# Krull dimensions of rings of holomorphic functions

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Abstract. We prove that the Krull dimension of the ring of holomorphic functions of a connected complex manifold is infinite iff it is $> 0$.

Let $R$ be a commutative ring. Recall that the *Krull dimension* $\dim(R)$ of $R$ is the supremum of lengths of chains of distinct proper prime ideals in $R$. Our main result is:

Theorem 1. Let $M$ be a connected complex manifold and $H(M)$ be the ring of holomorphic functions on $M$. Then the Krull dimension of $H(M)$ either equals 0 (iff $H(M) = \mathbb{C}$) or is infinite, iff $M$ admits a nonconstant holomorphic function $M \to \mathbb{C}$.

*Proof.* Our proof mostly follows the lines of the proof by Sasane [**S**], where Theorem 1 was proven in the case when $M$ is a domain in $\mathbb{C}$ (we note that Henricksen [**H**] was the first to prove that the ring of entire functions on $\mathbb{C}$ has infinite Krull dimension). We will use the Axiom of Choice in two ways: (a) to establish existence of certain maximal ideals and (b) to get the notion of the *ultralimit* of sequences of nonnegative real numbers. The latter depends on a choice of a nonprincipal ultrafilter $\omega$ on $\mathbb{N}$, which we fix once and for all. A nonprincipal ultrafilter on $\mathbb{N}$ can be regarded as a finitely-additive probability measure on $\mathbb{N}$ which vanishes on each finite subset and takes the value 0 or 1 on each subset of $\mathbb{N}$. Subsets of full measure are called *$\omega$-large*. The ultralimit of a sequence of (nonnegative) real numbers will be denoted

$$\omega\text{-}\lim_{k\to\infty} x_k,$$

it belong to $\mathbb{R}_+ \cup \{\infty\}$. For a sequence $(p_k)$ in a Hausdorff topological space $X$, one defines the ultralimit

$$\omega\text{-}\lim_{k\to\infty} p_k$$

as a point $a \in X$ such that for each neighborhood $U$ of $a$, the set $\{k : p_k \in U\}$ is $\omega$-large.

We will need several elementary properties of ultralimits:

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1. For every sequence $x_k \in \mathbb{R}_+$, the ultralmit

$$\omega\text{-}\lim_{k\to\infty} x_k \in [0, \infty]$$

exists.

2. If a sequence $(x_k)$ has the ordinary limit, we also have

$$\omega\text{-}\lim_{k\to\infty} x_k = \lim_{k\to\infty} x_k.$$

3.
$$\omega\text{-}\lim_{k\to\infty} \min(x_k, y_k) = \min(\omega\text{-}\lim_{k\to\infty} x_k, \omega\text{-}\lim_{k\to\infty} y_k).$$

The latter property will be only used in the case when

$$\omega\text{-}\lim_{k\to\infty} x_k = \omega\text{-}\lim_{k\to\infty} y_k = \infty$$

and we provide a quick proof in this situation: For each $c \in \mathbb{R}$ the sets

$$A = \{k \in \mathbb{N} : x_k \geq c\}, \quad B = \{k \in \mathbb{N} : y_k \geq c\}$$

are $\omega$-large. Therefore, their intersection $C = A \cap B$ is also $\omega$-large. However,

$$C = \{k : \min(x_k, y_k) \geq c\}.$$

Therefore,

$$\omega\text{-}\lim_{k\to\infty} \min(x_k, y_k) = \infty.$$

We refer the reader to [**DK**] and [**Go**] for a detailed treatment of ultrafilters and ultralimits.

Remark 2. An analytically inclined reader may prefer to use *Banach limits* of sequences of real numbers instead of ultralimits. The existence of a Banach limit depends upon the Hahn-Banach theorem which is a weak form of the Axiom of Choice.

Recall that a *valuation* on a unital ring $R$ is a map $\nu : R \to \mathbb{R}_+ \cup \{\infty\}$ such that:

1. $\nu(a + b) \geq \min(a, b)$,
2. $\nu(ab) = \nu(a) + \nu(b)$.
3. $\nu(a) = \infty \iff a = 0$.
4. $\nu(1) = 0$.

Below, given two sequences $x_k, y_k \in \mathbb{R}_+ \cup \{\infty\}$, we say that

$$y_k \gg x_k, \quad k \to \infty$$

iff

$$\omega\text{-}\lim_{k\to\infty} \frac{y_k}{x_k} = \infty$$

with the convention that $\frac{\infty}{\infty} = 1$.

Our main technical result is:

Proposition 3. Suppose that $R$ is a ring and $\nu_k$ is a sequence of valuations on $R$ for which there exists a sequence of elements $a_n \in R$ such that

$$\nu_k(a_n) \gg \nu_k(a_{n-1}), k \to \infty, \tag{1}$$

for every $n$. Then $\dim(R) = \infty$.

*Proof.* This proposition is implicit in Theorem 2.2 of [**S**], and our arguments will mostly follow his proof. For the following lemma, see Theorem 10.2.6 in [**Coh**] (see also Proposition 4.8 of [**Cla**] or Theorem 1 in [**K**]).

LEMMA 4. Let $I$ be an ideal in a commutative ring $A$ and $M \subset A \setminus I$ be a subset closed under multiplication. Then there exists an ideal $J \subset A$ containing $I$ and disjoint from $M$, so that $J$ is maximal with respect to this property. Furthermore, $J$ is a prime ideal in $A$.

Define the ideal $I < R$ by

$$I := \{a \in R | \exists k_0 = k_0(a) \text{ such that } \nu_k(a) \geq 1, \forall k > k_0\}.$$

Define the ideals $I_n < R$ by

$$I_n := \{a \in I | \nu_k(a) \gg \nu_k(a_{n-1})\}.$$

Then $a_n \in I_n$ and, hence, $I_n \neq 0$ for every $n$. Define the subsets

$$M_n := \{a \in I | \underset{k\to\infty}{\omega\text{-lim}} \frac{\nu_k(a)}{\nu_k(a_{n-1})} < \infty\};$$

they are closed under the multiplication. Then (1) implies that $I_{n+1} \subset I_n, M_n \subset M_{n+1}$ for all $n$.

Clearly, $I_n \cap M_n = \emptyset$, but $a_n \in I_n \cap M_{n+1}$. For each $n$ we let $\mathcal{J}_n$ denote the set of ideals $P \subset R$ such that $I_n \subset P, P \cap M_n = \emptyset$. By Lemma 4, every maximal element $P \in \mathcal{J}_n$ is a prime ideal.

LEMMA 5. $\mathcal{J}_n$ contains unique maximal element, which we will denote $P_n$ in what follows.

*Proof.* Suppose that $P', P''$ are two maximal elements of $\mathcal{J}_n$. We define the ideal $P = P' + P''$. Clearly, $P$ contains $I_n$. We claim that $P$ is disjoint from $M_n$. Indeed, given $p' \in P', p'' \in P''$, since $p' \notin M_n, p'' \notin M_n$, we have

$$\underset{k\to\infty}{\omega\text{-lim}} \frac{\nu_k(p')}{\nu_k(a_{n-1})} = \infty,$$

$$\underset{k\to\infty}{\omega\text{-lim}} \frac{\nu_k(p'')}{\nu_k(a_{n-1})} = \infty.$$

Since

$$\nu_k(p' + p'') \geq \min(\nu_k(p'), \nu_k(p'')),$$

we obtain

$$\underset{k\to\infty}{\omega\text{-lim}} \frac{\nu_k(p'+p'')}{\nu_k(a_{n-1})} \geq \underset{k\to\infty}{\omega\text{-lim}} \frac{\min(\nu_k(p'), \nu_k(p''))}{\nu_k(a_{n-1})} = \infty.$$

REMARK 6. This is the place in the proof where we really need ultralimits rather than lim sup.

Thus, $P \in \mathcal{J}_n$ and, in view of maximality of $P', P''$, we obtain

$$P' = P = P''. \quad \square$$

For each $n$ we define the ideal $Q_n := I_n + P_{n+1}$.

LEMMA 7. $Q_n \cap M_n = \emptyset$.

*Proof.* The proof is similar to the one of the previous lemma. Let $q = c + p$, $c \in I_n, p \in P_{n+1}$. Since $p \notin M_{n+1}$, $p \notin M_n$ as well. Therefore,

$$\omega\text{-}\lim_{k\to\infty} \frac{\nu_k(p)}{\nu_k(a_{n-1})} = \infty.$$

Since $c \in I_n$,

$$\omega\text{-}\lim_{k\to\infty} \frac{\nu_k(c)}{\nu_k(a_{n-1})} = \infty.$$

Therefore,

$$\omega\text{-}\lim_{k\to\infty} \frac{\nu_k(c+p)}{\nu_k(a_{n-1})} = \infty$$

as well. Thus, $q \notin M_n$. $\square$

COROLLARY 8. $Q_n \in \mathcal{J}_n$. In particular, $Q_n \subset P_n$.

*Proof.* It suffices to note that $I_n \subset Q_n$ according to the definition of $Q_n$. $\square$

LEMMA 9. $P_{n+1} \subset P_n$ and this inclusion is proper.

*Proof.* By the definition of $Q_n$ and Corollary 8, we have the inclusions

$$P_{n+1} \subset Q_n \subset P_n.$$

We now claim that $P_{n+1} \neq Q_n = I_n + P_{n+1}$. Recall that $a_n \in I_n \subset Q_n$ and $a_n \in M_{n+1}$, while $M_{n+1} \cap P_{n+1} = \emptyset$. Thus, $a_n \in Q_n \setminus P_{n+1}$. $\square$

We conclude that the ring $R$ contains an infinite chain of distinct prime ideals $P_n$ and, therefore, has infinite Krull dimension. Proposition 3 follows. $\square$

We will need the following classical result, see [**Con**, Ch. VII, Theorem 5.15]:

THEOREM 10. Let $D \subset \mathbb{C}$ be a domain, $c_j \in D$ be a sequence which does not accumulate anywhere in $D$ and $m_j$ be a sequence of positive integers. Then there exists a holomorphic function $g$ in $D$ which has zeroes only at the points $c_j$ and such that $m_j$ is the order of zero of $g$ at $c_j$, $j \in \mathbb{N}$.

We can now prove Theorem 1. If $M$ has only constant holomorphic functions then $H(M) = \mathbb{C}$ and, hence, $\dim H(M) = 0$. Thus, assume that $M$ admits a nonconstant holomorphic function $h : M \to \mathbb{C}$. We let $D$ denote the image of $h$. Pick a sequence $c_k \in D$ which converges to a point in $\hat{\mathbb{C}} \setminus D$ and which consists of regular values of $h$. (Here $\hat{\mathbb{C}}$ is the Riemann sphere.) For each $c_k$ the preimage $C_k := h^{-1}(c_k)$ is a complex submanifold in $M$; in each $C_k$ pick a point $b_k$. Define valuations

$$\nu_k : H(M) \to \mathbb{Z}_+ \cup \{\infty\}$$

by $\nu_k(f) := ord_{b_k}(f)$, the total order of $f$ at $b_k$, cf. [**Gu**, Chapter C, Definition 1].

For each $n \in \mathbb{N}$ define the sequence $m_{nk} = n^k$ and let $g_n$ be a holomorphic function on $D$ (as in Theorem 10) which has zero of the order $m_{nk}$ at $c_k$, $k \in \mathbb{N}$. Define $a_n := g_n \circ h \in H(M)$. Then for each $k$,

$$\lim_{k\to\infty} \frac{\nu_k(a_{n+1})}{\nu_k(a_n)} = \lim_{k\to\infty} \frac{(n+1)^k}{n^k} = \infty.$$

Therefore, by Proposition 3, $\dim(H(M)) = \infty$. $\square$

REMARK 11. 1. We refer the reader to Section 5.3 of [**Cla**] for further discussion of algebraic properties of rings of holomorphic functions.

2. For every Stein manifold $M$ (of positive dimension), the ring $H(M)$ has infinite Krull dimension. In particular, this applies to any noncompact connected Riemann surfaces (since every such surface is Stein, [**BS**]).

3. Noncompact connected complex manifolds $M$ of dimension $> 1$ can have $H(M) = \mathbb{C}$; for instance, take $M$ to be the complement to a finite subset in a compact connected complex manifold (of dimension $> 1$).

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