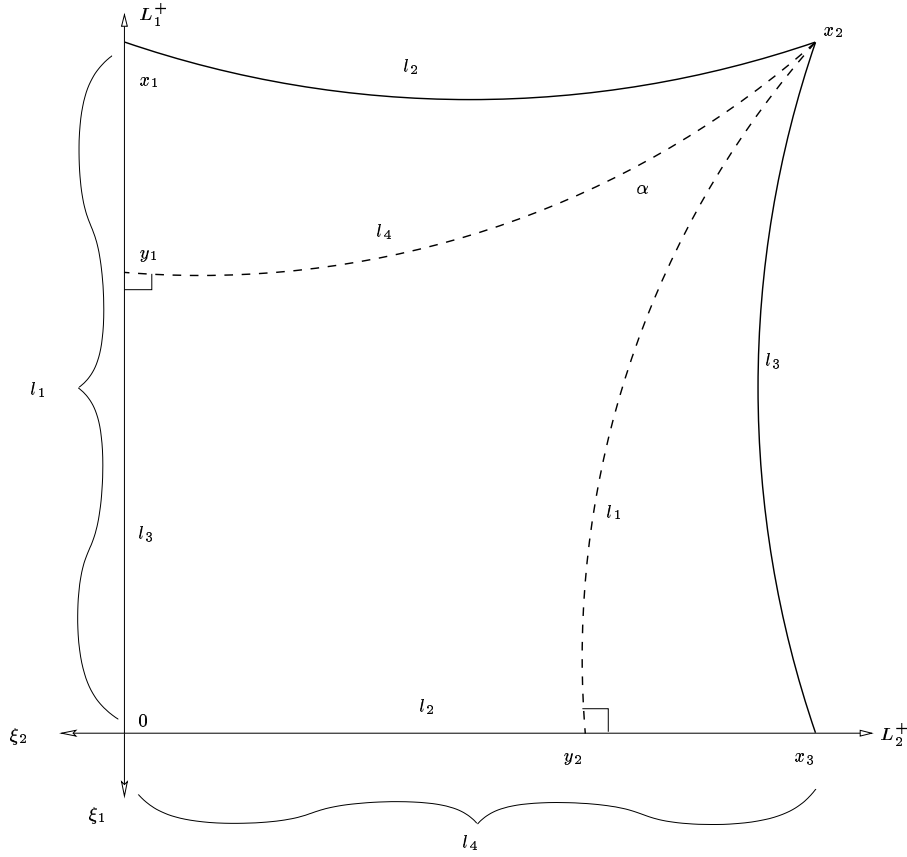


Figure 4: *Lambert quadrilateral.*

1. A fixes $P \cup \{p\}$ pointwise.
2. The derivative dA_p equals $e^t I$.

Suppose now that $n = 2$, \mathbb{R}^2 is the affine patch of $\mathbb{R}P^2$. Let $P \subset \mathbb{R}P^2$ be the projective line tangent to the unit vector $v \in T_0\mathbb{R}^2$, $p \in \mathbb{R}P^2 \setminus \mathbb{R}^2$ be the point at infinity so that the corresponding line l through the origin contains the unit vector w orthogonal to v . Then

$$A_{P,p,t} = A_{v,w,t} \in GL(2, \mathbb{R}) \subset PGL(3, \mathbb{R}).$$

The identity extension of this linear transformation to the element $\hat{A} \in GL(n, \mathbb{R}) \subset PGL(n+1, \mathbb{R})$ equals

$$A_{Q,q,t}$$

where Q is the projective hyperplane through the origin orthogonal to w , the point $q \in \mathbb{R}P^n \setminus \mathbb{R}^n$ corresponds to the line l as above.

Consider now a collection P_1, P_2, P_3, P_4 of projective hyperplanes in $\mathbb{R}P^n$ passing through the origin, so that the intersection

$$\bigcap_i P_i = S$$

is a codimension 2 projective hyperplane in $\mathbb{R}P^n$. We assume that the consecutive hyperplanes intersect at the angles $\frac{\pi}{4}$. For each P_i let $p_i \in \mathbb{R}P^n \setminus \mathbb{R}^n$ be the “dual point” i.e. the corresponding line l_i through the origin is orthogonal to P_i .

Remark 4.7. *Somewhat more invariantly, one can describe this setting as follows. We fix a positive definite bilinear form on $\mathbb{R}P^n$ so that the points p_i are dual to the hyperplanes P_i . Therefore the assumption that*

$$\bigcap_i P_i = S$$

is a codimension 2 projective hyperplane in $\mathbb{R}P^n$ implies that $\{p_1, \dots, p_4\}$ is contained in a projective line $s \subset \mathbb{R}P^n$ dual to S . We then are assuming that the points p_1, \dots, p_4 are cyclically ordered on s so that the distance between the consecutive points is $\pi/4$. (Note that $\mathbb{R}P^1 = s$ has length π .)

Then Theorem 4.6 can be restated as follows:

Theorem 4.8. *For each angle $\tau \in (-\frac{\pi}{4}, 0]$ there is a unique set of parameters $\vec{t} = (t_1, t_2, t_3, t_4) = (t_1, t_2, -t_2, -t_1)$ with $t_1 \geq 0, t_2 \leq 0$ so that the composition of the corresponding projective transformations*

$$A_{P_4, p_4, t_4} \circ \dots \circ A_{P_1, p_1, t_1}$$

equals the rotation R_τ around S by the angle τ , fixing S pointwise. Moreover, the function \vec{t} depends continuously on τ .

5 Bending

In this section we review the *bending deformation* of projective structures.

Recall that in the end of the previous section we defined projective transformations

$$A = A_{P,p,t} \in PGL(n+1, \mathbb{R})$$

corresponding to the triples (P, p, t) , where $P \subset \mathbb{R}P^n$ is a projective hyperplane, $p \in \mathbb{R}P^n \setminus P$ and $t \in \mathbb{R}$.

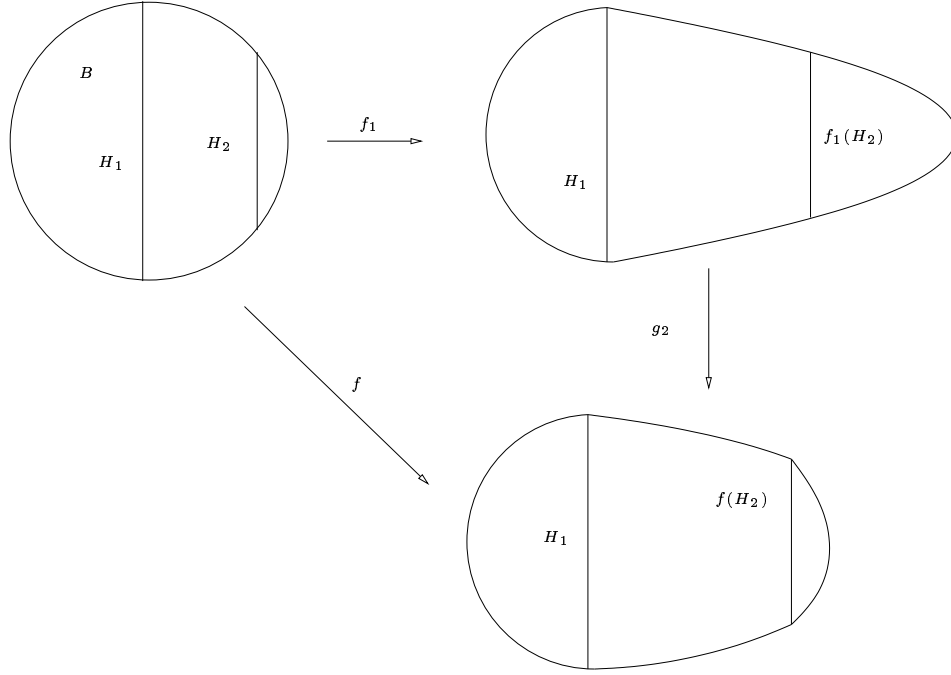


Figure 5: *Projective bending.*

Before proceeding with the general definition we start with a basic example of bending. Let B denote the open unit ball in \mathbb{R}^n , which we will identify with the hyperbolic n -space. Let H_1, \dots, H_k denote disjoint hyperbolic hypersurfaces in B so that H_i separates H_{i-1} from H_{i+1} , $i = 2, \dots, k-1$. We assume that H_i 's are cooriented in such a way that H_{i+1} is to the right from H_i , $i = 1, \dots, k$. Let H_i^\pm denote the half-space in B bounded by H_i and lying to the left (resp. right) from H_i . We set

$$B_i := H_i^+ \cap H_{i+1}^-, \quad i = 1, \dots, k-1$$

and

$$B_0 := H_1^-, B_k := H_k^+.$$

Let P_i denote the projective hyperplane containing H_i and let $p_i \in \mathbb{R}P^n$ denote the point dual to P_i with respect to the quadratic form where B is the unit ball. Choose real numbers t_1, \dots, t_k . Our goal is to *bend* B projectively in $\mathbb{R}P^n$ along the hypersurfaces H_i with the bending parameters t_i , $i = 1, \dots, k$. Let $A_i := A_{P_i, p_i, t_i}$, $i = 1, \dots, k$.

We will do bending inductively. First, let $f_1 : B \rightarrow \mathbb{R}P^n$ denote the map which is the identity on H_1^- and A_1 on H_1^+ . We then would like to bend $B_1 := f_1(B)$ along $f_1(H_2)$. The corresponding bending map g_2 is the identity on $f_1(H_2^-)$ and A'_2 on $f_1(H_2^+)$, where

$$A'_2 = A_1 \circ A_2 \circ A_1^{-1}.$$

Therefore the map

$$f_2 : B \rightarrow g_2(B_1)$$

equals to id on $B_0 = H_1^-$, to A_1 on $B_1 = H_1^+ \cap H_2^-$ and to $A_1 \circ A_2$ on H_2^+ . Continuing in the fashion inductively we eventually obtain the bending map

$$f : B \rightarrow \mathbb{R}P^n$$

so that the restriction $f|_{B_i}$ equals

$$A_1 \circ \dots \circ A_i, i = 1, \dots, k - 1,$$

and $f|_{B_0} = id$. The same construction works for an arbitrary locally finite collection \mathcal{H} of disjoint hyperplanes H_i . We then pick a component C_0 of $Y = B \setminus \cup_i H_i$ where the bending map f is the identity. Given a component C_k of Y we take the finite subcollection $\{H_1, \dots, H_k\}$ of hyperplanes in \mathcal{H} separating C_0 from C_k . We then repeat the above construction of bending map to define the restriction of bending to C_k .

Remark 5.1. *More generally, if (L, μ) is a measured codimension 1 totally-geodesic lamination in B , we can define projective bending with respect to this lamination. However, in view of Ratner's theorem, this generalization is not useful in the context of bendings of compact manifolds of dimension ≥ 3 .*

We now give the general definition of bending.

Let M be a projective manifold (or, more generally, an orbifold). Let $f : \tilde{M} \rightarrow \mathbb{R}P^n$ and $\rho : \Gamma = \pi_1(M) \rightarrow PGL(n + 1, \mathbb{R})$ be the developing map and the holonomy of M .

Let $L \subset M$ be a proper hypersurface (possibly contained in the boundary of M); let $\tilde{L} \subset \tilde{M}$ be the preimage of L in the universal cover of M .

We call the hypersurface L *flat* if it satisfies the following:

1. Each point $x \in L$ has a neighborhood $U \subset L$ so that the developing map sends U to an open subset of a projective hyperplane in $\mathbb{R}P^n$.
2. For a component $\tilde{L}_i \subset \tilde{L}$ let Γ_i be the stabilizer of \tilde{L}_i in Γ . Then the group $\rho(\Gamma_i)$ stabilizes a projective hyperplane $P_i = \text{Span}(f(\tilde{L}_i)) \subset \mathbb{R}P^n$.

We then require that for each \tilde{L}_i , the group $\rho(\Gamma_i)$ has an isolated fixed point $p_i \in \mathbb{R}P^n$ which is disjoint from P_i .

We define a *cooriented lamination* L in M as follows. Consider the union L of flat connected cooriented hypersurfaces L_i in M , which intersect the boundary of M transversally and so that for distinct L_i, L_j

$$L_i \cap L_j \cap \text{int}(M) = \emptyset.$$

In this paper we will be assuming that the collection of hypersurfaces L_i is locally finite in M , although one can make the discussion more general.

Definition 5.2. A transverse measure for L is a locally constant function $\mu : L \cap \text{int}(M) \rightarrow (0, \infty)$, $\mu : L_i \mapsto e^{t_i}$.

A measured cooriented lamination is the pair $\lambda = (L, \mu)$. The measured lamination $\lambda = (L, \mu)$ lifts to a cooriented measured lamination $\tilde{\lambda} = (\tilde{L}, \tilde{\mu})$ in \tilde{M} .

We now define the *bending deformation* c_λ of the projective structure c on M along the lamination λ . The structure c_λ will have the developing map f_λ satisfying the following properties:

For each component $H = \tilde{L}_i$ of \tilde{L} with the stabilizer Γ_i ; let P_i, p_i denote the projective hyperplane and a point in $\mathbb{R}P^n$ stabilized by $\rho(\Gamma_i)$ as above. Let H_-, H_+ denote the components of $\tilde{M} \setminus \tilde{L}$ to the left and to the right of H (with respect to the coorientation). We then require that there exists a projective transformation $g \in PGL(n+1, \mathbb{R})$ so that

$$f_\lambda|_{H_-} = g \circ f|_{H_-},$$

$$f_\lambda|_{H_+} = A_{P,p,t} \circ g \circ f|_{H_+}.$$

It is clear that the map f_λ with these properties exists and is unique up to postcomposition with projective transformations of $\mathbb{R}P^n$. By construction, f_λ is a local homeomorphism. Since $\tilde{\lambda}$ is Γ -invariant, it follows that the map f_λ is equivariant with respect to a homomorphism $\rho_\lambda : \Gamma \rightarrow PGL(n+1, \mathbb{R})$.

Thus the pair $(f_\lambda, \rho_\lambda)$ determines a projective structure c_λ on M .

The following simple lemma is used to ensure convexity of the projective structures on Gromov-Thurston examples.

Let H be a hyperplane in \mathbb{R}^n ; let H_\pm denote the closed half-spaces in \mathbb{R}^n bounded by H . Suppose that $D \subset H_-$ is a compact convex subset; let F denote the intersection $H \cap S$. Pick a point $p \in H_+$. Let $\Sigma = \text{Cone}_p(F)$ denote the (convex) cone with the vertex p and the base F .

Lemma 5.3. *Suppose that for each $x \in D$ the segment \overline{xp} crosses H inside F . Then the union $D \cup \Sigma$ is convex.*

Proof. Clearly, $D \cup \Sigma = \text{Cone}(p, D)$. Now convexity follows from Lemma 3.1. \square

Suppose that $p_i \in H_+$ is a sequence of points so that D, H, p_i satisfy all the above conditions. Assume that $\lim_i p_i = p \in \mathbb{R}P^n \setminus \mathbb{R}^n$; let $\Sigma \subset \mathbb{R}P^n$ denote the limit of the cones $\text{Cone}_{p_i}(F)$. Let $E \subset H_+$ denote a compact subset contained in the cone Σ so that

$$D \cap E = D \cap H = E \cap H = F.$$

Corollary 5.4. *$D \cup E$ is convex.*

Proof. By taking the limit, Lemma 5.3 implies that for each pair of points $x_1 \in \Sigma \setminus \{p\}$, $x_2 \in D$, the intersection $\overline{x_1 x_2} \cap H$ is contained in F . Since E is a convex subset of Σ ; it follows that $\overline{x_1 x_2} \subset D \cup E$. \square

Let P denote the projective closure of the hyperplane H .

Corollary 5.5. *For each $t \in \mathbb{R}$, the union*

$$A_{P,p,t}(E) \cup E$$

is convex.

Proof. The set $E_t = A_{P,p,t}(E)$ is clearly convex. By definition of the projective transformation $A_{P,p,t}$, the set E_t is contained in the cone Σ and $E_t \cap H = E_t \cap D = F$. \square

6 Construction of convex projective structures on Gromov-Thurston manifolds

Assume as before that the hyperbolic manifold M satisfies Assumption 2.2 and let M' be the manifold constructed (using pieces of M) as in section 2.

Theorem 6.1. *For each natural number $m \geq 8$ which is divisible by 4, the manifold M' admits a convex projective structure.*

Proof. The proof breaks in two steps:

1. We first *bend* the (hyperbolic) projective structure c on the manifold with boundary $N' \subset M'$ (see section 2) in order to obtain a new projective structure c_λ which has *flat boundary*. We then construct a projective structure a' on M' by gluing two copies of (N', c_λ) *via an order 2 rotation*.
2. We use the Theorem 3.9 to verify that (M', a') is convex.

Step 1. We begin by observing that since m is divisible by 4, the group D_m contains the dihedral subgroup D_4 . The fixed-point sets of the reflections contained in D_4 yield codimension 1 totally-geodesic submanifolds $L_0, \dots, L_4 \subset N'$, see figure 6. The angle between L_i, L_{i+1} equals $\frac{\pi}{4}$, $i = 0, \dots, 3$.

We assume that the submanifold $N' \subset M$ is chosen in such a way that $L_0 \subset \partial N'$. Then the boundary of N' is the union $L_0 \cup L'_0$.

We let c denote the projective structure on N' defined by the hyperbolic structure on N' . Since each L_j is totally-geodesic (in the hyperbolic manifold M) it is *flat* as a hypersurface in the projective manifold (N', c) . We define the cooriented lamination $L = L_1 \cup \dots \cup L_4$ by coorienting L_j 's as in figure 6. (Each arrow indicates the direction from left to right.)

We identify the hyperbolic space \mathbb{H}^n with the unit ball B in the Euclidean space $\mathbb{R}^n \subset \mathbb{R}P^n$, so that each flat hypersurface L_0, \dots, L_4, L'_0 corresponds (under the developing map) to a projective hypersurface $P_0, \dots, P_4, P'_0 \subset \mathbb{R}P^n$ passing through the codimension 2 hyperplane $Q \subset \mathbb{R}P^n$ containing the origin. The angles between P_i, P_{i+1} equal $\frac{\pi}{4}$ and the angle between P_0 and P'_0 is $\pi - \frac{\pi}{m}$.

For each $i = 1, \dots, 4$ let $p_i \in \mathbb{R}P^n \setminus \mathbb{R}^n$ be the point dual to P_i as defined in section 4.

We orient the 2-plane normal to Q in \mathbb{R}^n so that the orientation agrees with the above coorientation of L .

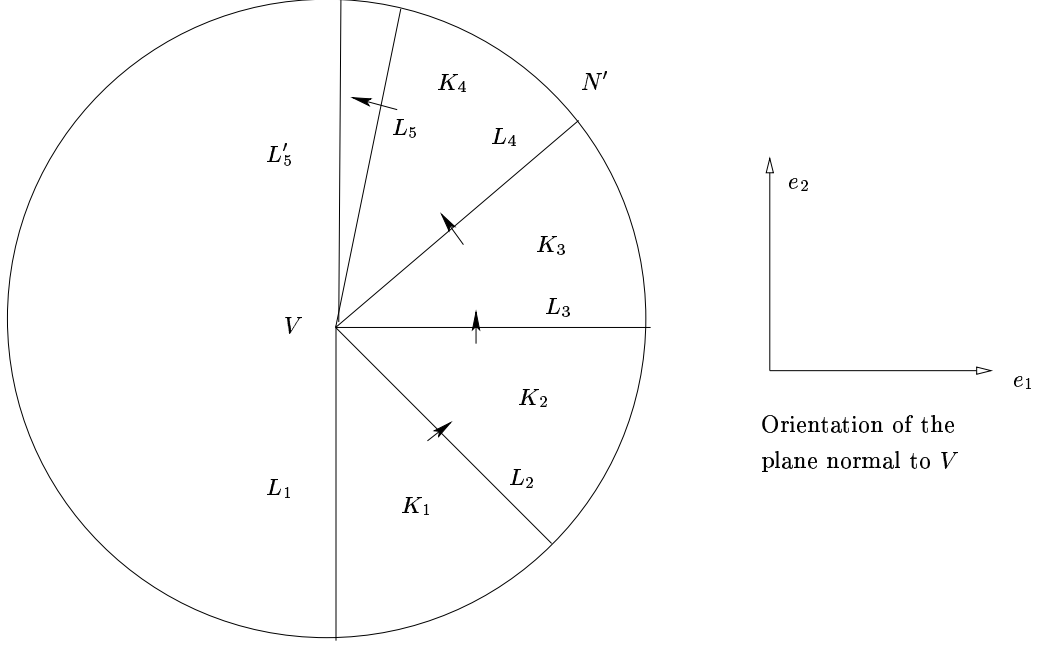


Figure 6: *Coorientation on L .*

Then, since $m > 4$, according to Theorem 4, there exist real numbers $t_1, t_2, t_3 = -t_2, t_4 = -t_1$ so that

$$A_{P_4, p_4, t_4} \circ \dots \circ A_{P_1, p_1, t_1} = R_{-\frac{\pi}{m}}, \quad (1)$$

is the rotation around Q by the angle $-\frac{\pi}{m}$.

The rotation $R_{-\frac{\pi}{m}}$ sends P'_0 to P_4 . Let μ denote the transverse measure to L defined by $L_i \mapsto e^{t_i}$. Let $\lambda := (L, \mu)$ and c_λ be the projective structure on N' obtained from c by bending along λ .

The equation (1) implies that the projective manifold (N', c_λ) has *flat boundary*. The boundary manifold is actually hyperbolic and is isometric to the boundary of the hyperbolic manifold N (which has geodesic boundary). Let $\theta' : (\partial N', c_\lambda) \rightarrow (\partial N', c_\lambda)$ denote the isometric involution which interchanges L_0 and L'_0 (see section 2). This involution corresponds to the order 2 rotation around Q in $\mathbb{R}P^n$.

Therefore, we put the projective structure a' on the manifold M' by gluing together two copies of (N', c_λ) via the isomorphism θ' of their boundaries. Then (M', a') admits an order 2 automorphism θ which fixes the codimension

2 submanifold V' pointwise and corresponds (under the developing map) to the order 2 rotation in $\mathbb{R}P^n$.

This concludes Step 1.

Step 2. Let N'_\pm denote the two copies of N' used to construct M' . Recall that (N', c) is tiled by $m - 1$ isometric copies O_j of the fundamental domain O of D_m (see section 2). The intersection $\cap_j O_j$ is a codimension 2 totally-geodesic submanifold $V \subset N'$.

Let $W_0^+ = L_0, W_1^+, \dots, W_{m-1}^+ \subset N'_+$ denote the flat hypersurfaces which appear as boundary components of the domains $O_j \setminus P$. For each j let $W_j^- := \theta(W_j^+) \subset M'$. Then for each j , the flat hypersurfaces with boundary W_j^-, W_j^+ match in M' to form a flat hypersurface (without boundary) $S_j \subset M'$ which is invariant under the automorphism $\theta : (M', a') \rightarrow (M', a')$.

The flat hypersurfaces S_j cut the projective manifold (M', a') into components, each of which is (projectively) isomorphic to the convex hyperbolic manifold with corners O .

We now pass to the universal cover $X = (\tilde{M}', \tilde{a}')$ of (M', a') . The codimension 2 submanifold V' lifts to X to a disjoint union of codimension two submanifolds, each of which is isomorphic to the open $n - 2$ -disk.

The inverse image $\tilde{S} \subset X$ of $S := \cup_j S_j$ is a union of flat hypersurfaces in X (called *walls*) which intersect along codimension 2 submanifolds above. The closure C'_j of each component of $X \setminus \tilde{S}$ is a convex subset, which is projectively isomorphic to the universal cover of the hyperbolic manifold with corners O . Thus we obtain a covering of X by closed subsets which are:

1. Codimension 0 strata (*cells*): Convex sets C'_j .
2. Codimension 1 strata: Facets $F'_i \subset C'_j$, which are $n - 1$ -dimensional intersections of C'_j with walls.
3. Codimension 2 strata: Components of the preimage of $P \subset M'$.

The nerve of this decomposition of X is a 2-dimensional cell complex Z , where every 2-cell has $2(m - 1) \geq 14$ edges.

By convexity, the links of vertices of Z do not contain any bigons. Thus Z satisfies the $C'(1/14)$ small cancellation condition.

We now analyze the unions of the adjacent cells. Suppose that C'_1, C'_2 share a facet F' .

Lemma 6.2. $C'_1 \cup C'_2$ is isomorphic to a proper convex subset of $\mathbb{R}P^n$.

Proof. Let $dev : X \rightarrow \mathbb{R}P^n$ denote the developing map of (M', a') . Then, by the construction of the projective structure a' , we can assume (after post-composing dev with a projective transformation) that:

$$B_i = dev(C'_i), i = 1, 2,$$

are relatively compact convex subsets of \mathbb{R}^n which are separated by a hyperplane $H \subset \mathbb{R}^n$ passing through the origin. Let P denote the projective closure of H . The intersection $B_1 \cap B_2$ is a facet Φ contained in H . Let H_i^+ denote the closed half-space in \mathbb{R}^n bounded by H and containing B_i , $i = 1, 2$. Let $p \in \mathbb{R}P^n \setminus \mathbb{R}^n$ be the point dual to H . Then there exists $t \in \mathbb{R}$ such that the union

$$U_t = B_1 \cup A_{P,p,t}(B_2)$$

is a convex set contained in the unit ball B with center at the origin. The Euclidean reflection τ in the hyperplane H preserves the union U_t .

Suppose that there exists a point $x_1 \in B_1$ such that the projective line l through x_1, p crosses H in a point y which is not in Φ . Then, by symmetry, l contains a point $x_2 = \tau(x_1) \in A_{P,p,t}(B_2)$. Thus the segment $\overline{x_1 x_2}$ contains a point $y \notin U_t$. Contradiction.

Therefore we can apply Corollary 5.4 to $B_1 \cup B_2$ (with $D := B_1, E := A_{P,p,t}(B_2)$) and conclude that $B_1 \cup B_2$ is convex. \square

We now *de-projectivize* the projective manifold X : We replace each cell C'_j with a convex cone \tilde{C}'_j , etc. The result is an affine $n+1$ -manifold \tilde{X} which is obtained by gluing convex cones \tilde{C}'_j . All the conditions of Theorem 3.9 are satisfied by \tilde{X} . It follows from Theorem 3.9 that \tilde{X} is isomorphic to a proper homogeneous open convex cone in \mathbb{R}^{n+1} . Thus X is isomorphic to a proper open convex subset of $\mathbb{R}P^n$ and therefore the manifold M' is a convex projective manifold. \square

Corollary 6.3. *The projective manifold M' is strictly convex.*

Proof. Since M' admits a metric of negative curvature, its fundamental groups are Gromov-hyperbolic. Since M' is convex, they are also strictly convex by Theorem 1.2. \square

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Department of Mathematics,
 University of California,
 Davis, CA 95616, USA,
 kapovich@math.ucdavis.edu