A path model for geodesics in Euclidean buildings and its applications to representation theory

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Abstract

In this paper we give a combinatorial characterization of projections of geodesics in Euclidean buildings to Weyl chambers. We apply these results to the representation theory of complex reductive Lie groups and to spherical Hecke rings associated with split nonarchimedean reductive Lie groups. Our main application is a generalization of the saturation theorem of Knutson and Tao for SL_n to other complex semisimple Lie groups.

1 Introduction

Let $\underline{G}$ be a split reductive algebraic group defined over $\mathbb{Z}$ and $\underline{G}^\vee$ be its Langlands' dual. In this paper we continue our study (which we began in [KLM3]) of interaction between the representation ring of the group $G^\vee := \underline{G}^\vee(\mathbb{C})$ and the *spherical Hecke algebra* $\mathcal{H}_G$ associated with the nonarchimedian group $G = \underline{G}(\mathbb{K})$, where $\mathbb{K}$ is a local field with discrete valuation.

In his papers [L1], [L2], P. Littelmann introduced a *path model* for the representations of complex reductive Lie groups $G^\vee$. The Littelmann path model gives a method to compute the structure constants of the representation ring of $G^\vee$ by counting certain piecewise linear paths, called LS paths.

In this paper we define a class of PL paths in Δ, a Weyl chamber of the Weyl group W of $G^\vee$. These paths will be called *Hecke paths* (see Definition 3.27) because of their connection with $\mathcal{H}_G$. We will prove that a path p in Δ is a Hecke path if and only if p is the projection into Δ of a geodesic segment in the Euclidean (Bruhat-Tits) building X associated with G. Thus, unlike LS paths which had to be *invented*, the Hecke paths appear very naturally as projections of geodesic segments. Hecke paths are defined by eliminating one of the axioms for LS paths, therefore *each LS path for $G^\vee$ is a Hecke path for G.*

The converse relation is more subtle and is discussed later in the introduction. To state our main results we need a definition of k_R, the *saturation factor* of the root

system R of the group $\underline{G}$. Let $\alpha_1, ..., \alpha_l \in R$ be the simple roots (corresponding to Δ). Let θ be the highest root and define positive integers $m_1, ..., m_l$ by

$$\theta = \sum_{i=1}^{l} m_i \alpha_i.$$

Then k_R is the least common multiple of the numbers m_i, $i = 1, ..., l$. We refer to section 2.3 for the computation of k_R.

Below, L is the character lattice of a maximal torus in $G^\vee$ (so that $\Delta \subset L \otimes \mathbb{R}$), $Q(R^\vee)$ is the root lattice of $R^\vee$. Our main result is the following theorem (see section 7), which, in a weaker form, has been conjectured by S. Kumar:

Theorem 1.1 (Saturation theorem). *Let $G^\vee$ and L be as above. Suppose that $\alpha, \beta, \gamma \in L$ are dominant characters such that $\alpha + \beta + \gamma \in Q(R^\vee)$ and that there exists $N \in \mathbb{N}$ so that*

$$(V_{N\alpha} \otimes V_{N\beta} \otimes V_{N\gamma})^{G^\vee} \neq 0.$$

Then for $k = k_R^2$ we have:

$$(V_{k\alpha} \otimes V_{k\beta} \otimes V_{k\gamma})^{G^\vee} \neq 0.$$

Here and in what follows V_λ is the irreducible representation of $G^\vee$ associated with the dominant weight λ of $G^\vee$. Also it will be convenient to introduce integers $n_{\alpha,\beta}(\gamma^*)$, the structure constants of the representation ring of the group $G^\vee$. Hence $n_{\alpha,\beta}(\gamma^*)$ are defined by the equation

$$V_\alpha \otimes V_\beta = \bigoplus_\gamma n_{\alpha,\beta}(\gamma^*) V_{\gamma^*}.$$

Here the right-hand side is the decomposition of the tensor product of the irreducible representations V_α and V_β into a direct sum of irreducible representations. We can then formulate the above theorem as

$$n_{N\alpha,N\beta}(N\gamma^*) \Rightarrow n_{k\alpha,k\beta}(k\gamma^*).$$

We will freely move back and forth between the symmetric formulation of saturation in Theorem 1.1 and the asymmetric formulation immediately above.

As an immediate corollary of Theorem 1.1 we obtain a new proof of the saturation theorem of A. Knutson and T. Tao [KT]:

Corollary 1.2. *Suppose that $R = A_l$, i.e. the semisimple part of $G^\vee$ is locally isomorphic to SL_{l+1}. Suppose that α, β, γ are dominant characters such that $\alpha + \beta + \gamma \in Q(R^\vee)$ and that there exists $N \in \mathbb{N}$ so that*

$$(V_{N\alpha} \otimes V_{N\beta} \otimes V_{N\gamma})^{G^\vee} \neq 0.$$

Then

$$(V_\alpha \otimes V_\beta \otimes V_\gamma)^{G^\vee} \neq 0.$$

Another proof of this theorem was given by H. Derksen and J. Weyman in [DW]. However the proofs of [KT] and [DW] do not work for root systems different from A_l.

Question 1.3. *Is it true that if G is a simple simply-laced group, then in Theorem 1.1 one can always take $k = 1$ and in the case of non-simply laced groups the smallest k which suffices is $k = 2$?*

The affirmative answer to this question is supported by the groups $Spin(5)$ and G_2 when we get the saturation constant $k = 2$ rather than 2^2 and 6^2 (see [KLM3]), as well as by a number of computer experiments with the exceptional root systems and the root systems D_l.

Using the results of [KLM2, KLM3] one can reformulate Theorem 1.1 as follows:

Theorem 1.4. *There exists a convex homogeneous cone $D_3 \subset \Delta^3$, defined by the* generalized triangle inequalities*, which depends only on the Weyl group W, so that the following hold:*

1. If a triple $(\alpha, \beta, \gamma) \in (\Delta \cap L)^3$ satisfies

$$(V_\alpha \otimes V_\beta \otimes V_\gamma)^{G^\vee} \neq 0,$$

then $(\alpha, \beta, \gamma) \in D_3$ and

$$\alpha + \beta + \gamma \in Q(R^\vee).$$

2. "Conversely", if $(\alpha, \beta, \gamma) \in k_R^2 \cdot L^3 \cap D_3$ and $\alpha + \beta + \gamma \in k_R^2 \cdot Q(R^\vee)$, then

$$(V_\alpha \otimes V_\beta \otimes V_\gamma)^{G^\vee} \neq 0.$$

We now outline the steps required to prove the Theorem 1.1. We will need several facts about Hecke rings $\mathcal{H}$. Let $\mathcal{O}$ denote the valuation ring of $\mathbb{K}$, then $K := \underline{G}(\mathcal{O})$ is a maximal compact subgroup in G. The lattice L defined above is the cocharacter lattice of a maximal torus $T \subset G$. The Hecke ring $\mathcal{H}$, as a $\mathbb{Z}$-module, is freely generated by the characteristic functions $\{c_\lambda : \lambda \in L \cap \Delta\}$. The multiplication on $\mathcal{H}$ is defined via the convolution product $\star$. Then the structure constants $m_{\alpha,\beta}(\gamma)$ of $\mathcal{H}$ are defined by

$$c_\alpha \star c_\beta = \sum_\gamma m_{\alpha,\beta}(\gamma) c_\gamma.$$

We refer the reader to [Gro], [KLM3] and to section 2.6 of this paper for more details.

Let $o \in X$ be the special vertex of the building X which is fixed by K. In section 2.5 we define the notion $d_\Delta(x, y)$ of the Δ-valued distance between points $x, y \in X$. Given a piecewise-geodesic path p in X we define its Δ-length as the sum of Δ-distances between the consecutive vertices.

The structure constants for $\mathcal{H}$ are related to the geometry of X via the following

Theorem 1.5 (See [KLM3], Theorem 8.12). *The number $m_{\alpha,\beta}(\gamma)$ is equal to the product of a certain positive constant by the number of geodesic triangles $T \subset X$ whose vertices are* special vertices *of X with the first vertex equal to o and whose Δ-side lengths are α, β, γ^*.*

Theorem 1.1 is essentially Statement 3, which follows from Statements 1 and 2, of the following

Theorem 1.6. *Set $\ell := k_R$.*

1. Suppose that $(\alpha, \beta, \gamma) \in D_3 \cap L^3$ and $\alpha + \beta + \gamma \in Q(R^\vee)$. Then the structure constants $m_{\cdot,\cdot}(\cdot)$ of the Hecke ring of the group G satisfy

$$m_{\ell\alpha,\ell\beta}(\ell\gamma) \neq 0.$$

2. Suppose that α, β, γ are dominant coweights of $\underline{G}$, such that $m_{\alpha,\beta}(\gamma) \neq 0$. Then

$$n_{\ell\alpha,\ell\beta}(\ell\gamma) \neq 0.$$

3. As a consequence of 1 and 2 we have: Suppose that $(\alpha, \beta, \gamma) \in D_3 \cap L^3$ and $\alpha + \beta + \gamma \in Q(R^\vee)$. Then

$$n_{\ell^2\alpha,\ell^2\beta}(\ell^2\gamma) \neq 0.$$

Remark 1.7. *a. Part 1 of the above theorem was proven in [KLM3]. Thus the point of this paper is to prove Part 2 of the above theorem.*

b. Examples in [KLM3] show that both implications

$$(\alpha, \beta, \gamma) \in D_3 \cap L^3, \alpha + \beta + \gamma \in Q(R^\vee) \Rightarrow m_{\alpha,\beta}(\gamma) \neq 0$$

and

$$m_{\alpha,\beta}(\gamma) \neq 0 \Rightarrow n_{\alpha,\beta}(\gamma) \neq 0$$

are **false** *for the groups G_2 and $SO(5)$. Therefore the dilation by k_R in both cases is necessary at least for these groups.*

More generally, we prove (Theorem 7.4)

Theorem 1.8. *Suppose that $\alpha, \beta, \gamma \in L$ are dominant weights so that one of them is the sum of minuscule weights. Then*

$$m_{\alpha,\beta}(\gamma) \Rightarrow n_{\alpha,\beta}(\gamma).$$

The proof of Part 2 of Theorem 1.6 proceeds as follows. In section 6.2 we prove a characterization theorem for folded triangles which implies:

Theorem 1.9. *There exists a geodesic triangle $T \subset X$ whose vertices are* special vertices *of X and whose Δ-side lengths are α, β, γ^* if and only if there exists a Hecke path $p : [0, 1] \to \Delta$ of Δ-length β so that*

$$p(0) = \alpha, p(1) = \gamma.$$

The directed segments π_α, the Hecke path p and the (reversed) directed segment π_γ fit together to form a "broken triangle", see Figure 1. Here and in what follows π_λ is the geodesic path parameterizing the directed segment $\overrightarrow{ox} = \lambda$.

Then, by combining theorems 1.5 and 1.9, we obtain

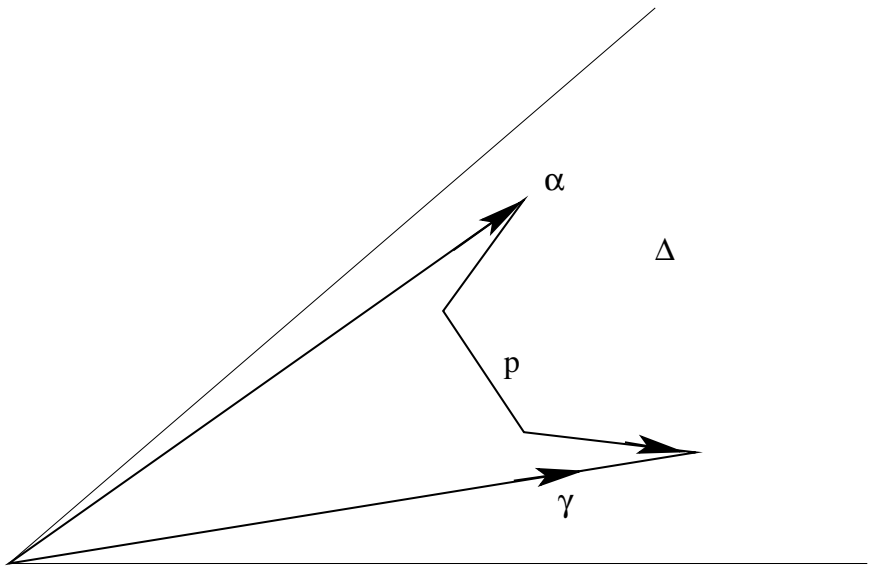

Figure 1: *A broken triangle.*

Theorem 1.10. *$m_{\alpha,\beta}(\gamma) \neq 0$ if and only if there exists a Hecke path $p : [0,1] \to \Delta$ of Δ-length β so that*

$$p(0) = \alpha, p(1) = \gamma.$$

This statement is an analogue of Littelmann's theorem which relates structure constants $n_{\alpha,\beta}(\gamma)$ of the representation ring with LS paths. The problem however is that not every Hecke path is an LS path (even for the group $SL(3)$).

In order to prove Theorem 1.1 our "path model" for the nonvanishing of the Hecke structure constants must be generalized to a model where Hecke paths are replaced by *generalized Hecke paths* (we also have to replace the LS-paths by *generalized LS paths*). More precisely, begin with a geodesic triangle $[x, y, z] \subset X$ whose vertices are special vertices of X and whose Δ-side lengths are β, γ^*, α. Now replace the geodesic segment $\overline{xy}$ with a certain piecewise-geodesic path $\tilde{p} \subset X$ connecting x and y in X, and which is contained in the 1-skeleton of a single apartment in X. We then show that the projection p of $\tilde{p}$ to Δ is a *generalized Hecke path.* The Δ-length of the path p still equals $d_\Delta(x, y) = \beta$.

The generalized Hecke paths have the advantage over Hecke paths that *their break points occur only at vertices of the building.* By using this observation we obtain

Theorem 1.11. *If p is a generalized Hecke path with $p(0) = 0$, then $k_R \cdot p$, the image of p under dilation by k_R, is a generalized LS path of Δ-length equal to $k_R \,\mathrm{length}_\Delta(p)$.*

Lastly, for *generalized LS paths* we prove, by modifying slightly Littelmann's arguments, the following:

Theorem 1.12. *$n_{\alpha,\beta}(\gamma) \neq 0$ if and only if there exists a generalized LS path q with Δ-length β so that*

$$\alpha + q(1) = \gamma$$

and the concatenation $\pi_\alpha * q$ is contained in Δ.

Theorem 1.1 is now obtained in 2 steps, the first of which is contained in [KLM3] and the second is at the heart of the present paper:

Step 1. It was shown in [KLM3, Theorem 9.17] that

$$n_{N\alpha,N\beta}(N\gamma) \neq 0 \Rightarrow m_{N\alpha,N\beta}(N\gamma) \neq 0.$$

Therefore, the vector $(N\alpha, N\beta, N\gamma^*)$ belongs to the homogeneous cone D_3. Hence, by Part 1 of Theorem 1.6 we conclude that $m_{\ell\alpha,\ell\beta}(\ell\gamma) \neq 0$.

Step 2. $m_{\ell\alpha,\ell\beta}(\ell\gamma) \neq 0$ implies existence of a geodesic triangle $[z, x, y] \subset X$ with the special vertices and Δ-side lengths $\ell\alpha, \ell\beta, \ell\gamma^*$. Then by projecting the corresponding path $\tilde{p}$ to Δ and dilating it by k_R we obtain a path $k\alpha + q(t)$ in Δ connecting $k\alpha$ to $k\gamma^*$, where $q(t)$ is a generalized LS path of Δ-length $k\beta$. (Recall that $k = \ell^2$.) Therefore, by appealing to Theorem 1.12, we see that

$$n_{k\alpha,k\beta}(k\gamma) \neq 0$$

which concludes the proof.

This paper is organized as follows. Preliminary material is discussed in section 2, where we review the concepts of Coxeter complexes, buildings and PL paths in buildings, as well as the *generalized distances* in buildings. In section 3 we define the notion of *chains*, which is essentially due to Littelmann. This concept allows one to define both LS paths and Hecke paths, as well as to relate Hecke paths and foldings of geodesic paths, which is discussed in the next section.

The main technical tool of this paper is the concept of *folding* of geodesics in a building into an apartment, which is done via retraction of the entire building to an apartment or chamber. [1] Properties of foldings are discussed in section 4. In section 4.3 we prove that the image of each geodesic segment in X under *folding* $f : X \to \Delta$ of X to a Weyl chamber, is a Hecke path, Theorem 4.16. We then prove a partial converse to this result, i.e. that each Hecke path which satisfies the *simple chain condition* can be unfolded in X. We also find a necessary and sufficient condition for unfolding of a path p which is *local*, i.e. it depends only on germs of the path p at its break-points.

In section 5 we review Littelmann's path model for the representation theory of complex semisimple Lie groups, in particular we discuss LS paths and generalized LS paths as well as raising and lowering operators.

We use *approximation of LS paths by paths satisfying the simple chain condition* to unfold LS paths in X, Theorem 6.1 in section 6.1. Although there are Hecke paths which are not LS paths, since the unfolding condition is local, by restricting the root system we reduce the general unfolding problem to the case of the LS paths. We thus establish that a path in Δ is unfoldable in X it and only if it is a Hecke path, this

[1] A similar idea is used by S. Gaussent and P. Littelmann in [GL], where they fold galleries in a building to galleries in an apartment.

is done in section 6.2, Theorem 6.5. The reader interested only in the proof of the saturation theorem can omit this section.

We prove the saturation theorem in section 7, Corollary 7.3. As we stated above, the idea of the proof is to replace Hecke paths with PL paths contained in the 1-skeleton of the Euclidean Coxeter complex. We show that dilation by k_R of such a path results in a *generalized LS path* which in turn suffices for finding nonzero invariant vectors in triple tensor products.

Hecke paths and folded galleries. It is interesting to ask what the relationship is between our paper using Hecke paths and the results of S. Gaussent and P. Littelmann in [GL] and others using positively folded galleries:

(a) *Hecke paths* correspond to the *positively folded galleries* defined in [GL]. However this correspondence can be established only **after** the folding–unfolding results of the present paper are proven. Therefore it appears that one cannot prove our results characterizing Hecke paths as projections of geodesic segments in building using the results of [GL] and vice versa.

(b) In [Sc], C. Schwer has used the gallery approach to compute the Hecke structure constants $m_{\alpha,\beta}(\gamma)$. In an earlier version of this paper we applied our theory to compute some Hecke structure coefficients. Thus both Hecke path and gallery models can be used to compute the structure constants of the spherical Hecke algebra.

(c) One can prove that

$$n_{\alpha,\beta}(\gamma) \neq 0 \Rightarrow m_{\alpha,\beta}(\gamma)$$

([KLM3, Theorem 9.17]) using both Hecke paths (as it is done in section 6.2) and positively folded galleries.

(d) There does not seem to be a direct way to carry over our proof of the main Theorem 1.1 or part 2 of Theorem 1.6 to a proof using positively folded galleries. It is clear from the above, that one of the critical steps in our argument is provided by Theorem 1.11 stating that dilation by k_R converts generalized Hecke paths to generalized LS paths. But there does not appear to be a way to "stretch" a folded gallery by the factor k_R — or, more generally, to produce an LS gallery from a positively folded gallery. This is an advantage of the Hecke paths, which are *geometric* objects, over positively folded galleries, which are *combinatorial* objects. Even when $k_R = 1$, there is no obvious (at least to us) reason why existence of a positively folded gallery would imply existence of an LS gallery. We note that if $k_R \neq 1$ then the existence of a positively folded gallery does not imply the existence of an LS gallery, there are counterexamples for $SO(5)$ and G_2.

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2 Definition and notation

2.1 Root systems and Coxeter complexes

A (discrete, nonnegatively curved) *Coxeter complex* is a pair (A, W), where A is either a Euclidean space or the unit sphere and W is a discrete reflection group acting on A. The *rank* of the Coxeter complex is the dimension of A.

Remark 2.1. *There are also Euclidean Coxeter complexes which are nondiscrete (i.e. the group W is nondiscrete, but has discrete linear part), as well as hyperbolic Coxeter complexes, where one uses reflection groups of isometries of the hyperbolic space. However we do not need them in the present paper.*

An isomorphism of Coxeter complexes $(A, W), (A', W')$ is an isometry $\iota : A \to A'$ so that

$$\iota W \iota^{-1} = W'.$$

Walls in (A, W) are fixed point sets of reflections $\tau \in W$. A point $x \in A$ is called *regular* if it does not belong to any wall and *singular* otherwise. The closure of each connected component of the set of regular points is called an *alcove* in the Euclidean case and a *chamber* in the spherical case.

In the case when W acts cocompactly on A, each alcove (resp. chamber) is a product (resp. join) of simplices and (A, W) determines structure of a *polysimplicial complex* on A. In general there exists a totally-geodesic subspace $A' \subset A$ which is W–invariant and such that A'/W is compact. Therefore each alcove in A is a product of simplices and a Euclidean subspace in A. Thus much of the discussion of Euclidean Coxeter complexes can be reduced to the case when A/W is compact.

Remark 2.2. *A triangulation of a fundamental alcove (chamber) in A determines a W–invariant simplicial complex on A. Thus we can always think of A as a simplicial complex.*

A *half-apartment* in A is the closure of a connected component of $A \setminus H$, where H is a wall in A.

"Most" Coxeter complexes are associated with root systems as we describe below. Suppose that R is a root system on a vector space V (i.e. each element $\alpha \in R$ is a linear functional on V). The *rank* of R is the number of simple roots in R, i.e. is the rank of the free abelian subgroup in V^* generated by R. Let A denote the Euclidean affine space corresponding to V. This data defines a finite Coxeter group W_{sph}, which is a reflection group generated by reflections in the hyperplanes $H_\alpha = \{x : \alpha(x) = 0\}$. *Weyl chambers* of W_{sph} are closures of the connected components of the complement to

$$\cup_{\alpha \in R} H_\alpha.$$

In what follows we fix a *positive Weyl chamber* Δ, it determines the subset of positive roots $R^+ \subset R$ and of *simple roots* $\Phi \subset R^+$. We also have the group of coweights $P(R^\vee)$ associated with R:

$$\lambda \in P(R^\vee) \iff \forall \alpha \in R, \quad \alpha(\lambda) \in \mathbb{Z}.$$

Let W_{aff} denote the *affine Coxeter group* determined by the above data, this group is generated by reflections in the hyperplanes (*affine walls*)

$$H_{\alpha,t} = \{\alpha(x) = t\}, t \in \mathbb{Z}.$$

The translation subgroup in W_{aff} is the *coroot lattice* $Q(R^\vee)$, it is generated by the coroots $\alpha^\vee, \alpha \in R$. The group $P(R^\vee)$ is the normalizer of W_{aff} in the group of translations V. Note that $V/P(R^\vee)$ is compact. The dimension of this quotient is the same as the dimension of V provided that $\mathrm{rank}(R)$ equals the dimension of V.

The *special vertices* of a Euclidean Coxeter complex are the points whose stabilizer in W_{aff} is isomorphic to W_{sph}. Equivalently, they are the points in the $P(R^\vee)$-orbit of the origin.

Remark 2.3. *If A/W_{aff} is compact, the vertices of (A, W_{aff}) are the vertices of the polysimplicial complex determined by W_{aff}.*

Let $A^{(0)}$ denote the *vertex set* of (A, W_{aff}), which consists of points of maximal intersection of walls in A. If R spans V^*, the set $A^{(0)}$ equals the vertex set of the polysimplicial complex in A defined by tessellation of A via alcoves of W_{aff}.

Given a Coxeter complex (A, W) and a point $x \in A$ we define a new Coxeter complex (S_x, W_x) where S_x is the unit tangent sphere at x and W_x is the stabilizer of x in W.

For a nonzero vector $\nu \in V$ we let $\bar{\nu} := \nu/|\nu|$ denote the *normalization* of ν. We define *rational* elements of the unit sphere S to be the unit vectors of the form

$$\eta = \bar{\nu}, \quad \nu \in P(R^\vee).$$

The next lemma follows immediately from compactness of $V/P(R^\vee)$:

Lemma 2.4. *Rational points are dense in S.*

Suppose that (A, W) is a Euclidean Coxeter complex. A *dilation* of (A, W) is a dilation h (i.e. a composition of translation and similarity $v \mapsto \lambda v, \lambda > 0$) in the affine space A so that

$$hWh^{-1} \subset W.$$

We let $Dil(A, W)$ denote the semigroup of dilations of the complex (A, W). We will refer to the number λ as the *conformal factor* of the dilation h.

Given a point $x \in A$ and a dilation $h \in Dil(A, W)$, we can define a new spherical Coxeter complex (S_x, W'_x) on the unit tangent sphere S_x at x via pull-back

$$W'_x := h^*(W_{t(x)}),$$

where $W_{h(x)}$ is the stabilizer of $h(x)$ in W.

Definition 2.5. *Suppose that W is a finite Coxeter group acting on a vector space V. Define a (nontransitive) relation $\sim_W$ on $V \setminus \{0\}$ by*

$$\mu \sim_W \nu \iff$$

$$\mu, \nu \text{ belong to the same Weyl chamber of } W.$$

We will frequently omit the subscript W in this notation.

Definition 2.6 (See [L2], page 514). *We say that nonzero vectors $\nu, \mu \in V$ satisfy $\nu \rhd_W \mu$ (for short, $\nu \rhd \mu$) if for each positive root α,*

$$\alpha(\nu) < 0 \Rightarrow \alpha(\mu) \leq 0.$$

Lemma 2.7. *Suppose that $\nu, \mu \in P(R^\vee)$, $w \in W = W_{aff}$ is such that $w(\nu) = \mu$. Then*

$$\mu - \nu \in Q(R^\vee).$$

Proof: The mapping w is a composition of reflections $\tau_i \in W$. Therefore it suffices to prove the assertion in case when w is a reflection τ. This reflection is a composition of a translation t and a reflection $\sigma \in W_o$. The translation t belongs to the translation subgroup $Q(R^\vee)$ of W, therefore it suffices to consider the case when $\tau = \sigma \in W_o$. Then $\tau = \tau_\beta$, where β is a root and we have

$$\mu - \nu = -\beta(\nu)\beta^\vee.$$

Since $\beta(\nu) \in \mathbb{Z}$ and $\beta^\vee \in Q(R^\vee)$, the assertion follows. □

2.2 Paths

Suppose that A, V, W_{aff}, etc., are as in the previous section.

Let $\tilde{\mathcal{P}}$ denote the set of all PL paths $p : [a, b] \to V$. We will be identifying paths that differ by orientation-preserving re-parameterizations $[a, b] \to [a', b']$. Accordingly, we will always (re)parameterize a PL path with the constant speed. We let $p'_-(t), p'_+(t)$ denote the derivatives of the function p from the left and from the right. The space $\tilde{\mathcal{P}}$ will be given the topology of uniform convergence.

If $p, q : [0, 1] \to A$ are PL paths in a simplicial complex such that $p(1) = q(0)$, we define their *composition* $r = p \cup q$ by

$$r(t) = \begin{cases} p(t), & t \in [0, 1], \\ q(t-1), & t \in [1, 2]. \end{cases}$$

Let $\mathcal{P} \subset \tilde{\mathcal{P}}$ denote the set of paths $p : [0, 1] \to V$ such that $p(0) = 0$. Given a path $p \in \mathcal{P}$ we let $p^* \in \mathcal{P}$ denote the *reverse* path

$$p^*(t) = p(1 - t) - p(1).$$

For a vector $\lambda \in V$ define a geodesic path $\pi_\lambda \in \mathcal{P}$ by

$$\pi_\lambda(t) = t\lambda, \quad t \in [0,1].$$

Given two paths $p_1, p_2 \in \mathcal{P}$ define their *concatenation* $p = p_1 * p_2$ by

$$p(t) = \begin{cases} p_1(2t), & t \in [0,1/2], \\ p_1(1) + p_2(2t-1), & t \in [1/2,1]. \end{cases}$$

Suppose that $p \in \mathcal{P}$ and $J = [a,b]$ is nondegenerate subinterval in $I = [0,1]$. We will use the notation $p|J \in \tilde{\mathcal{P}}$ to denote the function-theoretic restriction of p to $[a,b]$. We will use the notation $p|_J$ to denote the path in $\mathcal{P}$ obtained from $p|J$ by pre-composing $p|J$ with an increasing linear bijection $\ell : I \to J$ and post-composing it with the translation by the vector $-p(a)$.

Fix a positive Weyl chamber $\Delta \subset V$; this determines the set of positive roots $R^+ \subset R$, the set of simple roots $\Phi \subset R^+$. We define the subset $\mathcal{P}^+ \subset \mathcal{P}$ consisting of the paths whose image is contained in Δ.

For a path $p \in \mathcal{P}$ and a positive root $\alpha \in R^+$ define the *height* function

$$h_\alpha(t) = \alpha(p(t))$$

on $[0,1]$. Let $m_\alpha = m_\alpha(p) \in \mathbb{R}$ denote the minimum of h_α. Clearly $m_\alpha(p) \leq 0$ for all $p \in \mathcal{P}$. We define the set of "integral paths"

$$\mathcal{P}_{\mathbb{Z}} := \{p \in \mathcal{P} : \forall \alpha \in \Phi, m_\alpha(p) \in \mathbb{Z}\}.$$

More restrictively, we define the set $\mathcal{P}_{\mathbb{Z},loc}$ of paths $p \in \mathcal{P}$ which satisfy the following *local integrality condition*:

For each simple root $\alpha \in \Phi$ the function h_α takes integer values at the points of local minima.

2.3 The saturation factors associated to a root system

In this section we define and compute *saturation factors* associated with root systems. Let $o \in A$ be a special vertex, which we will identify with $0 \in V$.

Definition 2.8. *We define the* saturation factor k_R *for the root system R to be the least natural number k such that $k \cdot A^{(0)} \subset P(R^\vee) \cdot o$. The numbers k_R for the irreducible root systems are listed in the table (2).*

Note that the condition that $k \cdot A^{(0)} \subset P(R^\vee) \cdot o$ is equivalent to that each point of $k \cdot A^{(0)}$ is a special vertex.

Below we explain how to compute the saturation factors k_R following [KLM3]. First of all, it is clear that if the root system R is reducible and $R_1, ..., R_s$ are its irreducible components, then $k_R = LCM(k_{R_1}, ..., k_{R_s})$, where LCM stands for the

least common multiple. Henceforth we can assume that the system R is reduced, irreducible and has rank $n = \dim(V)$. Then the affine Coxeter group W_{aff} acts cocompactly on A and its fundamental domain (a *Weyl alcove*) is a simplex.

Let $\{\alpha_1, ..., \alpha_\ell\}$ be the collection of simple roots in R (corresponding to the positive Weyl chamber Δ) and θ be the highest root. Then

$$\theta = \sum_{i=1}^{\ell} m_i \alpha_i. \tag{1}$$

We have

Lemma 2.9 (See Section 2 of [KLM3]). $k_R = LCM(m_1, ..., m_n)$.

Below is the list of saturation factors:

Root system	θ	k_R
A_ℓ	$\alpha_1 + ... + \alpha_\ell$	1
B_ℓ	$\alpha_1 + 2\alpha_2 + ... + 2\alpha_\ell$	2
C_ℓ	$2\alpha_1 + 2\alpha_2 + ... + 2\alpha_{\ell-1} + \alpha_\ell$	2
D_ℓ	$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + ... + 2\alpha_\ell$	2
G_2	$3\alpha_1 + 2\alpha_2$	6
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	12
E_6	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 3\alpha_6$	6
E_7	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 3\alpha_5 + +3\alpha_6 + 4\alpha_7$	12
E_8	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 4\alpha_5 + +4\alpha_6 + 5\alpha_7 + 6\alpha_8$	60

(2)

2.4 Buildings

Our discussion of buildings follows [KL]. We refer the reader to [Br], [Ron], [Rou] for the more combinatorial discussion.

Fix a spherical or Euclidean (discrete) Coxeter complex (A, W), where A is a Euclidean space E or a unit sphere S and $W = W_{aff}$ or $W = W_{sph}$ is a discrete Euclidean or a spherical Coxeter group acting on A.

A metric space Z is called *geodesic* if any pair of points x, y in Z can be connected by a geodesic segment $\overline{xy}$.

Let Z be a metric space. A *geometric structure* on Z *modeled on* (A, W) consists of an atlas of isometric embeddings $\varphi : A \hookrightarrow Z$ satisfying the following compatibility condition: For any two charts φ_1 and φ_2, the transition map $\varphi_2^{-1} \circ \varphi_1$ is the restriction of an isometry in W. The charts and their images, $\varphi(A) = a \subset Z$, are called *apartments*. We will sometimes refer to A as the *model apartment*. We will require that there are *plenty of apartments* in the sense that any two points in Z lie in a common

apartment. All W-invariant notions introduced for the Coxeter complex (A, W), such as rank, walls, singular subspaces, chambers etc., carry over to geometries modeled on (A, W). If $a, a' \subset X$ are alcoves (in the Euclidean case) or chambers (in the spherical case) then there exists an apartment $A' \subset X$ containing $a \cup a'$: Just take regular points $x \in a, x' \in a'$ and an apartment A' passing through x and x'.

A geodesic metric space Z is said to be a $CAT(\kappa)$-space if *geodesic triangles in Z are "thinner" than geodesic triangles in a simply-connected complete surface of the constant curvature κ.* We refer the reader to [Ba] for the precise definition. Suppose that Z is a (non-geodesic) metric space with the discrete metric:

$$d(x, y) = \pi \iff x \neq y.$$

We will regard such a space as a $CAT(1)$ space as well.

Definition 2.10. *A* spherical building *is a $CAT(1)$-space modeled on a spherical Coxeter complex.*

Spherical buildings have a natural structure of polysimplicial piecewise spherical complexes. We prefer the geometric to the combinatorial view point because it appears to be more appropriate in the context of this paper.

Definition 2.11. *A Euclidean building is a CAT(0)-space modeled on a (discrete) Euclidean Coxeter complex.*

A building is called *thick* if every wall is an intersection of apartments. *Thickness* $t(H)$ of a wall H is the number of half-apartments in X bounded by H. For buildings associated with algebraic groups, the number $t(H)$ is independent of H and is called *thickness of the building.* A non-thick building can always be equipped with a natural structure of a thick building by reducing the Coxeter group.

Let $\mathbb{K}$ be a local field with a (discrete) valuation ν and valuation ring $\mathcal{O}$. Given a split reductive algebraic group $\underline{G}$ over $\mathbb{Z}$, and a nonarchimedean Lie group $G = \underline{G}(\mathbb{K})$ we can associate with it a Euclidean building (a Bruhat-Tits building) $X = X_G$. We refer the reader to [BT], [KLM3] and [Rou] for more detailed discussion of the properties of X. Here we only recall that:

1. X is thick and locally compact.

2. X is modeled on a Euclidean Coxeter complex (A, W_{aff}) whose dimension equals the rank of $\underline{G}$, and the root system is isomorphic to the root system of $\underline{G}$.

3. X contains a special vertex o whose stabilizer in G is $\underline{G}(\mathcal{O})$.

Example 2.12. *Let X be a (discrete) Euclidean building, consider the* spaces of directions $\Sigma_x X$. *We will think of this space as the space of germs of non-constant geodesic segments $\overline{xy} \subset X$. As a polysimplicial complex $\Sigma_x X$ is just the link of the point $x \in X$. The space of directions has the structure of a spherical building modeled on (S, W_{sph}), which is thick if and only if x is a special vertex of X, see [KL]. The same applies in the case when X is a spherical building.*

If X is a Euclidean building modeled on (A, W), for each point $x \in X$ the space of directions $\Sigma_x(X)$ has *two structures* of a spherical building:

1. The *restricted building structure* which is modeled on the Coxeter complex (S, W_x), where $S = S_x(A)$ is the unit tangent sphere at x and W_x is the stabilizer of x in the Coxeter group W. This building structure is thick.

2. The *unrestricted building structure* which is modeled on the Coxeter complex (S, W_{sph}), where $S = S_x(A)$ is the unit tangent sphere at x and W_{sph} is the linear part of the affine Coxeter group W_{aff}. This building structure is not thick, unless x is a special vertex.

Let B be a spherical building modeled on a spherical Coxeter complex (S, W_{sph}). We say that two points $x, y \in B$ are antipodal, if $d(x, y) = \pi$; equivalently, they are antipodal points in an apartment $S' \subset B$ containing both x and y. The quotient map $S \to S/W_{sph} \cong \Delta_{sph}$ induces a canonical projection $\theta : B \to \Delta_{sph}$ folding the building onto its model Weyl chamber. The θ-image of a point in B is called its *type*.

Remark 2.13. *To define $\theta(x)$ pick an apartment S' containing x and a chart $\phi : S \to S'$. Then $\theta(x)$ is the projection of $\phi^{-1}(x)$ to $S/W_{sph} \cong \Delta_{sph}$. We note that this is clearly independent of S' and ϕ.*

Lemma 2.14. *1. If $h : A \to A'$ is an isomorphism of apartments in B (i.e. ${\phi'}^{-1} \circ h \circ \phi \in W$) then $\theta \circ h = \theta$.*

2. If $x, x' \in B$ which belong to apartments A, A' respectively and $-x \in A, -x' \in A'$ are antipodal to x, x', then $\theta(x) = \theta(x')$ implies $\theta(-x) = \theta(-x')$.

Proof: (1) is obvious, so we prove (2). Pick an isomorphism $h : A \to A'$. Then (since $\theta(x) = \theta(x')$) there exists $w \in W \curvearrowright A'$ such that $w(h(x)) = x'$. Hence $w \circ h(-x) = -w \circ (x) = -x'$. The claim now follows from 1. $\square$

We will regard *n-gons* P in a building X as maps $\nu : \{1, ..., n\} \to X$, $\nu(i) = x_i$, where x_i will be the vertices of P. If $rank(X) \geq 1$ we can connect the consecutive vertices x_i, x_{i+1} by shortest geodesic segments $\overline{x_i x_{i+1}} \subset X$ thus creating a *geodesic polygon* $[x_1, x_2, \ldots, x_n]$ in X with the edges $\overline{x_i x_{i+1}}$. Observe that in case x_i, x_{i+1} are antipodal, the edge $\overline{x_i x_{i+1}}$ is not unique.

We say that two subsets F, F' in a building X are *congruent* if there exist apartments A, A' in X containing F, F' resp., and an isomorphism $A \to A'$ of Coxeter complexes which carries F to F'.

Convention 2.15. *Suppose that X is a spherical building. We will be considering only those geodesic triangles T in X for which the length of each geodesic side of T is $\leq \pi$.*

Let X, Y be buildings and $f : X \to Y$ a continuous map satisfying the following: For each alcove (in Euclidean case) or spherical chamber (in the spherical case) $a \subset X$, the image $f(a)$ is contained in an apartment of Y and the restriction $f|a$ is an isometry or similarity. Then we call f *differentiable* and define the *derivative* df of f as follows.

Given a point $x \in X$ and $y = f(x)$, the derivative df_x is a map $\Sigma_x(X) \to \Sigma_y(Y)$. For each $\xi \in \Sigma_x(X)$ let $\overline{xz} \subset X$ be a geodesic segment whose interior consists of regular points only and so that ξ is the unit tangent vector to $\overline{xz}$. Then f sends $\overline{xz}$ to a nondegenerate geodesic segment $\overline{yf(z)}$ contained in an apartment $A \subset Y$. Then we let $df_x(\xi) \in \Sigma_y(Y)$ be the unit tangent vector to $\overline{yf(z)}$.

We will be also using the above definition in the setting when a building Y is a Euclidean Coxeter complex (A, W) and $h \in Dil(A, W)$. Then after letting $X := h^*(A, W)$ we get an isometry $h : X \to Y$.

Convention 2.16. *Throughout the paper we will be mostly using roman letters x, y, z, etc., to denote points in Euclidean buildings and Greek letters ξ, η, ζ, etc., to denote points in spherical buildings. Sometimes however (e.g., in section 4.2) we will be working simultaneously with a spherical building X and its links $\Sigma_x(X)$, which are also spherical buildings. In this case we will use roman letters for points in X and Greek letters for points in $\Sigma_x(X)$.*

2.5 Generalized distances and lengths in buildings

Let (A, W) be a spherical or Euclidean Coxeter complex. The complete invariant of a pair of points $(x, y) \in A^2$ with respect to the action $W \curvearrowright A$, is its image $d_{ref}(x, y)$ under the canonical projection to $A \times A/W$. Following [KLM2] we call $d_{ref}(x, y)$ the *refined distance from x to y*. This notion carries over to buildings modeled on the Coxeter complex (A, W): For a pair of points (x, y) pick an apartment A' containing x, y and, after identifying A' with the model apartment A, let $d_{ref}(x, y)$ be the projection of this pair to $A \times A/W$.

If points ξ, η in a spherical building are antipodal we will use π for the refined distance $d_{ref}(\xi, \eta)$: This does not create much ambiguity since given apartment contains unique point antipodal to ξ.

In the case of Euclidean Coxeter complexes there is an extra structure associated with the concept of refined length. Given a Euclidean Coxeter complex (A, W_{aff}), pick a special vertex $o \in A$. Then we can regard A as a vector space V, with the origin $0 = o$. Let $\Delta \subset A$ denote a Weyl chamber of W_{sph}, the tip of Δ is at o.

Then following [KLM2], we define the Δ-*distance* between points of (A, W_{aff}) by composing d_{ref} with the natural forgetful map

$$A \times A/W_{aff} \to A/W_{sph} \cong \Delta.$$

To compute the Δ-distance $d_\Delta(x, y)$ we regard the oriented geodesic segment $\overline{xy}$ as a vector in V and project it to Δ. Again, the concept of Δ-distance, carries over to the buildings modeled on (A, W_{aff}).

Definition 2.17. *Let X be a thick Euclidean building. Define the set $D_n(X) \subset \Delta^n$ of Δ-side lengths which occur for geodesic n-gons in X.*

It is one of the results of [KLM2] that $D_n := D_n(X)$ is a convex homogeneous polyhedral cone in Δ^n, which depends only on (A, W_{sph}). The polyhedron D_3 in

Theorem 1.4 is the polyhedron $D_3(X)$. The set of *stability* inequalities defining D_n is determined in [KLM1] and [BS].

Theorem 2.18 (See Corollary 8.4 in [KLM3]). *Let X be a thick Euclidean building modeled on (A, W_{aff}). Suppose that $\alpha, \beta, \gamma \in P(R^\vee)$, $\alpha + \beta + \gamma \in Q(R^\vee)$ and $(\alpha, \beta, \gamma) \in D_3(X)$. Then there exists a geodesic triangle $T \subset X$ whose vertices are vertices of X and the Δ-side lengths are α, β, γ.*

Suppose that p is a PL path in a Euclidean building X. We say that p is a *billiard path* if for each $t, s \in [0, 1]$ the tangent vectors $p'(t), p'(s)$ have the same projection to the chamber Δ in the model apartment. If p is a path which is the composition

$$\overline{x_0 x_1} \cup ... \cup \overline{x_{m-1} x_m}$$

of geodesic paths, then the Δ-length of p is defined as

$$\text{length}_\Delta(p) := \sum_{i=1}^{m} d_\Delta(x_{i-1}, x_i)$$

where $d_\Delta(x, y)$ is the Δ-distance from x to y

Each PL path p admits a unique representation

$$p = p_1 \cup ... \cup p_n$$

as a composition of maximal billiard subpaths so that

$$\lambda_i = \text{length}_\Delta(p_i).$$

We define

$$\underline{\text{length}}_\Delta(p) := \underline{\lambda} = (\lambda_1, ..., \lambda_n).$$

Clearly, $\text{length}_\Delta(p)$ is the sum of the vector components of $\underline{\lambda}$.

2.6 The Hecke ring

In this section we review briefly the definition of spherical Hecke rings and their relation to the geometry of Euclidean buildings; see [Gro, KLM3] for more details.

Let $\mathbb{K}$ denote a locally compact field with discrete valuation v, and (necessarily) finite residue field. Let $\mathcal{O}$ be the subring of $\mathbb{K}$ consisting of elements with nonnegative valuation. Choose a uniformizer $\pi \in \mathcal{O}$.

Consider a connected reductive algebraic group $\underline{G}$ over $\mathbb{K}$. We fix a maximal split torus $\underline{T} \subset \underline{G}$ defined over $\mathcal{O}$. We put $G := \underline{G}(\mathbb{K}), K := \underline{G}(\mathcal{O})$ and $T := \underline{T}(\mathbb{K})$. We let $\underline{B} \subset \underline{G}$ be a Borel subgroup normalized by $\underline{T}$ and set $B := \underline{B}(\mathbb{K})$.

Let X denote the Bruhat-Tits building associated with the group $\underline{G}$; $o \in X$ is a distinguished special vertex stabilized by the compact subgroup K.

We also have free abelian cocharacter group $X_*(\underline{T})$ of $\underline{T}$ whose rank equals $\dim(\underline{T})$. This group contains the set of coroots $R^\vee$ of the group $\underline{G}$. The roots are the characters of $\underline{T}$ that occur in the adjoint representation on the Lie algebra of $\underline{G}$. The subset R^+ of the roots that occur in representation on the Lie algebra of $\underline{B}$ forms a positive system and the indecomposable elements of that positive system form a system of simple roots Φ. We let W denote the corresponding (finite) Weyl group.

The set of positive roots Φ determines a positive Weyl chamber Π^+ in $X_*(\underline{T})$, by

$$\Pi^+ = \{\lambda \in X_*(\underline{T}) : \ \langle \lambda, \alpha \rangle \geq 0, \alpha \in \Phi\}.$$

This chamber is a fundamental domain for the action of W on $X_*(\underline{T})$.

We define a partial ordering on Π^+ by $\lambda \succ \mu$ iff the difference $\lambda - \mu$ is a sum of positive coroots.

Definition 2.19. *The (spherical) Hecke ring $\mathcal{H} = \mathcal{H}_G$ is the ring of all locally constant, compactly supported functions $f : G \longrightarrow \mathbb{Z}$ which are K–biinvariant. The multiplication in $\mathcal{H}$ is by the convolution*

$$f \star g(z) = \int_G f(x) \cdot g(x^{-1}z) dx$$

where dx is the Haar measure on G giving K volume 1.

The ring $\mathcal{H}$ is commutative and associative. For $\lambda \in X_*(\underline{T})$ let c_λ be the characteristic function of the corresponding K-double coset $\lambda(\pi) \in K\backslash G/K$. Then the functions $c_\lambda, \lambda \in \Delta$ freely generate $\mathcal{H}$ as an (additive) abelian group. The structure constants for $\mathcal{H}$ are defined by the formula

$$c_\lambda \star c_\mu = \sum_\nu m_{\lambda,\mu}(\nu) c_\nu = c_{\lambda+\mu} + \sum m_{\lambda,\mu}(\nu) c_\nu, \tag{3}$$

where the last sum is taken over all $\nu \in \Pi^+$ such that $\lambda + \mu \succ \nu$ and therefore is finite.

It turns out that the structure constants $m_{\lambda,\mu}(\nu)$ are nonnegative integers which are polynomials in q with integer coefficients. These constants are determined by the geometry of the building as follows. Given α, β, γ let $\mathcal{T} = \mathcal{T}_{\alpha,\beta}(\gamma)$ denote the (finite) set of geodesic triangles $[o, x, y]$ in the building X which have the Δ-side lengths α, β, γ^*, so that y is the projection of the point

$$\gamma(\pi) \in T \subset G$$

into X under the map $g \mapsto g \cdot o$. Recall that γ as a cocharacter and therefore it defines a homomorphism $\mathbb{K}^* \to T$.

Theorem 2.20 (Theorem 9.11, [KLM3]). *$m_{\alpha,\beta}(\gamma)$ equals the cardinality of $\mathcal{T}$.*

3 Chains

3.1 Absolute chains

Let R be a root system on a Euclidean vector space V, $W = W_{sph}$ be the finite Coxeter group associated with R, let W_{aff} denote the affine Coxeter group associated to R. Our root system R is actually the *coroot system* for the one considered by Littelmann in [L2]. Accordingly, we will switch weights to coweights, etc. We pick a Weyl chamber Δ for W, this determines the positive roots and the simple roots in R. Let $-\Delta$ denote the negative chamber.

We get the Euclidean Coxeter complex (A, W_{aff}), where $A = V$ and the spherical Coxeter complex (S, W) where S is the unit sphere in V. By abusing notation we will also use the notation $\Delta, -\Delta$ for the positive and negative chambers in (S, W). We will use the notation $(A, W, -\Delta)$ for a Euclidean/spherical Coxeter complex with chosen negative chamber. More generally, we will use the notation (A, W_{aff}, a) for a Euclidean Coxeter complex with chosen alcove a.

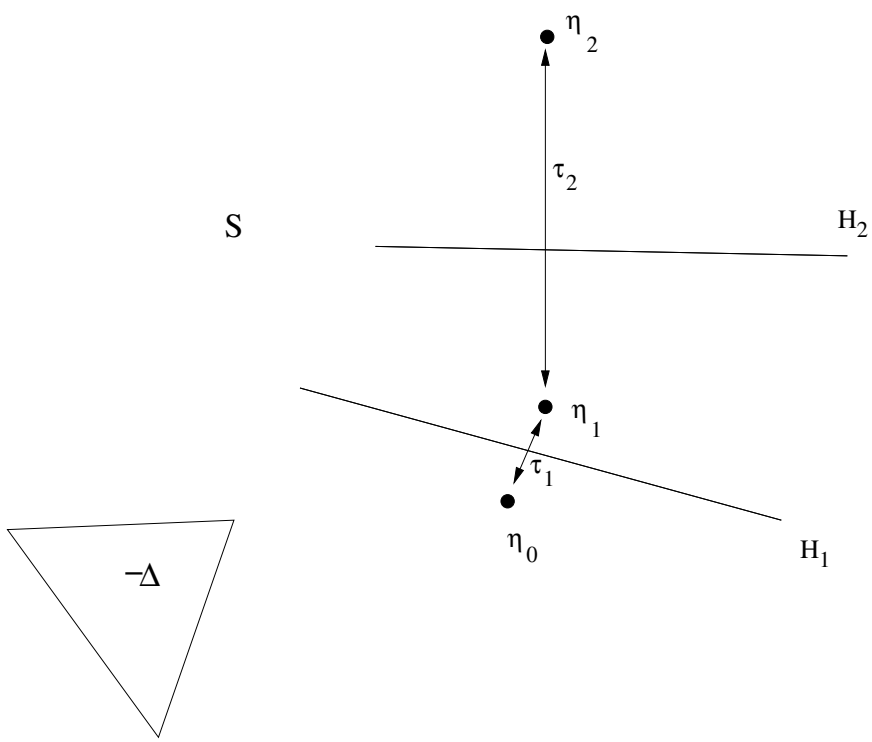

Figure 2: *A chain.*

Definition 3.1. *A W–*chain *in $(A, W, -\Delta)$ is a finite sequence $(\eta_0, ..., \eta_m)$ of nonzero vectors in V so that for each $i = 1, ..., m$ there exists a positive root $\beta_i \in R^+$ so that the corresponding reflection $\tau_i := \tau_{\beta_i} \in W$ satisfies*

1. $\tau_i(\eta_{i-1}) = \eta_i$.
2. $\beta_i(\eta_{i-1}) < 0$.

Sometimes we will refer to a chain as a $(V, W, -\Delta)$-chain to emphasize the choice of V, W and Δ. When the choice of W is clear we will frequently refer to W-chains as chains.

Remark 3.2. *One could call these chains (V, W, Δ)-chains instead, but this definition would not generalize to affine chains.*

Recall that τ_i is the reflection in the wall $H_i = \{\beta_i = 0\}$. More geometrically one can interpret the condition (2) by saying that the wall H_i separates the negative chamber $-\Delta$ from the vector η_i. In other words, the reflection τ_i moves the vector η_{i-1} "closer" to the positive chamber.

The concept of a chain generalizes naturally to Euclidean Coxeter complexes (A, W_{aff}). Pick an alcove a in A.

Definition 3.3. *An affine* chain *in (A, W_{aff}, a) is a finite sequence $(\eta_0, ..., \eta_m)$ of elements in A so that for each $i = 1, ..., m$ there exists a reflection $\tau_i \in W_{aff}$ such that*

1. *$\tau_i(\eta_{i-1}) = \eta_i$.*
2. *The hyperplane $H_i \subset A$ fixed by τ_i separates a from η_i.*

We now return to the chains as in definition 3.1. By restricting vectors η_i to have unit length we define chains in the spherical Coxeter complex, see Figure 2.

Definition 3.4. *The points η_i as in Definition 3.1 will be called* vertices *of the chain. We will say that the chain begins at η_0 and* ends *at η_m, or that this chain is* from η_0 to η_m. *We refer to a subsequence $(\eta_i, \eta_{i+1}, ..., \eta_m)$ as a* tail *of the chain.*

We will refer to the number m as the *length* of the chain. A chain (η_i) is called *simple* if it has length 1. Set

$$dist_W(\nu, \mu) = dist(\nu, \mu)$$

to be the maximal length m of a W-chain which begins at ν and ends at μ.

Given a chain $(\eta_0, ..., \eta_m)$ we define a *subdivision* of this chain to be a new chain in (A, W) which is still a chain from η_0 to η_m and which contains all the vertices of the original chain.

The concept of chain determines a partial order on the W-orbits in V:

Definition 3.5 (See [L2], page 509). *For a pair of nonzero vectors $\nu, \mu \in V$ which belong to the same W-orbit, write $\nu \geq_W \mu$ (or simply $\nu \geq \mu$) if there exists a W-chain from ν to μ. Accordingly, $\nu > \lambda$ if $\nu \geq \lambda$ and $\nu \neq \lambda$.*

Lemma 3.6. *Suppose that $\nu \geq \mu$ and α is a positive root such that $\alpha(\mu) \leq 0$. Then*

$$\nu \geq \tau_\alpha(\mu).$$

Proof: If $\alpha(\mu) = 0$ then $\tau_\alpha(\mu) = \mu$ and there is nothing to prove. Thus we assume that $\alpha(\mu) < 0$. Consider a chain $(\nu = \nu_0, ..., \nu_s = \mu)$. Then, since $\alpha(\mu) < 0$, we also have $\alpha(\tau_\alpha(\mu)) > 0$ and thus we get a longer chain

$$(\nu = \nu_0, ..., \nu_s = \mu, \tau(\mu)). \quad \square$$

The word metric d_W on the finite Coxeter group W defines the *length function*

$$\ell : W \cdot \lambda \to \mathbb{N}$$

by

$$\ell(\mu) := \min\{d_W(w, 1) : w \in W, w^{-1}(\mu) \in \Delta\}.$$

Lemma 3.7. *If $\nu > \mu$ then $\ell(\nu) > \ell(\mu)$.*

Proof: It suffices to prove the assertion in the case when

$$\mu = \tau(\nu), \tau = \tau_\beta,$$

where β is a positive root, $\beta(\nu) < 0, \beta(\mu) > 0$. If $W \cong \mathbb{Z}/2$ the assertion is clear, so we suppose that it is not the case. Then we can embed the Cayley graph Γ of W as a dual graph to the tessellation of V by the Weyl chambers of W. Suppose that $\nu \in w^{-1}(\Delta)$, then the wall $H_\beta = \{\beta = 0\}$ separates $w^{-1}(\Delta)$ from Δ, where $w \in W$ is the shortest element such that $w(\nu) \in \Delta$. Let $p : [0, 1] \to \Gamma$ denote the shortest geodesic from 1 to w in Γ. The path p crosses the wall H at a point $x = p(T)$. We construct a new path q by

$$q|[0, T] = p|[0, T], \quad q|[T, 1] = w \circ p|[T, 1].$$

The path q connects $1 \in \Delta$ to the Weyl chamber $\tau w(\Delta)$ containing μ. This path has a break-point at x, which is not a vertex of the Cayley graph. Therefore, by eliminating the backtracking of q at x, we obtain a new path which connects 1 to $\tau w(\Delta)$ and whose length is one less than the length of p. $\square$

Corollary 3.8. *The length of a chain in V does not exceed the diameter of the Cayley graph of W.*

Corollary 3.9. *Suppose that $\nu \in \Delta$ and $\nu \geq \mu$. Then $\mu = \nu$.*

Proof: Since $\nu \in \Delta$, $\ell(\nu) = 0$. Hence by Lemma 3.7, $\ell(\mu) = 0$, which implies that $\mu = \nu$. $\square$

Lemma 3.10. *Suppose that $\nu = w(\mu) \neq 0$ for some $w \in W$ and $\mu \rhd \nu$. Then $\nu \geq \mu$.*

Proof: Let Δ be the positive Weyl chamber. Suppose that Δ_0 is a chamber containing ν and Δ' is a chamber containing μ. These chambers are nonunique, but we can choose them in such a way that for all $\xi \in \Delta_0, \xi' \in \Delta'$

$$\xi' \rhd \xi.$$

In other words, if a wall separates Δ' from Δ then it also separates Δ_0 from Δ. Let

$$(\Delta_0, \Delta_1, ..., \Delta_m = \Delta')$$

be a gallery of Weyl chambers, i.e. for each i, $\Delta_i \cap \Delta_{i+1}$ is a codimension 1 face F_i of Δ_i, Δ_{i+1}. We choose this gallery to have the shortest length, i.e. so that the number m is minimal. Let H_i be the wall containing F_i and τ_i be the reflection in H_i. We claim that the sequence

$$\nu = \eta_0, \eta_1 = \tau_1(\eta_0), ..., \mu = \eta_m = \tau_m(\eta_{m-1}),$$

after deletion of equal members, is a chain.

Our proof is by induction on m. If $m = 0$ and $\nu = \mu$, there is nothing to prove. Suppose the assertion holds for $m - 1$, let us prove it for m. We claim that for all points $\xi_1 \in \Delta_1, \xi_0 \in \Delta_0$, $\xi_1 \rhd \xi_0$. Indeed, otherwise the wall H_1 does not separate Δ_0 from Δ, but separates Δ_1 from Δ. Then H_1 does not separate Δ_m from Δ either. Thus, as in the proof of Lemma 3.7, we can replace the gallery $(\Delta_0, \Delta_1, ..., \Delta_m)$ with a shorter gallery connecting Δ_0 to Δ', contradicting minimality of m. Now, clearly,

$$\eta_0 \geq \eta_1, \eta_0 \rhd \eta_1.$$

Therefore, by the induction

$$\eta_0 \geq \eta_1 \geq \eta_m \Rightarrow \nu = \eta_0 \geq \eta_m = \mu. \quad \square$$

Remark 3.11. *The converse to the above lemma is false for instance for the root system A_2. See Figure 3, where $\eta_0 \geq \eta_1$ but $\eta_0 \not\rhd \eta_1$.*

As a corollary of Lemma 3.10 we obtain:

Lemma 3.12. *Let $\nu \in V \setminus \{0\}$ and let μ be the unique vector in $W \cdot \nu$ which belongs to Δ. Then $\nu \geq \mu$.*

Proof: Clearly, $\mu \rhd \nu$. Then the assertion follows from Lemma 3.10. $\square$

Definition 3.13 (Maximality condition). *We say that a chain $(\eta_0, \eta_1, ..., \eta_m)$ is* maximal *if it cannot be* subdivided *into a longer W-chain. Equivalently, $dist(\eta_i, \eta_{i+1}) = 1$ for each $i = 0, ..., m - 1$.*

Lemma 3.14. *Suppose that $\nu \geq \mu$, and there exists a simple root β such that $\tau_\beta(\nu) = \mu$ and $\beta(\nu) < 0$. Then $dist(\nu, \mu) = 1$.*

Proof: Consider a chain from ν to μ, i.e. a sequence of vectors $\nu = \nu_0, \nu_1, ..., \nu_s = \mu$ and positive roots $\beta_1, ..., \beta_s$ so that

$$\nu_i = \tau_{\beta_i}(\nu_{i-1}) \quad \text{and} \quad \beta_i(\nu_{i-1}) < 0, i = 1, ..., s.$$

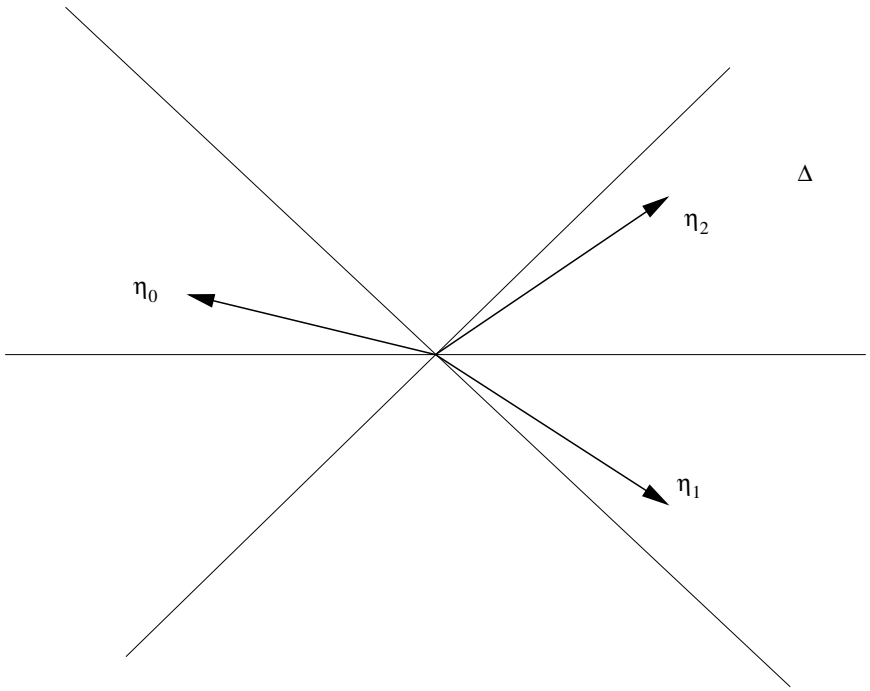

Figure 3: *A chain.*

Then

$$\mu - \nu = -\beta(\nu)\beta^\vee,$$

and

$$\mu - \nu = \sum_{i=1}^{s} -\beta_{i-1}(\nu_i)\beta_i^\vee.$$

Thus

$$\beta = \sum_{i=1}^{s} \frac{\langle \nu_i, \beta_i^\vee \rangle}{\langle \nu, \beta^\vee \rangle} \beta_{i-1}$$

i.e. the simple root β is a positive linear combination of positive roots. It follows that $s = 1$ and $\beta_1 = \beta$. □

3.2 Relative chains

1. Chains relative to a root subsystem.

Let R be a root system on V with the set of simple roots Φ, W be the corresponding Weyl group. Let $\Phi' \subset \Phi$ be a subset, $W' \subset W$ the corresponding reflection subgroup and Δ' the positive chamber for W', defined by the property that all simple roots $\alpha \in \Phi'$ are nonnegative on Δ'. Thus we will be (frequently) considering $(V, W', -\Delta')$-chains rather than $(V, W, -\Delta)$-chains. In this paper we will be using subgroups W' which are stabilizers of points x in (A, W_{aff}) (where $W = W_{sph}$). Then we will think of a relative (V, W')-chain as a chain in the tangent space T_xA of the point x.

Lemma 3.15. *Given a nonzero vector $\eta \in V$ there exists a W'-chain $(\eta = \eta_0, ..., \eta_m)$ with $\eta_m \in \Delta'$, so that the chain (η_i) is maximal with respect to the original root system R.*

Proof: The proof is by induction on the number $r = r(\eta)$ of simple roots in Φ' which are negative on the vector η. If $r(\eta) = 0$ then $\eta \in \Delta'$ and there is nothing to prove. Suppose the assertion holds for all η with $r(\eta) \leq k$. Let η be such that $r(\eta) = k + 1$. Pick a root $\beta \in \Phi'$ such that $\beta(\eta) < 0$. Then for the vector

$$\zeta := \tau_\beta(\eta)$$

we have:

$$\zeta = \eta - \beta(\eta)\beta^\vee.$$

Clearly, $\beta(\zeta) > 0$. Thus the pair (η, ζ) is a W'-chain; this chain is maximal as a W-chain by Lemma 3.14, since it is defined using a simple reflection in W.

For each simple root $\alpha \in \Phi' \setminus \{\beta\}$ which is nonnegative on η

$$\alpha(\zeta) = \alpha(\eta) - \beta(\eta)\alpha(\beta^\vee) \geq \alpha(\eta) \geq 0.$$

Therefore $r(\zeta) < r(\eta)$ and we are done by the induction. □

Lemma 3.16. *Suppose that $W' \subset W$ is a reflection subgroup as above. For any two vectors α, δ the following are equivalent:*

1. *There exists β, γ so that $\alpha \geq_{W'} \beta \sim_W \gamma \geq_{W'} \delta$.*
2. *There exists β so that $\alpha \geq_{W'} \beta \sim_W \delta$.*
3. *There exists γ so that $\alpha \sim_W \gamma \geq_{W'} \delta$.*

Here $\sim$ is the relation from Definition 2.5.

Proof: It is clear that $2 \Rightarrow 1$ and $3 \Rightarrow 1$. We will prove that $1 \Rightarrow 2$, since the remaining implication is similar. We have chains

$$(\alpha = \eta_0, ..., \eta_m = \beta), \quad \eta_i = \tau_i(\eta_{i-1}),\ i = 1, ..., m,$$

$$(\gamma = \eta'_0, ..., \eta'_s = \delta), \quad \eta'_i = \tau'_i(\eta'_{i-1}),\ i = 1, ..., s.$$

Then we can extend the first chain to

$$(\alpha = \eta_0, ..., \eta_m = \beta, \tau'_1(\beta) = \eta_{m+1}, ..., \tau'_s(\eta_{m+s-1}) = \eta_{m+s} =: \epsilon).$$

After discarding equal members of this sequence we obtain a chain from α to ϵ. Since β, γ belong to the same chamber, the vectors ϵ and $\delta = \tau'_s \circ ... \circ \tau'_1(\gamma)$ also belong to the same chamber. Therefore we obtain

$$\alpha \geq \epsilon \sim \delta. \quad \square$$

Definition 3.17. *We will write $\alpha \gtrsim_{W'} \delta$ if one of the equivalent conditions in the above lemma holds. We will frequently omit the subscript W' in this notation when the choice of the subgroup W' is clear or irrelevant.*

The reason for using the relation $\sim$ with respect to W rather than its subgroup W' (as may seem more natural) is that we will be taking limits under which the subgroup W' is increasing (but the limit is still contained in W). Such limits clearly preserve the relation $\gtrsim_{W'}$, but not the relation defined using $\sim_{W'}$.

The next corollary immediately follows from Lemma 3.16:

Corollary 3.18. $\alpha \gtrsim \delta \iff -\delta \gtrsim -\alpha$.

Lemma 3.19. *Suppose that $\nu \gtrsim \mu$ and $\alpha \in \Phi$ is such that $\alpha(\nu) < 0$, $\alpha(\mu) \geq 0$. Then*

$$\tau_\alpha(\nu) \gtrsim \mu.$$

Proof: Let λ be such that

$$\nu \geq \lambda \sim \mu.$$

Then $\alpha(\lambda) \geq 0$ and it follows that $\lambda \neq \nu$, i.e. $\nu > \lambda$. By applying [L2, Lemma 4.3] we get

$$\tau_\alpha(\nu) \geq \lambda \sim \mu. \quad \square$$

2. Chains relative to positive real numbers (a-chains in the sense of Littelmann).

Let a be a positive real number and let $\nu, \mu \in V$ be nonzero vectors in the same W-orbit.

Definition 3.20 (P. Littelmann, [L2]). An a-chain *for (ν, μ) is a chain $(\lambda_0, \ldots, \lambda_s)$ which starts at ν, ends at μ and satisfies*

(i) For each $i > 0$ we have

$$t_i := \beta_i(a\lambda_{i-1}) \in \mathbb{Z},$$

where $\lambda_i = \tau_{\beta_i}(\lambda_{i-1})$ as in the Definition 3.1.

(ii) For each i, dist$(\lambda_{i-1}, \lambda_i) = 1$.

Remark 3.21. *Our root system R is the coroot system for the one considered by Littelmann.*

Our goal is to give this definition a somewhat more geometric interpretation. In particular, we will see that the concept of an a-chain is a special case of the concept of a chain relative to a root subsystem.

The root system R defines an affine Coxeter complex (A, W_{aff}) on A. Let $x \in P(R^\vee)$ be a special vertex; set $x_i := x + a\lambda_i$, $\tau_i := \tau_{\beta_i}$, $i = 0, ..., s$. Thus $t_i = \beta_i(x_{i-1})$. Note that $t_i \in \mathbb{Z}$ iff $\beta_i(x_{i-1}) \in \mathbb{Z}$.

Lemma 3.22. *$\beta_i(x_0) \in \mathbb{Z}$ for each $i = 1, 2, ..., s$.*

Proof: It suffices to consider the case $x = 0$. We have:

$$x_{i-1} = \tau_{i-1}(x_{i-2}) = x_{i-2} - \beta_{i-1}(x_{i-2})\beta_{i-1}^\vee = x_{i-2} - t_{i-1}\beta_{i-1}^\vee = \ldots = x_0 - \sum_{j=1}^{i-1} t_j \beta_j^\vee.$$

Hence

$$t_i = \beta_i(x_{i-1}) = \beta_i(x_0) - \sum_{j=1}^{i-1} t_j \beta_i(\beta_j^\vee).$$

Since $t_i \in \mathbb{Z}$ and $\beta_i(\beta_j^\vee) \in \mathbb{Z}$ for all j, it follows that $\beta_i(x_0) \in \mathbb{Z}$. □

We define the integers $k_i := \beta_i(x_0)$ and the affine walls

$$H_i := H_{\beta_i, k_i} = \{v \in V : \beta_i(v) = k_i\}.$$

The reflection σ_i in the wall H_i belongs to the group W_{aff}, its linear part is the reflection $\tau_i \in W_{sph}$, $i = 1, ..., s$.

The argument in the proof of the above lemma can be easily reversed and hence we get

Corollary 3.23. *The integrality condition (i) is equivalent to the assumption that the point x_0 lies on the intersection of walls H_i of the Euclidean Coxeter complex (A, W_{aff}), where each H_i is parallel to the reflection hyperplane of τ_i. Equivalently, the W-chain $(\lambda_0, ..., \lambda_s)$ is actually a W'-chain, where $W' = W_{x_0}$ is the stabilizer of x_0 in W_{aff}.*

Therefore, identify the vectors λ_i with vectors in the tangent space $V' := T_{x_0}(A)$, let $\Delta' \subset V'$ denote the Weyl chamber of W' which contains the (parallel transport of the) positive chamber Δ. We obtain

Proposition 3.24. *Littelmann's definition of an a-chain is equivalent to the conjunction of*

1. *$(\lambda_0, ..., \lambda_s)$ is a chain in $(V', W', -\Delta')$.*
2. *This chain is maximal as a W-chain.*

Thus the choice of the real number a amounts to choosing a Coxeter subcomplex (V', W') in (V, W_{sph}). The reader will also note the discrepancy between (1) and (2): The chain condition refers to the restricted Coxeter complex (V', W'), while the maximality condition refers to the unrestricted one, (V, W_{sph}). This is the key difference between *LS paths* and *Hecke paths*.

Remark 3.25. *Note that both conditions (1) and (2) are vacuous if x_0 is a special vertex in the Euclidean Coxeter complex (equivalently, if $a\nu$ is a coweight): If*

$$(\lambda_0, \lambda_1, ..., \lambda_s)$$

is a W'-chain, since $W' \cong W_{sph}$, we can subdivide this chain to get a longer chain

$$(\lambda_0 = \lambda'_0, \lambda'_1, ..., \lambda'_{m-1}, \lambda'_m = \lambda_s)$$

between λ_0 and λ_s which satisfies the unit distance condition $dist(\lambda'_i, \lambda'_{i+1}) = 1$ for all i.

3.3 Hecke paths

The goal of this section is to introduce a class of PL paths which satisfy a condition similar to Littelmann's definition of LS (and generalized LS) paths. These paths (*Hecke paths*) play the role in the problem of computing structure constants for spherical Hecke algebras which is analogous to the role that LS paths play in the representation theory of complex semisimple Lie groups.

Let $(A, W_{aff}, -\Delta)$ be a Euclidean Coxeter complex corresponding to a root system R, with fixed negative chamber $-\Delta$. Let $p \in \tilde{\mathcal{P}}$ be a path equal to the composition

$$\overline{x_1 x_2} \cup ... \cup \overline{x_{n-1} x_n}.$$

For each vertex $x = x_i, i = 2, ..., n-1$ we define the unit tangent vectors $\xi, \eta \in S_x$ to the segments $\overline{x_i x_{i-1}}, \overline{x_i x_{i+1}}$.

Definition 3.26. *We say that the path p* satisfies the chain condition *if for each $x = x_i, i = 2., ..., n-1$ there exists a unit vector μ so that*

1.

$$-\xi \geq_{W_x} \mu, \textit{in the spherical Coxeter complex } (S_x, W_x, \Delta).$$

2. $\mu \sim \eta$ in the (unrestricted) spherical Coxeter complex (S, W_{sph}), i.e. for each root $\alpha \in R$ we have

$$\alpha(\mu) \geq 0 \iff \alpha(\eta) \geq 0.$$

In other words, for each $t \in [0, 1]$ we have

$$p'_-(t) \gtrsim_{W_{p(t)}} p'_+(t).$$

Intuitively, at each break-point $p(t)$ the path p "turns towards the positive chamber". In what follows we will use the notation $\mathcal{P}_{chain}$ for the set of all paths $p \in \mathcal{P}$ satisfying the chain condition.

Definition 3.27. *A path $p \in \tilde{\mathcal{P}}$ is called a* Hecke path *if it is a billiard path which satisfies the chain condition, i.e. for each t*

$$p'_-(t) \geq_{W_{p(t)}} p'_+(t).$$

Below is example of a class of Hecke paths. Suppose that $p \in \mathcal{P}$ and for each $t \in [0, 1]$ either p is smooth at t or there exists a reflection $\tau \in W_{p(t)}$ so that the derivative of τ equals τ_β, $\beta \in R^+$ and

1. $d\tau_{p(t)}(p'_-(t)) = p'_+(t)$.
2. $\beta(p'_-(t)) < 0$, $\beta(p'_+(t)) > 0$.

Then p is a Hecke path with the length of each chain in $(S_{p(t)}, W_{p(t)})$ equal 0 or 1. See Figure 4.

Definition 3.28. *We say that p satisfies the* simple chain *condition if at each breakpoint $x = p(t)$ the chain can be chosen to be simple, i.e. of length* 1.

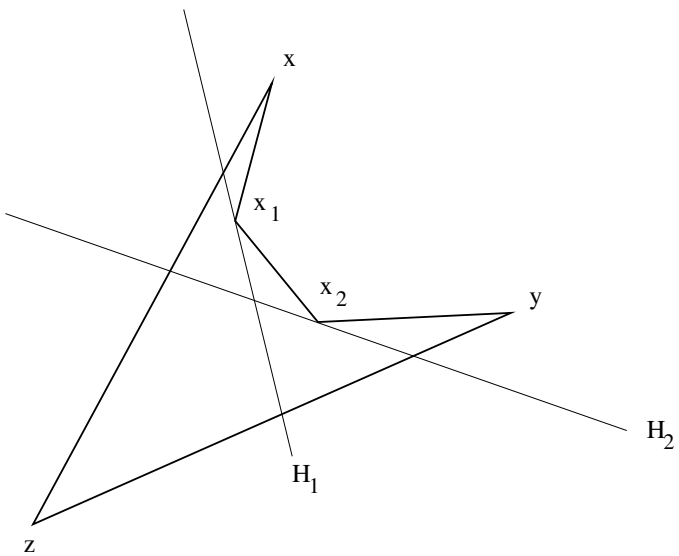

Figure 4: *A billiard path satisfying the simple chain condition.*

In what follows we will also need

Definition 3.29. *Suppose that p is a path satisfying the chain condition. We call p a* generalized Hecke path *if each geodesic segment in p (regarded as a vector) is an integer multiple of some $w\varpi$ where $w \in W$ and $\varpi = \varpi_i$ is one of the fundamental coweights.*

3.4 A compactness theorem

Pick $\epsilon > 0$. We define the subset $\mathcal{P}_{m,\epsilon} \subset \mathcal{P}$ consisting of paths p with

$$\underline{\text{length}}_\Delta(p) = \underline{\lambda} = (\lambda_1, ..., \lambda_m)$$

so that for each i,

$$\epsilon \leq |\lambda_i| \leq \epsilon^{-1}. \tag{4}$$

Theorem 3.30. *For each $\epsilon > 0$ the set $\mathcal{P}_{chain,m,\epsilon} := \mathcal{P}_{chain} \cap \mathcal{P}_{m,\epsilon}$ is compact in $\mathcal{P}_{m,\epsilon}$.*

Proof: Suppose that $p \in \mathcal{P}_{chain}$ is a concatenation of m billiard paths p_i. Then the number of breaks in the broken geodesic p_i is bounded from above by a constant

c equal to the length of a maximal chain in the Bruhat order of the finite Weyl group W_{sph}, see Corollary 3.8. This immediately implies that the subset $\mathcal{P}_{chain,m,\epsilon}$ is precompact in $\mathcal{P}$. What has to be proven is that this subset is closed.

Let $\lambda_{ij} \in V$ be vectors of nonzero length so that

$$p_i = \pi_{\lambda_{i1}} * \pi_{\lambda_{i2}} * ... * \pi_{\lambda_{in}}$$

are Hecke paths, $i = 1, 2, ...$ We suppose that

$$\lim_{i\to\infty} \lambda_{ij} = 0, j = 2, ..., n-1$$

and

$$\lim_{i\to\infty} \lambda_{i1} = \lambda_{1\infty}, \lim_{i\to\infty} \lambda_{in} = \lambda_{\infty,n}$$

are nonzero vectors. It is clear that

$$\lim_i p_i = p_\infty := \pi_{\lambda_{\infty,n}} * \pi_{\lambda_{\infty,n}}.$$

Lemma 3.31. *Under the above conditions the path p_∞ is again a Hecke path.*

Proof: Let $x := \lambda_{\infty,n}$. We need to check that the unit vectors $\bar{\lambda}_{\infty,n}, \bar{\lambda}_{\infty,n}$ satisfy

$$\bar{\lambda}_{\infty,n} \geq_{W_x} \bar{\lambda}_{\infty,n}.$$

Here and below, W_x is the stabilizer of $x \in V$ in the Coxeter group W_{aff}.

Let x_{ij} denote the break-point of p_i which is the concatenation point between $\lambda_{i,j}$ and $\lambda_{i,j+1}$. Then

$$\bar{\lambda}_{i,j} \geq_{W_{x_{ij}}} \bar{\lambda}_{i,j+1},$$

$\lim_i x_{ij} = x$. If $\sigma_{ij} \in W_{aff}$ is a reflection fixing x_{ij}, then, up to a subsequence,

$$\sigma_{\infty,j} = \lim_i \sigma_{ij} \in W_{aff}$$

fixes the point x. Therefore it follows from the definition of a chain that

$$\bar{\lambda}_{\infty,j} \geq_{W_x} \bar{\lambda}_{\infty,j+1}$$

for each j. By putting these inequalities together we obtain

$$\bar{\lambda}_{\infty,1} \geq_{W_x} \bar{\lambda}_{\infty,n}. \quad \square$$

Suppose now that p_i, q_i are Hecke paths,

$$p_i = \pi_{\lambda_{i1}} * \pi_{\lambda_{i2}} * ... * \pi_{\lambda_{in}}, q_i = \pi_{\mu_{i1}} * \pi_{\mu_{i2}} * ... * \pi_{\mu_{im}}$$

and the concatenation $r_i := p_i * q_i$ satisfies the chain condition. We suppose that $r_\infty = p_\infty * q_\infty$ is the limit of the sequence of paths r_i and the paths p_∞, q_∞ are not constant.

Lemma 3.32. *Under the above assumptions, the path r_∞ also satisfies the chain condition.*

Proof: Lemma 3.31 implies that the paths p_∞, q_∞ are Hecke. Therefore it suffices to verify the chain condition at the concatenation point $x = p_\infty(1)$. Up to passing to a subsequence, we have:

$$\lim_{i\to\infty} \lambda_{ij} = \lambda_{\infty,j}, \lim_{i\to\infty} \mu_{ij} = \mu_{\infty,j},$$
$$\lim_{i\to\infty} \bar{\lambda}_{ij} = \bar{\lambda}_{\infty,j}, \lim_{i\to\infty} \bar{\mu}_{ij} = \bar{\mu}_{\infty,j}.$$

Suppose that

$$\lim_i \lambda_{ij} = 0, j = k+1, ..., n, \quad \lim_i \mu_{ij} = 0, j = 1, ..., l,$$

and

$$\lambda_{\infty,k} \neq 0, \quad \mu_{\infty,l+1} \neq 0.$$

By Lemma 3.31,

$$\bar{\lambda}_{\infty,k} \geq_{W_x} \bar{\lambda}_{\infty,n}, \quad \bar{\mu}_{\infty,1} \geq_{W_x} \bar{\mu}_{\infty,l+1}.$$

On the other hand, it is clear that

$$\bar{\lambda}_{\infty,n} \gtrsim_{W_x} \bar{\mu}_{\infty,1}.$$

Therefore, by Lemma 3.16, we get

$$\bar{\lambda}_{\infty,k} \gtrsim_{W_x} \bar{\mu}_{\infty,l+1}. \quad \square$$

We now can finish the proof Theorem 3.30. Suppose that $p_i = p_{i1} * ... * p_{im}$ is a sequence of paths in $\mathcal{P}_{chain,m,\epsilon}$, where each p_{ij} is a billiard path, and p_∞ is the limit of this sequence. Each sequence of billiard paths $(p_{ij})_{i\in\mathbb{N}}$ converges to a billiard path $p_{\infty,j}$ which is a Hecke path according to Lemma 3.31. Consider now a concatenation point x of the subpaths $p_\infty := p_{\infty,j}, q_\infty := p_{\infty,j+1}$. The paths p_∞, q_∞ are nonconstant by the inequality (4). Therefore we can apply Lemma 3.32 to conclude that the path p_∞ satisfies chain condition at the point x. $\square$

Remark 3.33. *It is easy to see that the assumption that the length of each billiard subpath p_{ij} is bounded away from zero, is necessary in Theorem 3.30.*

4 Folding

The key tool for proving the main results of this paper is *folding* of polygons in a building X into apartments and Weyl chambers. The folding construction replaces a geodesic segment $\tilde{p}$ in X with a PL path p in an apartment. This construction was used in [KLM3] to construct various counter-examples. The reader will note that the folding construction used in the present paper is somewhat different from the one in [KLM3].

4.1 Folding via retraction

Suppose that X is a Euclidean or spherical building modeled on the Coxeter complex (A, W), we identify the model apartment A with an apartment $A \subset X$; let $a \subset A$ be an alcove (or a chamber in the spherical case). Recall that the *retraction, or folding, to an apartment* $f = Fold_{a,A} : X \to A$ is defined as follows (see for instance [Rou]):

Given a point $x \in X$ choose an apartment A_x containing x and a. Then there exists a (unique) isomorphism $\phi : A_x \to A$ fixing $A \cap A_x$ pointwise and therefore fixing a as well. We let $f(x) := \phi(x)$. It is easy to see that $f(x)$ does not depend on the choice of A_x. Observe that f is an isometry on each geodesic $\overline{xy}$, where $y \in a$.

The retraction can be generalized as follows.

Suppose that X is a Euclidean building, a is an alcove with a vertex v (not necessarily special). Let $\Delta \subset A$ denote a Weyl chamber with the tip o. Choose a dilation $h \in Dil(A, W)$ which sends v to o. Let $\mathbb{P} : A \to \Delta$ denote the natural projection which sends points $x \in A$ to $W_{sph} \cdot x \cap \Delta$, where W_{sph} is the stabilizer of o in W_{aff}.

We define a folding $g = Fold_{v,h,\Delta} : X \to \Delta$ as the composition

$$\mathbb{P} \circ h \circ Fold_{a,A}.$$

The mapping g will be called a *folding* into a Weyl chamber. Observe that g (unlike $Fold_{a,A}$) does not depend upon the choice of the alcove a, therefore it will be denoted in what follows $g = Fold_{v,h,\Delta}$. Note that

$$Fold_{v,k\circ h,\Delta} = k \circ Fold_{v,h,\Delta}.$$

In case when $h = id$ we will abbreviate $Fold_{v,h,\Delta}$ to $Fold_{\Delta}$.

Remark 4.1. *The folding maps are Lipschitz and differentiable. The restriction of the retraction $Fold_{a,A}$ to each chamber (alcove) is a congruence of two chambers (alcoves).*

If $h = id$ (and thus $v = o$) one can describe $f = Fold_{\Delta}$ as follows. Given a point $x \in X$ find an apartment A_x through v, x and a Weyl chamber $\Delta_x \subset A_x$ with the tip v. Let $\phi : \Delta_x \to \Delta$ be the unique isometry extending to an isomorphism of Coxeter complexes $A_x \to A$. Then $f(x) = \phi(x)$.

Suppose now that X is a Euclidean building, $x \in X$; we give the link $\Sigma_x(X)$ structure of an unrestricted spherical building Y. Let R denote the corresponding root system. Let δ be a chamber in Y and $\xi, \mu \in \delta$. Let $f : X \to \Delta$ be a folding of X to a Weyl chamber. Let x', ξ', μ' denote the images of x, ξ, μ under f and df_x. Then

Lemma 4.2. *1. $d_{ref}(\xi, \mu) = d_{ref}(\xi', \mu')$.*

2. For each $\alpha \in R$,

$$\alpha(\mu') \geq 0 \iff \alpha(\eta') \geq 0.$$

Proof: The restriction $df|_\delta$ is an isometry which is the restriction of an isomorphism of spherical apartments. This proves (1). To prove (2) observe that df sends δ to a spherical Weyl chamber δ' in $\Sigma_{x'}X$. $\square$

Let f be a folding of X into an apartment or a chamber.

Lemma 4.3. *For each geodesic segment $\overline{xy} \subset X$ its image $f(\overline{xy})$ is a broken geodesic, i.e. it is a concatenation of geodesic segments.*

Proof: We give a proof in the case of a folding into an apartment and will leave the other case to the reader. Let $A' \subset X$ denote an apartment containing the geodesic segment $\overline{xy}$. Let $a_1, ..., a_m$ denote the alcoves (or chambers in the spherical case) in A' covering $\overline{xy}$, set $\overline{x_i x_{i+1}} := \overline{xy} \cap a_i$. For each a_i there exists an apartment A_i containing the alcoves a and a_i. The restriction of the retraction f to A_i is an isometry. It is now clear that the path $f(\overline{xy})$ is a composition of the geodesic paths $f(\overline{x_i x_{i+1}})$. $\square$

We let $x'_i = f(x_i)$ denote the break points of $f(\overline{xy})$. For each x'_i let ξ'_i, η'_i denote the unit tangent vectors in $T_{x'_i}A$ which are tangent to the segments $\overline{x'_i x'_{i-1}}, \overline{x'_i x'_{i+1}}$ respectively.

Lemma 4.4. *The broken geodesic $f(\overline{xy})$ is a* billiard path*, i.e. for each break point x'_i the vectors $\xi' := \xi'_i, \eta' := \eta'_i$ satisfy*

$$\exists w \in W_{x'_i} : w(\xi') = -\eta'.$$

Proof: We again present a proof only in the case of a folding into an apartment. Let $Y = \Sigma_{x_i}X$ denote the spherical building which is the space of directions of X at x_i. Let Δ_Y denote the Weyl chamber of this building and $\theta : Y \to \Delta_Y$ the canonical projection. The directions $\xi = \xi_i$ and $\eta = \eta_i$ of the segments $\overline{x_i x_{i-1}}, \overline{x_i x_{i+1}}$ are antipodal in the building Y. Since the folding f is an isomorphism of the apartments $A_i \to A, A_{i+1} \to A$, and

$$df(\eta) = \eta', \quad df(\xi) = \xi',$$

we see that

$$\theta(\xi) = \theta(\xi'), \quad \theta(\eta) = \theta(\eta').$$

The assertion now follows from Lemma 2.14, part (2). $\square$

Lemma 4.5. *Suppose that X is a Euclidean building, $f = Fold_{a,A} : X \to A$ and $g = Fold_{z,h,\Delta}$ are foldings to an apartment and a chamber respectively. Then for each PL path p in X we have:*

1. $\underline{\text{length}}_\Delta(f(p)) = \underline{\text{length}}_\Delta(p)$.

2. $\underline{\text{length}}_\Delta(g(p)) = k \cdot \underline{\text{length}}_\Delta(p)$*, where $k > 0$ is the conformal factor of the dilation h.*

Proof: We will prove the first assertion since the second assertion is similar. It suffices to give a proof in the case when p is a billiard path. Then, analogously to the proof of Lemma 4.4, there exists a representation of p as a composition of geodesic subpaths

$$p = p_1 \cup ... \cup p_m$$

so that the restriction of f to each p_i is a congruence. Therefore

$$\text{length}_\Delta(p_i) = \text{length}_\Delta(f(p_i))$$

and hence

$$\text{length}_\Delta(p) = \sum_i \text{length}_\Delta(p_i) = \text{length}_\Delta(f(p)). \quad \square$$

Derivative of the retraction. We assume that $rank(X) \geq 1$. We identify the model apartment A with an apartment in X. Pick $a \subset A$ which is an alcove (in the Euclidean case) or a chamber (in the spherical case). Given a point $x' \in X$ choose an apartment (A', W') through a and x' and let $\phi : A \to A'$ denote the inverse to the retraction $f = Fold_{a,A} : A' \to A$. Set $x = f(x')$ and let W'_x denote the stabilizer of x' in W'. Then the link $Y = \Sigma_{x'}(X)$ has a natural structure of a thick spherical building modeled on (S, W'_x). It is easy to see that (S, W'_x) is independent of the choice of A'. Observe that if x' is antipodal to a regular point $y \in a$ then x' is regular itself and therefore $W'_x = \{1\}$. We next define a chamber $s \subset S$:

Given a regular point $y \in a \setminus \{x'\}$ and a geodesic segment $\overline{x'y}$, let $\zeta = \zeta(y)$ denote the unit tangent vector to $\overline{x'y}$ at x'. Then the set

$$\{\zeta(y) : y \text{ is a regular point in } a\},$$

is contained in a unique spherical chamber $s \subset S$. (If x' is antipodal to some $y \in int(a)$ then $s = S$.)

Set $f' := \phi \circ f = Fold_{a,A'}$.

Lemma 4.6. *The derivative $d_{x'}f' : Y \to S$ equals $Fold_{s,S}$.*

Proof: Given $\eta \in Y$, find an alcove (or a spherical chamber) c so that $\eta \in \Sigma_{x'}c$. Then there exists an apartment $A_\eta \subset X$ containing both a and c. Let S_η denote the unit tangent sphere of A_η at x'. Then $\eta \in S_\eta$ and $s \subset S_\eta$. Now it is clear from the definition that

$$df'(\eta) = Fold_{s,S}(\eta)$$

since both maps send S_η to S and fix s pointwise. $\square$

Folding of polygons. Suppose now that X is a building and $\tilde{P} = [\tilde{z}, \tilde{x}_1, \ldots, \tilde{x}_n]$ is a geodesic polygon in X. Pick an apartment $A \subset X$ which contains $\overline{\tilde{z}\tilde{x}_1}$ and an alcove $a \subset A$ which contains $\tilde{z}$. Let $\Delta \subset A$ denote a Weyl chamber (in case X is Euclidean) with the tip o. Let f be a folding of X of the form

$$f = Fold_{a,A}$$

or

$$f = Fold_{\tilde{z},h,\Delta},$$

where h is a dilation sending $\tilde{z}$ to the tip of Δ. We will then apply f to $\tilde{P}$ to obtain a *folded polygon* $P := f(\tilde{P})$ in A or Δ respectively.

Observe that the restriction of f to the edges $\overline{\tilde{z}\tilde{x}_1}$ and $\overline{\tilde{x}_n\tilde{z}}$ of P is an isometry or a similarity. The restriction of f to the path

$$\tilde{p} = \overline{\tilde{x}_1\tilde{x}_2} \cup ... \cup \overline{\tilde{x}_{n-1}\tilde{x}_n}$$

preserves the *type* of the unit tangent vectors, cf. Lemma 4.5. We will be using foldings into apartments and chambers to transform geodesic polygons in X into *folded polygons.*

In the special case when $\tilde{P} = T$ is a triangle (and thus $n = 2$), the folded triangle $P = f(T)$ has two geodesic sides $\overline{zw_1} := f(\overline{\tilde{z}\tilde{x}_1}), \overline{x_2 z} := f(\overline{\tilde{x}_2\tilde{z}})$ and one *broken side* $p := f(\overline{\tilde{x}_1\tilde{x}_2})$, so we will think of $f(T)$ as a *broken triangle.*

The next proposition relates folding into a Weyl chamber with the concept of folding of polygons used in [KLM3]. Let $P = [o, x_1, x_2, \dots, x_n]$ be a polygon in Δ. Triangulate P from the vertex o into geodesic triangles $T_i = [x_i, x_{i+1}, o]$. Suppose that $\tilde{P} \subset X$ is a geodesic polygon

$$\tilde{P} = [o, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n], \tilde{x}_1 = x_1,$$

triangulated into geodesic triangles $\tilde{T}_i = [\tilde{x}_i, \tilde{x}_{i+1}, o]$, where each $\tilde{T}_i$ is contained in an apartment A_i. Assume that for each i there exists a congruence

$$\phi_i : \tilde{T}_i \to T_i$$

i.e. an isometry sending $\tilde{x}_j$ to x_j $(j = i, i+1)$ which extends to an isomorphism of Coxeter complexes $\phi_i : A_i \to A$.

Proposition 4.7. *Under the above assumptions, for each i, $Fold_\Delta|\tilde{T}_i = \phi_i|\tilde{T}_i$.*

Proof: Let $\Delta_i \subset A_i$ denote the preimage of Δ under ϕ_i. Then each Δ_i is a Weyl chamber, hence $\phi_i|\Delta_i = Fold_\Delta|\Delta_i$, by the alternative description of $Fold_\Delta$ given earlier in this section. □

The following lemma shows that unfolding of polygons is a *local problem.* Suppose that $T = [o, x_1, ..., x_n] \subset A$ is a geodesic polygon so that $x_i \neq o$ for each i. For each $i = 2, ..., n-1$ we define the unit vectors

$$\xi_i, \eta_i, \zeta_i \in \Sigma_{x_i} A$$

which are tangent to the segments $\overline{x_i x_{i-1}}, \overline{x_i x_{i+1}}, \overline{x_i o}$. Define thick spherical buildings $Y_i := \Sigma_{x_i}(X)$. By combining the above proposition with [KLM3, Condition 7.5] we obtain

Lemma 4.8. *The polygon T can be unfolded in X to a geodesic triangle $\tilde{T}$ whose vertices project to o, x_1, x_n if and only if for each $i = 2, ..., n-1$, there exists a triangle $[\tilde{\xi}_i, \tilde{\zeta}_i, \tilde{\eta}_i] \subset Y_i$ so that*

$$d_{ref}(\tilde{\xi}_i, \tilde{\zeta}_i) = d_{ref}(\xi_i, \zeta_i),$$

$$d_{ref}(\tilde{\eta}_i, \tilde{\zeta}_i) = d_{ref}(\eta_i, \zeta_i),$$

$$d(\tilde{\xi}_i, \tilde{\eta}_i) = \pi.$$

We will eventually obtain a characterization of the broken triangles in Δ which are foldings of geodesic triangles in X as *billiard triangles satisfying the chain condition*, see section 6. The goal of the next section is to give a *necessary* condition for a broken triangle to be unfolded; we also give a partial converse to this result.

4.2 Converting folded triangles in spherical buildings into chains

Suppose that X is a (thick) spherical or Euclidean building modeled on (A, W). Consider a triangle $\tilde{T} = [x, y, z] \subset X$ with

$$\beta := d_{ref}(x, y), \quad \gamma := d_{ref}(y, z).$$

Assume that A is embedded in X so that it contains x and z. Let $a \subset A$ be a spherical chamber or a Euclidean alcove containing z. In the spherical case we regard a as the *negative chamber* in A, let Δ denote the positive chamber $-a$. We have the retraction $f := Fold_{a,A} : X \to A$.

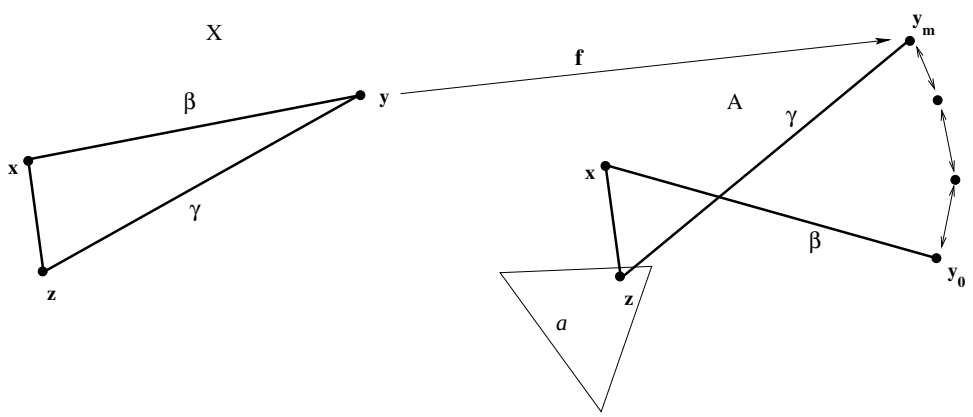

Figure 5: *Converting geodesic triangle to a chain.*

Theorem 4.9. *There exists a (A, W, a)–chain $(y_0, ..., y_m)$ such that $y_m = f(y)$, $d_{ref}(x, y_0) = \beta$, $d_{ref}(y_m, z) = \gamma$. (In the case when X is a Euclidean building the above chain is an affine chain.) See Figure 5.*

Proof: We prove the assertion for the spherical buildings as the Euclidean case is completely analogous. (This is also the only case when this theorem is used in the present paper.)

Our proof is by induction on the rank of the building. Consider first the case when $\text{rank}(X) = 0$ (i.e. $A = S^0$ is the 2-point set). If y and x are both distinct from z, then $f(y) \neq z$. This implies that $f(y) = x$ and we take

$$y_0 := z, y_1 := y, m = 1.$$

In the remaining cases we will use the chain $y_0 = f(y) = y_m$.

Suppose now that $\mathrm{rank}(X) = r \geq 1$ and the assertion holds for all (spherical) buildings of rank $r-1$, let's prove it for buildings of rank r.

We let $\tilde{p} : [0, c] \to \overline{xy}$ denote the unit speed parametrization of $\overline{xy}$ and set $p := f(\tilde{p})$. *We assume for now that* $z \notin \overline{xy}$.

As in the proof of Lemma 4.3, we "triangulate" the geodesic triangle $\tilde{T}$ into geodesic triangles $\tilde{T}_i := [z, \tilde{x}_i, \tilde{x}_{i+1}]$, where the points $\tilde{x}_i = \tilde{p}(t_i)$, $i = 1, ..., n$, are chosen so that each triangle $\tilde{T}_i$ is contained in an apartment $A_i \subset X$ and the map f restricts to an isometry $f : \tilde{T}_i \to f(\tilde{T}_i) \subset A$. Here $\tilde{x}_0 := x, \tilde{x}_{n+1} = y$. Observe that each side of $\tilde{T}_i$ has positive length.

Let $\tilde{S}_i := \Sigma_{\tilde{x}_i}(A_i)$ denote the unit tangent sphere at $\tilde{x}_i$. Define $\tilde{s}_i$ to be the (unique) chamber in $\tilde{S}_i$ containing all the directions of the geodesic segments from $\tilde{x}_i$ to the interior of a. This determines the *positive chamber* $\tilde{\Delta}_i = -\tilde{s}_i \subset S_i$.

Set

$$\tilde{\eta}_i := \tilde{p}'(t_i), \tilde{\xi}_i := -\eta_i,$$

and let $\tilde{\zeta}_i \in \tilde{S}_i$ denote the unit tangent vector to $\overline{\tilde{x}_i z}$.

Now, applying the retraction f to all this data, we obtain:

1. The folded triangle $T = f(\tilde{T})$ which has two geodesic sides $\overline{zx}, \overline{zy_m}$ (where $y_m = f(y)$), and the broken side represented by the path $p = f(\tilde{p})$. In particular, $d_{ref}(y_m, z) = d_{ref}(y, z)$ (as required by the theorem).

2. The vertices $x_i = p(t_i) = f(\tilde{x}_i)$ of the broken geodesic p.

3. Unit tangent vectors $\xi_i = df(\tilde{\xi}_i), \eta_i = df(\tilde{\eta}_i), \zeta_i = df(\tilde{\zeta}_i)$ in $\Sigma_x A$. These vectors are tangent to the segments $\overline{x_i x_{i-1}}, \overline{x_i x_{i+1}}, \overline{x_i z}$ respectively.

4. The positive chamber $\Delta_i = df(\tilde{\Delta}_i)$ and the negative chamber $s_i = df(\tilde{s}_i)$ in the spherical Coxeter complex $(S_i = \Sigma_{x_i}(A), W_i = W_{x_i})$. The negative chamber contains the directions tangent to the geodesic segments from x_i to the chamber $a \subset A$.

Our goal is to convert the broken side p of T into a chain in A by "unbending" the broken geodesic p to a geodesic segment in A. See Figure 6.

Lemma 4.10. *The path p satisfies the following:*

1. *The metric lengths of p and $\tilde{p} = \overline{xy}$ are the same.*
2. $p'(0) = \tilde{p}'(0)$.
3. *At each break-point x_i there exists an (S_i, W_i, s_i)-chain from $-\xi_i$ to η_i.*

Proof: The first two assertions are clear from the construction. Let's prove the last statement. For each i and the point $\tilde{v} = \tilde{x}_i$ we have the spherical building $Y := \Sigma_{\tilde{v}}(X)$ which has rank $r-1$. This building contains the antipodal points

$$\tilde{\xi}_i, \tilde{\eta}_i$$

and the point $\tilde{\zeta}_i$. We form the geodesic triangle $\tau = [\tilde{\xi}_i, \tilde{\eta}_i, \tilde{\zeta}_i] \subset Y$, where we use an arbitrary shortest geodesic in Y to connect ξ_i to η_i. Therefore ξ_i, η_i, ζ_i are vertices of the broken geodesic triangle $df(\tau) \subset S_i$.

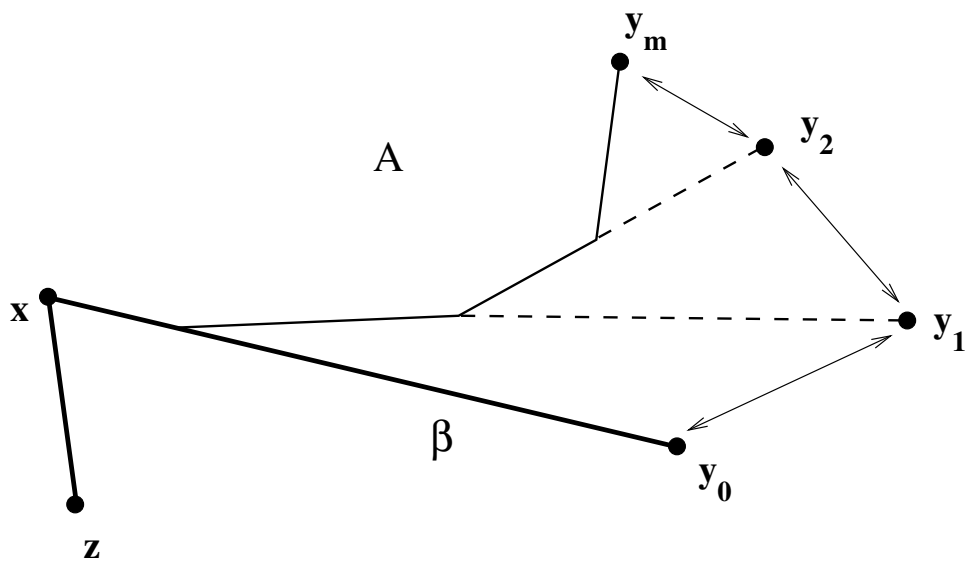

Figure 6: *Forming a chain by unbending.*

As in Lemma 4.6, we use the isomorphism $S_i \to \tilde{S}_i$ (sending s_i to $\tilde{s}_i$) to identify these apartments. Under this identification, $df : Y \to S_i$ is the retraction $Fold_{s_i,S_i}$ of Y to the apartment S_i. Thus $df(\tau)$ is a folded triangle in S_i.

Hence, by the (rank) induction hypothesis, for each i there exists a chain

$$(-\xi_i, ..., \eta_i)$$

in the spherical Coxeter complex (S_i, W_i, s_i). □

Lemma 4.11. *For each path $p : [0, c] \to A$ satisfying the conclusion of Lemma 4.10, there exists a point $y' \in A$ such that*

$$d_{ref}(x, y') = \beta$$

and

$$y' \geq p(c)$$

in $(A, W, -\Delta)$.

Proof: We use the second induction, on the number n of vertices in the broken geodesic p. Set $u := p(c)$.

The metric length of the path p equals the metric length of the path $\tilde{p} = \overline{xy}$, the tangent directions of these paths at x are the same. Therefore, if $n = 0$ (and hence the path p is geodesic) there is nothing to prove, one can simply take $y' = u$.

Assume that the assertion holds for all $n \leq N - 1$, let's prove it for N. We treat the path p as the composition

$$p|[0, t_N] \cup \overline{x_N u}$$

Our goal is to replace the geodesic subpath $\overline{x_N u}$ with a geodesic path $w(\overline{x_N u})$, where $w \in W$ is fixing x_N, so that:

1. $\overline{x_{N-1}x_N} \cup w(\overline{x_N u})$ is a geodesic segment.

2. There exists an $(A, W, -\Delta)$-chain between $w(u)$ and u. Then we would be done by the induction on n. Indeed, the new path

$$p|[0, t_N] \cup w(\overline{x_N u})$$

has one less break-point and still satisfies the conclusion of Lemma 4.10. Thus, by the induction hypothesis, there exists $y' \in A$ so that

$$y' \geq w(u) \geq u \Rightarrow y' \geq u,$$

$$d_{ref}(x, y') = d_{ref}(x, y).$$

Construction of w. Recall that there exists an $(S_N, W_N, -\Delta_N)$-chain

$$(-\xi_N = \nu_0, ..., \nu_k = \eta_N),$$

hence we have a sequence of reflections $r_1, ..., r_k \in W_N$ (fixing walls $H_i \subset S_N$, $i = 1, ..., k$) so that:

$$r_i(\nu_{i-1}) = \nu_i, i = 1, ..., k,$$

and each wall H_i separates ν_i from the negative chamber s_N. We extend each reflection r_i from S_N to a reflection r_i in A, and each H_i to a wall H_i in A.

We therefore define the following points in A:

$$y_k := u, y_{k-1} := r_k(y_k), y_{k-2} := r_{k-1}(y_{k-1}), \ldots, y_0 := r_1(y_1).$$

Note that the directions ν_i are tangent to the segments $\overline{x_N y_i}$. Thus for each i, the wall H_i separates the point y_i from the negative chamber $a \subset A$ and the sequence

$$(y_0, ..., y_k = u)$$

forms a chain. We set $w = r_1 \circ ... \circ r_k$. The vector ν_0 is antipodal to $-\xi_N$, hence the path

$$p|[0, T_N] \cup w(\overline{x_N u})$$

is geodesic at the point $x_N = p(T_N)$. □

This concludes the proof of Theorem in the case when $z \notin \overline{xy}$.

We now consider the special case when the above proof has to be modified: The triangle T is degenerate, i.e. $z \in \overline{xy}$, but the alcove a is such that $y \notin A$. Thus the folding $Fold_{a,A}$ is not an isometry on T. Then the tangent direction $\tilde{\zeta}_i$ is not defined when $x_i = z$. Note that $x_i = z$ then is the only break-point in the broken side of T'.

In this case we replace the vertex z with an arbitrary point z' in the interior of a and repeat the above arguments. The chains constructed in the process will be independent of the choice of z' and thus, after taking the limit $z' \to z$, we obtain a chain as required by the assertion of Theorem. □

Corollary 4.12 (Cf. [KLM3], Theorem 8.2, Part 4). *Suppose that X is a Euclidean building. Assume that $\alpha := d_\Delta(z,x), \beta := d_\Delta(x,y), \gamma := d_\Delta(y,z)$ are in $P(R^\vee)$ and x, y, z are special vertices of X. Then*

$$\alpha + \beta + \gamma \in Q(R^\vee).$$

Proof: Let $(y_0, ..., y_m)$ be an affine chain given by Theorem 4.9. We regard the point x as the origin o in A; thus we will regard z, y_0, y_m as vectors in V. Then, according to Lemma 2.7,

$$y_m - y_0 \in Q(R^\vee).$$

Consider the vectors $\beta' := y_0 - x, \gamma' := z - y_m, \alpha' := x - z$ in $P(R^\vee)$. By the definition of Δ-length,

$$\alpha' \in W_{sph}\alpha, \quad \beta' \in W_{sph}\beta, \quad \gamma' \in W_{sph}\gamma.$$

Therefore, by applying Lemma 2.7 again we see that the differences

$$\alpha - \alpha', \quad \beta - \beta', \quad \gamma - \gamma'$$

all belong to $Q(R^\vee)$. Since

$$\alpha' + \beta' + \gamma' = y_0 - y_m \in Q(R^\vee),$$

the assertion of lemma follows. □

The following simple proposition establishes a partial converse to Theorem 4.9:

Proposition 4.13. *Suppose that X is a thick spherical building and, as before, the point z belongs to a negative chamber $a = -\Delta$. Then, for each simple chain (y_0, y_1) such that $d_{ref}(x, y_0) = \pi, d_{ref}(z, y_1) = \beta$, there exists a point $y \in X$ so that*

$$d_{ref}(y, z) = \gamma \quad \text{and} \quad d_{ref}(x, y) = \pi.$$

Remark 4.14. *Recall that $d_{ref}(x, y) = \pi$ means that the points x and y are antipodal.*

Proof: Let $\tau(y_0) = y_1$ where τ is a reflection in a wall $H \subset A$ as in the definition of a chain. Let $A = A^- \cup A^+$ be the union of half-apartments, where A^- is bounded by H and contains a. By the definition of a chain, $y_1 \in A^-, y_0 \in A^+$ and hence the antipodal point $x = -y_0$ belongs to A^+.

Since X is thick, there exists a half-apartment $B^- \subset X$ which intersects A along H. Define the apartment $B := A^- \cup B^-$; then there exists an isomorphism of Coxeter complexes

$$\phi : A \to B, \phi|A^- = id.$$

We set $y := \phi(y_0)$.

Since ϕ is an isomorphism of Coxeter complexes which fixes z, it preserves the refined distance to the point z and hence

$$d_{ref}(y, z) = d_{ref}(y_1, z) = \gamma.$$

The union $C := A^+ \cup B^-$ is also an apartment in X. Then there exists an isomorphism $\psi : B^- \to A^-$ so that

$$\psi \circ \phi | A^+ = \tau | A^+.$$

The isomorphism ψ extends to an isomorphism $\rho : C \to A$ fixing A^+ pointwise and hence fixing the point x. Therefore

$$d_{ref}(x, y) = d_{ref}(x, y_0) = \pi. \quad \square$$

4.3 Folding polygons in Euclidean buildings

Our next goal is to show that each folding transforms certain PL paths in Euclidean buildings to paths satisfying the chain condition.

Suppose that X is a Euclidean building with model apartment (A, W_{aff}), $\Delta \subset A$ is the positive Weyl chamber with the tip o. Consider a PL path $\tilde{p} : [0, c] \to \tilde{A}$, which is parameterized with the unit speed, where $\tilde{A} \subset X$ is an apartment. We assume that for each $t \in [0, c]$

$$\tilde{p}'_-(t) \sim_{W_{sph}} \tilde{p}'_+(t),$$

for instance, $\tilde{p}$ could be a geodesic path.

Thus the path $\tilde{p}$ *trivially satisfies the chain condition.* Let $g : X \to \Delta$ be a folding into Δ, $g = Fold_{z,h,\Delta}$ for a certain $z \in A$ and h. Recall that the folding g is the composition of three maps:

$$g = \mathbb{P}_\Delta \circ h \circ f, \quad f = Fold_{a,A},$$

where a is an alcove in A containing z, $h \in Dil(A, W_{aff})$ is a dilation sending z to the point o. Consider the structure of a Coxeter complex on A given by the pull-back

$$h^*(A, W_{aff}).$$

We thus get a new (typically non-thick) building structure for X, the one modeled on $h^*(A, W_{aff})$.

Definition 4.15. *We say that a path $\tilde{p}$ is* generic *if it is disjoint from z and from the codimension 2 skeleton of X and the break-points of $\tilde{p}$ are disjoint from the codimension 1 skeleton of X, where X is regarded as a building modeled on $h^*(A, W_{aff})$.*

The main result of this section is

Theorem 4.16. *The folded path $p = g(\tilde{p})$ satisfies the chain condition.*

Proof: The proof of this theorem is mostly similar (except for the projection $\mathbb{P}$ which causes extra complications) to the proof of Theorem 4.9 in the previous section. We will prove Theorem 4.16 in two steps: We first establish it for the paths $\tilde{p}$ which are *generic*. Then we use the *compactness theorem* to prove it in general.

Proposition 4.17. *The conclusion of Theorem 4.16 holds for* generic *paths $\tilde{p}$.*

Proof: If a point $\tilde{x} = \tilde{p}(t)$ is a regular point of X, then

$$dg_{\tilde{x}} : \Sigma_{\tilde{x}}(X) \to \Sigma_x(A), \quad x = g(\tilde{x})$$

is an isometry. Thus the path p trivially satisfies the chain condition at the point x.

Therefore we assume that $\tilde{x}$ is a singular point of X. Since $\tilde{p}$ is assumed to be generic, this point lies on exactly one wall of X; moreover, $\tilde{p}$ is geodesic near $\tilde{x}$.

We first analyze what happens to the germ of $\tilde{p}$ at $\tilde{x}$ under the retraction f. We suppose that the restriction of f to the germ $(\tilde{p}, \tilde{x})$ is not an isometry (otherwise there is nothing to discuss). Let $\tilde{\zeta} \in \Sigma_{\tilde{x}}(X)$ denote the tangent to the geodesic segment $\overline{\tilde{x}z}$. Let $\eta \in \Sigma_{\tilde{x}}(X \cap \tilde{A})$ be the tangent vector $p'(t)$, $\tilde{\xi} := -\tilde{\eta}$ (this vector is also tangent to the path p). Set $\beta := d_{ref}(\tilde{\zeta}, \tilde{\eta})$.

We obtain the triangle $\tau = [\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}]$ in $\Sigma_{\tilde{x}}(X)$. The derivative of the retraction f at $\tilde{x}$ is a retraction of the spherical building $\Sigma_{\tilde{x}}(X)$ into its apartment S, after identification of S with the sphere $S_{x'}(A)$, $x' := f(\tilde{x})$ (see Lemma 4.6). Define the following elements of S:

$$\eta' := df_{\tilde{x}}(\tilde{\eta}), \quad \xi' := df_{\tilde{x}}(\tilde{\xi}), \quad \zeta' := df_{\tilde{x}}(\tilde{\zeta}).$$

Therefore, according to Theorem 4.9, the folded triangle $\tau' = df_x(\tau) \subset S$ yields an $(S, W_{x'}, -\Delta_{x'})$-chain[2]

$$(\mu' = -\xi', \eta')$$

such that $d_{ref}(\zeta, \eta') = d_{ref}(\tilde{\zeta}, \tilde{\eta}) = \beta$.

Here $-\Delta_{x'}$ is a chamber in $(S, W_{x'})$ which contains the unit tangent vector to the segment $\overline{x'z'}$ where $z' \in a$ is a regular point.

Remark 4.18. *Note that our assumptions on $\tilde{p}$ imply that $(S, W_{x'})$ has a unique wall. If the corresponding wall in A does not pass through z, then the negative chamber $-\Delta_{x'}$ in $(S, W_{x'})$ is uniquely determined by the condition that it contains the direction tangent to $\overline{x'z}$.*

Consider now the effect of the rest of the folding g on the path $\tilde{p}$ at $\tilde{x}$. Let $x := g(\tilde{x})$. We identify S with the unit tangent sphere at the point x.

The dilation h clearly preserves the chain condition at x' (since it acts trivially on the unit tangent sphere). The restriction of the projection $\mathbb{P} = \mathbb{P}_\Delta$ to the germ of $hf(\tilde{p})$ at $hf(\tilde{x})$ is necessarily an isometry (since $\tilde{p}$ is generic), hence it is given by an element $w \in W = W_{sph}$. This element transforms the above chain to another $(S, W_x, -\Delta_x)$–chain, where

$$\Delta_x := d(w \circ h)(\Delta_{x'}).$$

What is left to verify is that the positive chamber Δ_x in this complex contains a translate of the positive chamber Δ. In case when x belongs to the interior of Δ, the

[2]Which is necessarily simple since $\tilde{p}$ is assumed generic.

segment $\overline{ox}$ is not contained in any wall and thus the negative chamber $-\Delta_x$ has to contain the initial direction of the segment $\overline{xz}$ (see the remark above). However this initial direction belongs to $-\Delta$ and thus Δ_x contains Δ.

Consider the exceptional case when x is on the boundary of Δ. It then belongs to a unique wall H in the Coxeter complex (A, W_{aff}) and this wall passes through the origin o. Rather than trying to use Theorem 4.9 to verify the chain condition at x, we give a direct argument. Let η, ξ be the unit vectors which are the images of η', ξ' under

$$d(w \circ h) : \Sigma_{x'}(A) \to \Sigma_x(A).$$

Since the path p is entirely contained in Δ, the vector $p'_-(t)$ points *outside* of Δ and the vector $p'_+(t)$ points *inside*. The reflection σ in the wall H sends the vector $-\xi = p'_-(t)$ the vector $\eta = p'_+(t)$. It is then clear that the (simple) chain condition is satisfied at the point x.

Lastly, we consider the points $\tilde{x} = \tilde{p}(t)$ for which f is an isometry on the germ of $\tilde{p}$ at $\tilde{x}$. The point $x = g(\tilde{x})$ belongs to a face of Δ contained in a wall H, and this is the only wall of (A, W_{aff}) which passes through x. Then, necessarily, the germ of the path $hf(\tilde{p})$ at $hf(\tilde{x})$ is a geodesic. We now simply repeat the arguments of the exceptional case in the above proof (see also the proof of Proposition 4.19) to see that p satisfies the chain condition at x. □

We are now ready to prove Theorem 4.16 for arbitrary paths $\tilde{p}$. We will do so by approximating the path $\tilde{p}$ via *generic paths*. Let λ be an arbitrary vector in $\tilde{A}$. We let $\tilde{q}_\lambda := \tilde{p} + \lambda$ denote the translation of the path $\tilde{p}$ by the vector λ. It is clear, from the dimension count, that for an open and dense set of vectors λ, the path $\tilde{q}_\lambda$ is *generic*.

Since the folding g is continuous,

$$p = g(\tilde{p}) = \lim_{\lambda \to 0} g(\tilde{q}_\lambda).$$

By the Proposition 4.17, each $g(\tilde{q}_\lambda)$ satisfies the chain condition. Observe that the Δ-lengths of the paths $\tilde{p} + \lambda$ are independent of λ. Since f and $\mathbb{P}$ preserve Δ-lengths of PL paths and the dilation h changes them by a fixed amount, we can apply the compactness theorem (Theorem 3.30) to conclude that the limiting path p satisfies the chain condition as well. □

We now verify that, at certain points, the folded path p satisfies the *maximal chain condition*.

Proposition 4.19. *Under the assumptions of Theorem 4.16 let $\tilde{x} = \tilde{p}(t)$ be such that the folding f restricts to an isometry on the germ $(\tilde{p}, \tilde{x})$. Then the path $p = g(\tilde{p})$ satisfies the* maximal chain condition *at $x = g(\tilde{x})$.*

Proof: Our proof follows Littelmann's arguments in his proof of the PRV Conjecture, see [L1]. We fold the path $q := hf(\tilde{p})$ into Δ inductively.

We subdivide the interval $[0, c]$ as

$$0 = t_0 < t_1 < ... < t_k = c$$

such that $[t_i, t_{i+1}]$ are maximal subintervals so that $q|[t_i, t_{i+1}]$ is contained in a Weyl chamber of W_{sph}.

We first apply to q an element $w_0 \in W_{sph}$ which sends $q([0, t_1])$ into Δ, so we can assume that this subpath belongs to Δ. Assume that the restriction of f to the germ $(\tilde{p}, \tilde{p}(t_1))$ is an isometry. Let μ', η' be the vectors $q'_-(t_1), q'_+(t_1)$. Then

$$\mu' \sim \eta',$$

see Lemma 4.2. Set $x := q(t_1)$. The image η of the vector η' under $\mathbb{P}$ is obtained as

$$dw_1(\eta'),$$

where $w_1 \in W_{sph}$ fixes the point x and η is the unique vector in the W_x-orbit of $\eta' \in S_x$ which points inside Δ. Below we describe w_1 as a composition of reflections.

Let R' denote the root subsystem in R generated by the set of simple roots Φ' which vanish at the point x. Let Δ' denote the positive chamber for W_x defined via Φ'. Then the vector η can be described as the unique vector in the W_x-orbit of η' (now, regarded as a vector in $V = T_o(A)$) which belongs to the interior of Δ'. According to Lemma 3.15,

$$w_1 = \tau_m \circ .. \circ \tau_1, \text{ where for each } i, \quad \tau_i = \tau_{\beta_i}, \beta_i \in \Phi',$$

so that the sequence of vectors

$$(\eta_0 = \eta', \eta_1 := \tau_1(\eta_0), ..., \eta_m = \tau_m(\eta_{m-1}) = \eta),$$

is a chain in $(S, W_x, -\Delta)$ which is maximal as a chain in $(S, W_{sph}, -\Delta)$.

We therefore apply the identity transformation to the path $q|[0, t_1]$ and the element w_1 to the path $q|[t_1, c]$ to transform the path q to the new path

$$q_1 = q|[0, t_1] \cup w_1 \circ q|[t_1, c].$$

Clearly, $\mathbb{P}(q) = \mathbb{P}(q_1)$ and η is the unit vector tangent to $q_1|[t_1, c]$ at x. The above arguments therefore show that q_1 satisfies the *maximal chain condition* at the point x.

We then proceed to the next point t_2, $q_1(t_2)$ belongs to the boundary of Δ and we transform q_1 to q_2 by

$$q_2|[0, t_2] = q_1|[0, t_2], \quad q_2|[t_2, c] = w_2 \circ q_1|[t_2, c],$$

where w_2 is a certain element of W_{sph} fixing $q_1(t_2) = q_2(t_2)$. Therefore $\mathbb{P}(q_2) = \mathbb{P}(q_1) = \mathbb{P}(q)$ and we repeat the above argument. $\square$

Definition 4.20. *Suppose that $P = \overline{zx} \cup p \cup \overline{yz}$ is a polygon in A, where $p : [0,1] \to A$ is a PL path such that $p(0) = x, p(1) = y$. We say that P* satisfies the chain condition *(resp. simple chain condition, resp. maximal chain condition) if its subpath p satisfies the chain condition (resp. simple chain condition, resp. maximal chain condition).*

Therefore, as an application of Theorem 4.16 we obtain

Corollary 4.21. *Suppose that $T = [\tilde{z}, \tilde{x}, \tilde{y}] \subset X$ is a geodesic triangle, $\tilde{z}$ is a special vertex which belongs to an alcove $a \subset A$. Let $\Delta \subset A$ be a Weyl chamber with the tip $\tilde{z} = o$ and $P = Fold_\Delta(T)$ be the folding of T into Δ. Then the folded triangle P satisfies the chain condition.*

A converse to this corollary will be proven in Theorem 6.9; the following is a *partial* converse to the corollary 4.21 (which is essentially contained in [KLM3, Lemma 7.7]):

Corollary 4.22. *Let $\Delta \subset A$ be a Weyl chamber with the tip of o in X. Suppose that a polygon $P = [o, x_1, ..., x_n] \subset \Delta$, satisfies the* simple chain *condition (at each vertex $x_i, 0 < i < n$) and*

$$p = \overline{x_1 x_2} \cup ... \cup \overline{x_{n-1} x_n}$$

is a billiard *path. Then P unfolds to a geodesic triangle $T \subset X$, i.e. $Fold_\Delta(T) = P$.*

Proof: Let $f := Fold_\Delta$. We run the argument from the proof of Theorem 4.9 in the reverse; the reader will observe that our argument is essentially the same as in the proof of the Transfer Theorem in [KLM2]. Triangulating the polygon P from the vertex o we obtain geodesic triangles $P_i = [o, x_i, x_{i+1}]$, $i = 1, 2, ..., n-1$. Let $\xi_i, \zeta_i, \eta_i \in \Sigma_{x_i}(A)$ denote the unit tangent vectors to the segments $\overline{x_i x_{i-1}}, \overline{x_i o}, \overline{x_i x_{i+1}}$ respectively.

We unfold P inductively. Set $T_1 := P_1$; let $A_1 := A$, this apartment contains the triangle T_1. Set $\tilde{x}_1 := x_1, \tilde{x}_2 := x_2$,

Suppose that we have constructed apartments $A_i \subset X$ and flat triangles $T_i = [o, \tilde{x}_i, \tilde{x}_{i+1}] \subset A_i$, $i = 1, ..., m-1$, so that T_i is congruent to P_i $(i = 1, ..., m-1)$ and

$$\angle(\tilde{\xi}_i, \tilde{\eta}_i) = \pi, i = 1, ..., m-1.$$

Here $\tilde{\xi}_i, \tilde{\eta}_i, \tilde{\zeta}_i$ are directions in $\Sigma_{\tilde{x}_i}(X)$ which correspond to the directions ξ_i, η_i, ζ_i under the congruences $T_i \to P_i$. Our goal is to produce a flat triangle $T_m \subset A_m \subset X$ so that the above properties still hold.

Since we have a simple chain $(-\xi_m, \eta_m)$ in (S_{x_m}, W_{x_m}), it follows from Proposition 4.13 that there exists a point $\tilde{\eta}_m \in \Sigma_{\tilde{x}_m}(X)$ so that

$$d(\tilde{\eta}_m, \tilde{\xi}_m) = \pi, \quad d(\tilde{\eta}_m, \zeta_m) = d_{ref}(\eta_m, \zeta_m).$$

Let A_m denote an apartment in X which contains $\overline{o\tilde{x}_m}$ and such that $\tilde{\eta}_m$ is tangent to A_m. Construct a geodesic segment $\overline{\tilde{x}_m \tilde{x}_{m+1}} \subset A_m \subset X$ whose metric length equals the one of $\overline{x_m x_{m+1}}$ and whose initial direction is $\tilde{\eta}_m$. This defines a flat triangle

$$T_m = [o, \tilde{x}_m, \tilde{x}_{m+1}] \subset A_m.$$

It is clear from the construction that the triangle T_m is congruent to P_m, in particular, $d_{ref}(o, x_3) = d_{ref}(o, y_3)$. Observe also that

$$\overline{\tilde{x}_{m-1}\tilde{x}_m} \cup \overline{\tilde{x}_m\tilde{x}_{m+1}}$$

is a geodesic segment (because $\angle(\tilde{\eta}_m, \tilde{\xi}_m) = \pi$). See Figure 7.

Therefore, by induction we obtain a geodesic triangle $T = [o, x_1, \tilde{x}_n] \subset X$, which is triangulated (from o) into flat geodesic triangles T_i which are congruent to P_i's. We claim that $f(T) = P$. For each i the folding f sends the triangle T_i to P_i, according to Proposition 4.7. Therefore $f(T) = P$.

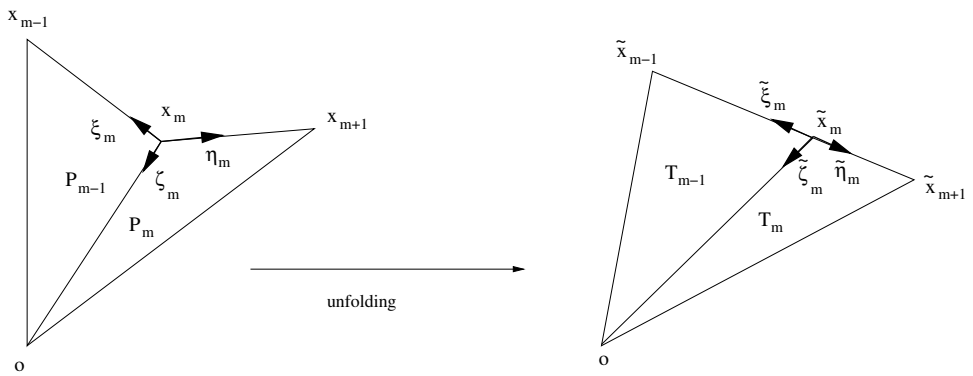

Figure 7: *Unfolding a broken triangle.*

As in the proof of Theorem 4.9, the argument has to be modified in case when $x_i = o$ for some i, $x_i = p(t_i)$. Then the vector $\mu = p'_-(t_i)$ belongs to the negative chamber $-\Delta$ and the vector $\lambda = p'_+(t_i)$ belongs to the positive chamber Δ. Since p is a billiard path, there exists $w \in W_{sph}$ which sends μ to λ. Now the chain and billiard conditions imply that x_i is the only break-point in p. Thus we can take

$$T := \overline{ox_1} \cup p([0, t_i]) \cup w^{-1}p([t_i, 1]) \cup w^{-1}(\overline{ox_n}).$$

This degenerate geodesic triangle (it is contained in the geodesic through the points $x_1, w(x_n)$) folds to P under the projection $\mathbb{P}_\Delta : A \to \Delta$. □

The same argument as above proves the following generalization of Corollary 4.22

Corollary 4.23. *Suppose that P is a polygon in Δ which is the composition*

$$\overline{ox} \cup p \cup q \cup \overline{yo}.$$

Assume that paths p, q satisfy the simple chain condition. Then there exists a polygon $\tilde{P} \subset X$ of the form

$$\overline{ox} \cup \tilde{p} \cup \tilde{q} \cup \overline{\tilde{y}o}$$

so that $f(\tilde{P}) = P$, $f(\tilde{p}) = p, f(\tilde{q}) = q$, $f(\tilde{y}) = y$, and $\tilde{p}, \tilde{q}$ are geodesic paths.

We now use our analysis of the folded triangles (polygons) to relate them to the Littelmann triangles (polygons).

5 Littelmann polygons

5.1 LS paths

Let R be a root system on a Euclidean vector space V, $W = W_{sph}$ be the finite Coxeter group associated with R, let W_{aff} denote the affine Coxeter group associated to R. This root system R is actually the *coroot system* for the one considered by Littelmann in [L2]. Accordingly, we will switch weights to coweights, etc. We pick a Weyl chamber Δ for W, this determines the positive roots and the simple roots in R. We get the Euclidean Coxeter complex (A, W_{aff}), where A is the affine space corresponding to V. Given $x \in A$ let W_x denote the stabilizer of x in W_{aff}.

Suppose we are given a vector $\lambda \in \Delta \subset V$, a sequence of real numbers

$$\underline{a} = (a_0 = 0 < a_1 < ... < a_r = 1),$$

and a sequence of vectors in $W\lambda$

$$\underline{\nu} = (\nu_1, ..., \nu_r), \quad \text{so that} \quad \nu_1 > ... > \nu_r$$

with respect to the order in Definition 3.5.

Definition 5.1. *The pair* $(\underline{\nu}, \underline{a})$ *is called a* real (billiard) path *of the* Δ*-length* λ.

Definition 5.2. *(P. Littelmann, [L2]) A real path of* Δ*-length* λ *is called* rational *if* λ *is a coweight and all numbers* a_i *a rational.*

Remark 5.3. *Littelmann uses the notion* path of type λ *rather than of the* Δ*-length* λ.

Set $a'_i := a_i - a_{i-1}$, $i = 1, 2, ..., r$. The data $(\underline{\nu}, \underline{a})$ determines a PL path $p \in \mathcal{P}$ whose restriction to each interval $[a_{i-1}, a_i]$ is given by

$$p(t) = \sum_{k=1}^{i-1} a'_k \nu_k + (t - a_{i-1})\nu_i, t \in [a_{i-1}, a_i]. \tag{5}$$

Our interpretation of real and rational paths is the one of a broken (oriented) geodesic L in V. Each oriented geodesic subsegment of L is parallel to a positive multiple of an element of $W\lambda$, thus L is a billiard path. The break points of the above path are the points

$$x_1 = a_1\nu_1, \ldots, x_i = x_{i-1} + a'_i\nu_i, \ldots$$

Since $\sum_i a'_i = 1$, is clear that

$$\text{length}_\Delta(L) = \lambda,$$

in the sense of the definition in section 2.5. This justifies our usage of the name Δ-length λ in the above definitions, rather than Littelmann's notion of *type*.

Observe that given a PL path $p(t) \in \mathcal{P}$ (parameterized with the constant speed) one can recover the nonzero vectors $\nu_i \in V$ and the numbers a_i and a'_i.

Definition 5.4 (P. Littelmann, [L2].). *A rational path $p(t)$ is called an* LS path[3] *if it satisfies further* integrality condition*:*

For each $i = 1, ..., s-1$ there exists an a_i-chain for the pair (ν_i, ν_{i+1}) (in the sense of Definition 3.20).

Observe that, since

$$\eta \geq \tau \iff -\tau \geq -\eta,$$

it follows that p is an LS path if and only if p^* is.

Theorem 5.5. *(P. Littelmann, [L2, Lemma 4.5]) Each LS path belongs to $\mathcal{P}_{\mathbb{Z},loc}$.*

Our next goal is to give a more geometric interpretation of the LS paths. Suppose that $p \in \mathcal{P}$ is a billiard path given by the equation (5), with the vertices

$$0 = x_0, x_1, ..., x_r.$$

At each vertex point $x_i, 0 < i < r$, we have unit tangent vectors ξ_i, μ_i which are tangent to the segments $\overline{x_i x_{i-1}}, \overline{x_i x_{i+1}}$. Note that at each vertex $x_i, 0 < i < r$ we have the restricted and unrestricted spherical Coxeter complexes; the positive chamber Δ in V determines positive chambers Δ_i in the restricted spherical complexes (S_{x_i}, W_{x_i}).

Theorem 5.6. *A billiard path $p(t)$ of Δ–length $\lambda \in P(R^\vee)$ is an LS path if and only if it is a Hecke path which satisfies the* maximal chain condition *(cf. Definition 3.26): At each vertex $x_i, 0 < i < r$ there exists a $(S_{x_i}, W_{x_i}, -\Delta_i)$-chain between $-\xi_i$ and μ_i and this chain is maximal as a (S_{x_i}, W)-chain.*

Proof: Recall that given a nonzero vector $v \in V$, $\bar{v}$ denotes its normalization $v/|v|$.

It is easy to see (and left to the reader) that if $p(t)$ is a satisfies the above chain condition and $\lambda = \text{length}_\Delta(p)$ is a coweight, then all numbers a_i are rational.

Consider the first break point $x_1 = x_0 + a_1\nu_1$ of the broken geodesic path $p(t)$. Observe that

$$\bar{\nu}_1 = -\xi_1, \bar{\nu}_2 = \mu_1 \in S_{x_1}.$$

According to Proposition 3.24, existence of an a_1-chain for the pair (ν_1, ν_2) is equivalent to existence of an $(S_{x_1}, W_{x_1}, \Delta_1)$-chain, which is maximal in the unrestricted Coxeter complex,

$$(\bar{\nu}_1 = \eta_{1,0}, \eta_{1,1}, ..., \eta_{1,s_1} = \bar{\nu}_2).$$

Thus the path p satisfies the maximal chain condition at the first break point x_1 if and only if it satisfies the *integrality condition* as in Definition 5.4, at the point x_1.

We now proceed to the next break point $x_2 = x_1 + a_2'\nu_2$. We identify normalized vectors $\bar{\nu}_1, \bar{\nu}_2$ with unit vectors in S_{x_1}. Note that if $p(t)$ is an LS path of the Δ-length λ, then there exists an element $w_1 \in W_{x_1}$ which sends $\bar{\nu}_1$ to $\bar{\nu}_2$. The same is true if p is a Hecke path.

[3] a Lakshmibai-Seshadri path

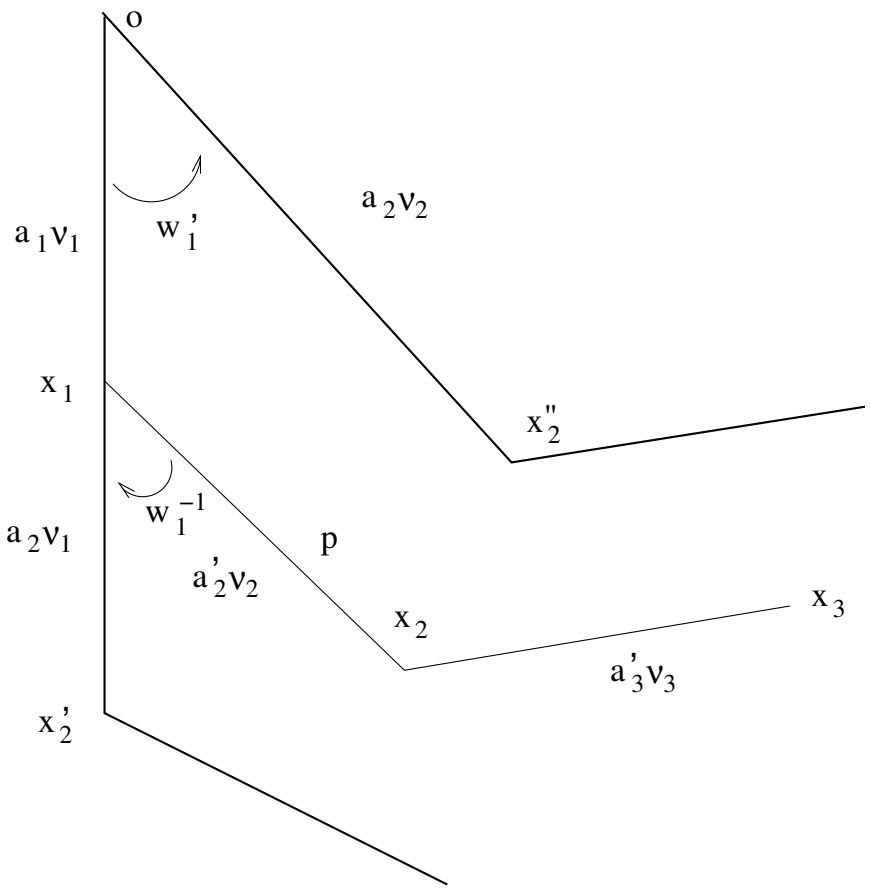

Figure 8: *Unbending a path.*

Set $x_2' := w_1^{-1}(x_2)$. Observe that, in both cases of an LS path and a Hecke path,

$$\overline{x_0x_1} \cup \overline{x_1x_2'}$$

is a geodesic segment $\overline{x_0x_2'}$; the corresponding directed segment represents the vector $a_2\nu_1$. Let $w_1' \in W_{sph}$ denote the linear part of w_1. Set $x_2'' := w_1'(x_2')$. We translate the vectors $\bar{\nu}_2, \bar{\nu}_3$ to the unit tangent sphere $S_{x_2''}$. The directed segment $\overrightarrow{x_0x_2''}$ represents the vector $a_2\nu_2$. See Figure 8.

We are now again in position to apply Proposition 3.24 with $a = a_2$: There exists a maximal chain

$$(\bar{\nu}_2 = \eta_{2,0}, \eta_{2,1}, ..., \eta_{2,s_2} = \bar{\nu}_3)$$

if and only if there exists an a_2-chain for the pair (ν_2, ν_3). The product $w_1 \circ (w_1')^{-1}$ is a translation in W_{aff} which carries x_2'' back to x_2. Therefore it induces an isomorphism of the restricted Coxeter spherical complexes

$$(S_{x_2''}, W_{x_2''}) \to (S_{x_2}, W_{x_2})$$

which carries positive chamber to positive chamber. Hence this translation sends the chain $(\eta_{2,i})$ to a maximal chain in (S_{x_2}, W_{x_2}).

We continue in this fashion: On the i-th step we "unbend" the broken geodesic

$$\overline{x_0x_1} \cup ... \cup \overline{x_{i-1}x_i}$$

to a directed geodesic segment $\overrightarrow{x_0x_i'}$ representing the vector $a_i\nu_1$, then apply an appropriate element $w_{i-1}' \in W_{sph}$ to transform segment $\overline{x_0x_i'}$ to $\overline{x_0x_i''}$; finally, appeal to Proposition 3.24 to establish equivalence between the maximal chain condition and the LS path axioms. □

As a corollary of Theorem 5.6 we obtain

Corollary 5.7. *Let $T = [z, x, y] \subset X$ be a geodesic triangle and $f = Fold_{z,h,\Delta}$ be a folding into the Weyl chamber. Set $\beta := d_\Delta(x, y)$. Assume that $T' = f(T)$ is such that $f(x), f(y)$ and all break-points of the broken geodesic $f(\overline{xy})$ are special vertices. Then $f(\overline{xy})$ is an LS path of the Δ-length $k\beta$. Here k is the conformal factor of the dilation h.*

5.2 Root operators

With each simple root $\alpha \in \Phi$, Littelmann [L2] associates *raising and lowering* root operators e_α and f_α acting $\mathcal{P}$ as follows.

Recall that given a path $p(t)$ and a root α we have the height function $h_\alpha(t) := \alpha(p(t))$. The number m_α is the minimal value of h_α on $[0, 1]$.

If $m_\alpha > -1$ then e_α is not defined on p. Otherwise let t_1 be the minimal t for which $h_\alpha(t) = m_\alpha$ and let $t_0 \in [0, t_1]$ be maximal such that $h_\alpha(t) \geq m_\alpha + 1$ for all $t \in [0, t_0]$.

The operator e_α will not change the path p for $t \in [0, t_0]$ and, as far as $[t_1, 1]$ is concerned, the path $p|[t_1, 1]$ will change only by a translation in W_{aff} along the line L_α parallel to the vector $\alpha^\vee$. Thus it remains to describe the path $q = e_\alpha(p)$ on $[t_0, t_1]$. If h_α were not to have any local minima on $[t_0, t_1]$ then $q|_{[t_0,t_1]}$ would be obtained by the reflection

$$q|_{[t_0,t_1]} := \tau_\alpha \circ p|_{[t_0,t_1]}$$

and we would set

$$q := p|_{[0,t_0]} * \tau_\alpha \circ p|_{[t_0,t_1]} * p|_{[t_1,1]}.$$

(Here we treat the paths resulting from the restriction of p to subintervals of $[0, 1]$ as elements of $\mathcal{P}$, according to the convention in section 2.)

This is the definition of e_α of [L1], however the definition of e_α which we will need in this paper is the more refined one of [L2]. Call a subinterval $[s, u] \subset [t_0, t_1]$ a *spike* if it is a maximal interval satisfying

$$h_\alpha(s) = h_\alpha(u) = \min(h_\alpha\big|[s, u]).$$

Thus $h_\alpha\big|[t_0, t_1]$ is decreasing on the complement to the union of *spikes*. The restriction of q to each *spike* is obtained from p by a translation along L_α. The restriction to

each subinterval disjoint from a *spike* is obtained by a reflection. To be more precise, subdivide the interval $[t_0, t_1]$ into

$$[t_0, s_1] \cup [s_1, s_2] \cup ... \cup [s_k, t_1],$$

where the *spike* and *non-spike* intervals alternate. Observe that $[t_0, s_1], [s_k, t_1]$ are not *spikes*. Then

$$q := p|_{[0,t_0]} * \tau_\alpha(p|_{[t_0,s_1]}) * p|_{[s_1,s_2]} * ... * \tau_\alpha(p|_{[s_k,t_1]}) * p|_{[t_1,1]}.$$

Note that the operator e_α changes the geometry of the path p by an isometry near every point $p(t)$ which is neither a point of local minimum for h_α nor is a point where $h_\alpha(t) = m_\alpha - 1$. Otherwise the local change is done by a "bending" with respect to a hyperplane parallel to H_α. These hyperplanes are not necessarily walls of W_{aff}. However, if all local minimal values of h_α belong to $\mathbb{Z}$, these hyperplanes are indeed walls and we obtain:

For each path $p \in \mathcal{P}_{\mathbb{Z},loc}$, for each simple root α, the path $q = e_\alpha(p)$ satisfies the following: The interval $[0, 1]$ can be subdivided into subintervals $[s_i, s_{i+1}]$ such that the restriction $q|[s_i, s_{i+1}]$ is obtained from the restriction of p by post-composition with an element of W_{aff}.

The lowering operators f_α are defined analogously to the raising operators, we refer the reader to [L2] for the precise definition. (See however Property 1 below.) At this stage we note only that f_α is undefined on p iff $m_\alpha > h_\alpha(1) - 1$. Let $\mathcal{E}$ be the semigroup generated by e_α's, $\mathcal{F}$ be the semigroup generated by f_α's and $\mathcal{A}$ be the semigroup generated by all root operators. The semigroups contain the identity operator by default. For each $\phi \in \mathcal{A}$ let $Dom(\phi)$ denote the domain of ϕ.

Remark 5.8. *In fact, Littelmann extends the operators f_α, e_α to the entire $\mathcal{P}$ by declaring $f_\alpha(p) = \mathbf{0}$ for all p for which f_α is undefined. However we will not need this extension in the present paper.*

Below we list certain properties of the root operators. Most of them are either clear from the definition or are proven in [L2]. Most proofs that we present are slight modifications of the arguments in [L2].

Property 1. (P. Littelmann, [L2, Lemma 2.1 (b, e)].)

$$e_\alpha \circ f_\alpha(p) = p, \quad \text{if} \quad p \in Dom(f_\alpha),$$

$$f_\alpha \circ e_\alpha(p) = p, \quad \text{if} \quad p \in Dom(e_\alpha),$$

$$e_\alpha(p^*) = (f_\alpha(p))^*, \quad (e_\alpha(p))^* = f_\alpha(p^*),$$

the latter could be taken as the definition of f_α.

Property 2. (See [L2, Lemma 2.1].) For each $p \in Dom(e_\alpha) \cap \mathcal{P}$,

$$m_\alpha(e_\alpha(p)) = m_\alpha(p) + 1,$$

$$p \in Dom(e_\alpha^N) \iff N < |m_\alpha|.$$

Property 3. Suppose that p is a path in $\mathcal{P}_\mathbb{Z}$ which does not belong to the domain of any $e_\alpha, \alpha \in \Phi$. Then p is contained in Δ. Indeed, for each simple root α we have to have $m_\alpha(p) > -1$. Since $p \in \mathcal{P}_\mathbb{Z}$, $m_\alpha(p) = 0$. Thus $p \in \mathcal{P}^+$.

Property 4. (See [L2, Proposition 3.1 (a, b)].) For each $\alpha \in \Phi$, $Dom(f_\alpha) \cap \mathcal{P}_\mathbb{Z}$ is open and $f_\alpha|\mathcal{P}_\mathbb{Z}$ is continuous.

Property 5. (See [L2, §7, Corollary 1 (a)].) Let $p \in \mathcal{P}^+$ and ϕ be a composition of lowering operators defined on p. Then $\phi(p) \in \mathcal{P}_\mathbb{Z}$.

Property 6. Combining Properties 4 and 5 we conclude that for each $\mathbf{f} \in \mathcal{F}$, $Dom(\mathbf{f}) \cap \mathcal{P}^+$ is open and $\mathbf{f}|\mathcal{P}^+$ is continuous.

Property 7. (See [L2, Corollary 3, Page 512].) p is an LS path of the Δ-length λ if and only if there exists $\mathbf{f} \in \mathcal{F}$ such that

$$p = \mathbf{f}(\pi_\lambda)$$

Property 8. (See [L2, Corollary 2(a), page 512].) The set of LS paths of the given Δ-length is stable under $\mathcal{A}$.

Property 9. Suppose that $p \in \mathcal{P}$, $t \in [0, 1]$, $\alpha \in \Phi$ and $x := p(t)$ satisfy

$$\alpha(x) \in \mathbb{Z}, \quad p'_-(t) \gtrsim_{W_x} p'_+(t).$$

Then the path $q = e_\alpha(p)$ also satisfies

$$q'_-(t) \gtrsim_{W_y} q'_+(t),$$

for $y = q(t)$.

Proof: If $h_\alpha(t) \neq m_\alpha, m_\alpha - 1$, the germs of the paths p and q at t differ by a translation. Thus the conclusion trivially holds in this case. The same argument applies if $h_\alpha(t) = m_\alpha - 1$ and

$$\alpha(p'_-(t)) \geq 0, \alpha(p'_+(t)) \leq 0.$$

The nontrivial cases are:

1. $h_\alpha(t) = m_\alpha - 1$, $\alpha(p'_-(t)) \leq 0$, $\alpha(p'_+(t)) \leq 0$. In this case the assertion follows from Lemma 3.6 with $\nu = p'_-(t), \mu = p'_+(t)$.

2. $h_\alpha(t) = m_\alpha$, $\alpha(p'_-(t)) \leq 0$, $\alpha(p'_+(t)) \geq 0$. In this case the assertion follows from Lemma 3.19 with $\nu = p'_-(t), \mu = p'_+(t)$. □

Property 10. Suppose that $p = p_1 * p_2$ where $p_1 \in \mathcal{P}^+$. Then for each $e \in \mathcal{E}$ defined on p we have

$$e(p) = p_1 * e(p_2).$$

Proof: It suffices to prove this for $e = e_\alpha$, $\alpha \in \Phi$. In the latter case it follows directly from the definition of the operator e_α. $\square$

The next property is again clear from the definition:

Property 11. Suppose that

$$\mathbf{e} = e_{\beta_m} \circ ... \circ e_{\beta_1}$$

where $\beta_i \in \Phi$, $p \in Dom(\mathbf{e})$. Set

$$p_i := e_{\beta_i} \circ ... \circ e_{\beta_1}(p), i = 1, ..., m.$$

Then for each $T \in [0, 1]$, the sequence of vectors

$$(p'_+(T), (p_1)'_+(T), ..., (p_m)'_+(T)),$$

after deleting equal members, forms a chain.

Lemma 5.9. *Given a path p there are only finitely many operators $\mathbf{e} \in \mathcal{E}$ which are defined on p.*

Proof: Break the path p as the concatenation

$$p_1 * ... * p_s.$$

of geodesic paths each of which is contained in a single alcove and let $T_i \in [0, 1]$ be such that $p(T_i) = p_i(1/2)$; set $T_0 := 0$. Then for each $e_\alpha \in \mathcal{E}$ defined on p there exists i such that the derivatives of $e_\alpha(p)$ and p at T_i are not the same. Moreover,

$$q = e_\alpha(p) = q_1 * ... * q_s,$$

where each q_i is a geodesic path contained in an alcove. Consider the vector

$$L(p) := (\ell(p'(0)), \ell(p'(T_1)), ..., \ell(p'(T_s))) \in (\mathbb{N} \cup \{0\})^{s+1},$$

where $\mathbb{N}^{s+1}$ is given the lexicographic order and ℓ is the length function on W_{sph}-orbits induced from the word metric on W_{sph} as in Lemma 3.7. Then, by combining Lemma 3.7 and Property 11 above, for each $\alpha \in \Phi$,

$$L(e_\alpha(p)) < L(p).$$

Lemma follows. $\square$

5.3 Generalized LS paths

In this paper we will need two generalizations of the concept of an LS path; the first one will be needed for the proof of the saturation theorem (section 7), the second will be used in section 6.1 for the proof of the unfolding theorem. Although we will use the name *generalized LS path* for both generalizations, it will be clear from the context which generalization is being referred to.

The first generalization, $\mathcal{LS}_1$. [4]

Suppose we are given a collection of LS paths p_i of the Δ-length $\lambda_i \in \Delta \cap P(R^\vee)$, $i = 0, ..., m$. We will use the notation

$$\underline{\lambda} = (\lambda_0, ..., \lambda_m)$$

and

$$\lambda := \sum_{i=0}^{m} \lambda_i.$$

Remark 5.10. *Actually, for our main application it will suffice to consider λ_i's which are multiples of the fundamental coweights ϖ_i. Therefore such paths are automatically* generalized Hecke paths *as defined in Definition 3.29.*

Definition 5.11. *The concatenation*

$$p = p_0 * p_1 * ... * p_m$$

will be called a generalized LS path *with* $\underline{\text{length}}_\Delta(p) = \underline{\lambda}$*, if for each* $i = 0, ..., m-1$

$$p_i'(1) \gtrsim p_{i+1}'(0).$$

The set of such generalized LS paths will be denoted $\mathcal{LS}_1$.

This definition is a very special case of the one used by Littelmann in [L3] under the name of a *locally integral concatenation.*

Recall that according to the definition of Δ-length,

$$\lambda = \text{length}_\Delta(p).$$

Observe that each LS path p satisfies the above definition, since for each i

$$p_i'(1) \geq p_{i+1}'(0).$$

Example 5.12. *Suppose that u, v are dominant coweights. Then $p = \pi_u * \pi_v$ is a generalized LS path.*

[4] This generalization of LS paths will be used in the proof of the saturation theorem.

The second generalization $\mathcal{LS}_2$. [5]

Suppose that $p_1, p_2 \in \mathcal{P}$ appear as

$$p_1 = \tilde{p}_1|_{[0,a]}, \quad 0 < a < 1,$$

$$p_2 = \tilde{p}_2|_{[b,1]}, \quad 0 < b < 1,$$

where $\tilde{p}_1, \tilde{p}_2$ are LS paths, $a, b \in \mathbb{Q}$. (See section 2.2 for the definition of $\tilde{p}_1|_{[0,a]}$ and $\tilde{p}_2|_{[b,1]}$.) Define the path $p := p_1 * p_2$. Assume that

$$p_1'(1) \rhd p_2'(0);$$

in other words, if t is such that $p(t) = p_1(1)$ then

$$p_-'(t) \rhd p_+(t).$$

Definition 5.13. *The concatenation p will be called a* generalized LS path *if the* concatenation point $p_1(1)$ *is a regular point*[6] *of* (A, W_{aff}) *and* $p(1) \in P(R^\vee)$.

The set of such generalized LS paths will be denoted $\mathcal{LS}_2$.

Example 5.14. *Suppose that $u \in \Delta, v \in V$ are such that $u, v \in P(R^\vee) \otimes \mathbb{Q}$, $u + v \in P(R^\vee) \cap \Delta$ and the head of the vector u is a regular point in (A, W_{aff}). Then $p = \pi_u * \pi_v \in \mathcal{LS}_2$.*

This definition is again a very special case of the one given by Littelmann in [L2, 5.3]. Littelmann does not assume that $p_1(1)$ is regular, but instead imposes certain chain conditions at this point.

Properties of generalized LS paths:

Property 0. If $p \in \mathcal{LS}_1$ then p^* is also in $\mathcal{LS}_1$.

Proof: Represent p as a concatenation $p_1 * ... * p_m$ of LS paths as in the Definition 5.11. Then

$$p^* = (p_m^*) * ... * (p_1^*),$$

where each path p_i^* is again an LS path. Now the assertion follows from Lemma 3.18. □

Property 1. $\mathcal{LS}_2$ is stable under the root operators; see [L2, Lemma 5.6, 2-nd assertion]. In particular, suppose that p is as in Example 5.14. Then for each $\mathbf{f} \in \mathcal{F}$ (defined on p), $\mathbf{f}(p)$ is a generalized LS path of the Δ-length $\lambda = u + v$.

Property 2. $\mathcal{LS}_1$ is stable under the root operators.

Proof: Suppose that $p = p_1 * ... * p_m$ is a concatenation of LS paths as above and e_α is a raising operator. In particular, for each i we have a vector u_i so that

$$p_i'(1) \geq u_i \sim p_{i+1}'(0).$$

[5] This notion of generalized LS path will be used only to unfold Hecke paths.

[6] I.e. it does not belong to any wall.

For each i, $e_\alpha(p_i)$ is again a Littelmann path. Therefore $e_\alpha(p)$ is a concatenation of LS paths $q_1 * ... * q_m$. We have to verify that for each i there is a vector $v_i \in V$ so that

$$q_i'(1) \geq v_i \sim q_{i+1}'(0).$$

This however follows from the Property 9 in the previous section. To check that $\mathcal{LS}_1$ is preserved by f_α we use that $q \in \mathcal{LS}_1 \iff q^* \in \mathcal{LS}_1$ and

$$f_\alpha(p) = (e_\alpha(p^*))^*. \quad \square$$

Property 3. $\mathcal{LS}_1$ and $\mathcal{LS}_2$ are contained in $\mathcal{P}_{\mathbb{Z},loc}$. For $\mathcal{LS}_1$ it is immediate since the set of LS paths is contained in $\mathcal{P}_{\mathbb{Z},loc}$ (see Theorem 5.5). For $\mathcal{LS}_2$ it is a special case of [L2, Lemma 5.5].

Property 4. Suppose that $p \in \mathcal{LS}_1$. Then there exists an element $\mathbf{e} \in \mathcal{E}$ defined on p such that $q = \mathbf{e}(p) \in \mathcal{P}^+$.

Proof: If $m_{\alpha_1}(p) \leq -1$ then we apply a power $e_{\alpha_1}^{k_1}$ to p so that $q_1 := e_{\alpha_1}^{k_1}(p)$ satisfies $m_{\alpha_1}(q_1) > -1$. However, since $e_{\alpha_1}^{k_1}(p) \in \mathcal{LS}_1 \subset \mathcal{P}_{\mathbb{Z},loc}$, it follows that $q_1 \in \mathcal{P}_{\mathbb{Z},loc}$ and so $m_{\alpha_1}(q_1) = 0$. We then apply a power of e_{α_2} to q_1, etc. According to Lemma 5.9, this process must terminate. Therefore, in the end we obtain a path

$$\mathbf{e}(p) = q$$

which does not belong to the domain of any raising operator. Since $q \in \mathcal{P}_{\mathbb{Z},loc}$ it follows that q is entirely contained in Δ. $\square$

Recall, see [Bo, Chapter VI, section 10] that if a root system R spans V then each dominant coweight $\lambda \in \Delta$ is a positive integer combination

$$\lambda = \sum_{i=1}^{l} n_i \varpi_i,$$

where ϖ_i are fundamental coweights. This assertion (as it stands) is false without the above assumption on V. In the general case we have

$$\lambda = \lambda' + \sum_{i=1}^{l} n_i \varpi_i,$$

where $\lambda' \in V'$, $n_i \in \mathbb{N} \cup \{0\}$. As an alternative the reader can restrict the discussion to semisimple groups only, when $V' = 0$.

Convention 5.15. *From now on we will be assuming that in Definition 5.11*

$$\lambda_j = k_j \varpi_j, \quad k_j \in \mathbb{N},$$

for each $j = 1, ..., m$, where ϖ_j is the j-th fundamental coweight, and

$$\lambda_0 \in V'.$$

Then the subpath p_0 is necessarily geodesic.

Lemma 5.16. *Suppose that $p \in \mathcal{LS}_1 \cap \mathcal{P}^+$ is a generalized LS path with* $\underline{\text{length}}_\Delta(p) = \underline{\lambda}$. *Then*

$$p = \pi_{\lambda_0} * ... * \pi_{\lambda_m}.$$

Proof: Represent p as the concatenation of maximal LS subpaths, $p = p_0 * p_1 ... * p_m$. The geodesic subpath p_0 clearly equals π_{λ_0}. Since p_1 is an LS path and $p_1'(0) \in \Delta$, we see that p_1 is a geodesic path (see Corollary 3.9) which therefore equals π_{λ_1}. Moreover, because

$$p_1'(1) \geq u_1 \sim p_2'(0),$$

it follows that $u_1 = p_1'(1)$ and thus $p_1'(1) = \lambda_1 \sim p_2'(0)$. Let $x_1 := p_1(1)$; this point lies on the boundary face of Δ which does not contain λ_2. Note that the vector $p_2'(0)$, regarded as an element of $T_{x_1}(A)$, points inside the Weyl chamber Δ (for otherwise p is not contained in Δ). On the other hand, since $\lambda_1 \sim p_2'(0)$, the vector $p_2'(0)$ belongs either to Δ or to the Weyl chamber

$$\tau_{\beta_2}(\Delta)$$

adjacent to Δ. Since $p \in \mathcal{P}^+$, it is clear that $p_2'(0) \in \Delta$. Thus p_2 is the geodesic path π_{λ_2}. Continuing in this fashion we conclude that

$$p = \pi_{\lambda_0} * \pi_{\lambda_1} * ... * \pi_{\lambda_m}. \quad \square$$

Theorem 5.17. *Suppose that p is a generalized LS path in the sense of $\mathcal{LS}_1$ with* $\underline{\text{length}}_\Delta(p) = \underline{\lambda}$ *(satisfying convention 5.15). Then there exists $\mathbf{f} \in \mathcal{F}$ such that*

$$p = \mathbf{f}(\pi_{\lambda_0} * ... * \pi_{\lambda_m})$$

Proof: If $p \in \mathcal{P}^+$ then we are done. Otherwise, by combining Lemma 5.16 with the Property 4, we find an $\mathbf{e} \in \mathcal{E}$,

$$\mathbf{e} = e_{\alpha_1}^{k_1} \circ ... e_{\alpha_n}^{k_n}$$

such that $p \in Dom(\mathbf{e})$ and $\mathbf{e}(p) = q \in \mathcal{P}^+$. Therefore

$$q = \pi_{\lambda_0} * \pi_{\lambda_1} * ... * \pi_{\lambda_m}$$

and thus the composition

$$\mathbf{f} = f_{\alpha_n}^{k_n} \circ ... f_{\alpha_1}^{k_1}$$

satisfies $p = \mathbf{fe}(p) = \mathbf{f}(\pi_{\lambda_0} * \pi_{\lambda_1} * ... \pi_{\lambda_m})$. $\square$

5.4 Path model for the representation theory of Lie groups

Suppose that

$$p(t) \in \mathcal{LS}_1 \cup \mathcal{LS}_2$$

is a generalized LS path with $\text{length}_\Delta(p) = \beta$ and

$$\underline{\text{length}}_\Delta(p) = \underline{\lambda}.$$

Suppose that $\alpha \in P(R^\vee)$ is such that $\alpha + p(t)$ is contained in Δ. Then α and p define a polygon

$$P := \overline{oy_0} \cup (p + \alpha) \cup \overline{y_n o} \subset \Delta$$

where $\alpha = \overrightarrow{oy_0}$, $y_n = \alpha + p(1)$. Let γ denote the vector $\overrightarrow{oy_n}$; then γ is also a dominant coweight. Recall that the contragredient dominant coweight $\gamma^* \in \Delta$ is obtained by projecting the vector $-\gamma$ to the Weyl chamber Δ by the projection $\mathbb{P} : V \to \Delta$.

Definition 5.18. *1. A polygon P above will be called a* (broken) Littelmann polygon *with the Δ-side lengths α, β, γ^*.*

2. If $p(t)$ is an LS path then P will be called a (broken) Littelmann triangle *with the Δ-side lengths α, β, γ^*.*

Pick a lattice L such that

$$Q(R^\vee) \subset L \subset P(R^\vee).$$

Then there exists a unique connected semisimple complex Lie group $G^\vee$ with the root system $R^\vee$ and the character lattice L of the maximal torus $T^\vee \subset G^\vee$. Recall that irreducible representations V of $G^\vee$ are parameterized by their dominant weights, $V = V_\lambda$, $\lambda \in \Delta \cap L$.

Pick a path $q \in \mathcal{P}^+$ such that $q(1) = \beta$. Then, according to [L2, Decomposition formula, Page 500] we have

Theorem 5.19 ([L2]). *The tensor product $V_\alpha \otimes V_\beta$ contains V_γ as a subrepresentation if and only if there exists a path $p \in \mathcal{F}(q)$ such that $\pi_\alpha * p \in \mathcal{P}^+$ and $\pi_\alpha * p(1) = \gamma$.*

Remark 5.20. *Littelmann works with simply-connected group $G^\vee$ and weights α, β, γ in $P(R^\vee)$. The statement for non-simply-connected groups trivially follows from the simply-connected case.*

In particular, since p is an LS paths of the Δ-length β if and only if $p \in \mathcal{F}(\pi_\beta)$, it follows that

Theorem 5.21 ([L1], [L2]). *The tensor product $V_\alpha \otimes V_\beta$ contains V_γ as a subrepresentation if and only if there exists a (broken) Littelmann triangle in $\Delta \subset V$, with the Δ-side-lengths α, β, γ^*.*

In other words, $V_\gamma \subset V_\alpha \otimes V_\beta$ if and only if there exists an LS path p of Δ-length β such that

1. $\pi_\alpha * p \in \mathcal{P}^+$.
2. $p(1) + \alpha = \gamma$.

We will apply Theorem 5.19 as follows. Represent the vector β as the integer linear combination of fundamental coweights

$$\beta = \sum_{i=1}^{n} k_i \varpi_i.$$

We reorder the fundamental coweights so that $k_i > 0$ for all $i = 1, ..., m$ and $k_i = 0, i \geq m+1$. Set $\lambda_i := k_i \varpi_i, 1 \leq i \leq m$ and let $\underline{\lambda} = (\lambda_1, ..., \lambda_m)$. Therefore the path

$$\pi_{\underline{\lambda}} := \pi_{\lambda_1} * ... \pi_{\lambda_m}$$

belongs to $\mathcal{LS}_1$ and $\pi_{\underline{\lambda}}(1) = \beta$. Then

Corollary 5.22. *The tensor product $V_\alpha \otimes V_\beta$ contains V_γ as a subrepresentation if and only if there exists a generalized LS path p so that*

1. $\underline{\text{length}}_\Delta(p) = \underline{\lambda}$.
2. $\pi_\alpha * p \in \mathcal{P}^+$.
3. $\pi_\alpha * p(1) = \gamma$.

Proof: Set

$$q = \pi_{\underline{\lambda}}.$$

According to Theorems 5.17 and Property 2 of generalized LS paths (section 5.3), $p \in \mathcal{P}$ is a generalized LS path with $\underline{\text{length}}_\Delta(p) = \underline{\lambda}$ if and only if $p \in \mathcal{F}(q)$. Now the assertion follows from Theorem 5.19. □

Combining Corollary 4.21, Theorem 5.6 and Theorem 5.21 we obtain

Corollary 5.23. *Suppose that X is a thick Euclidean building modeled on the Coxeter complex (A, W_{aff}). Let $\alpha, \beta, \gamma^* \in L$ be dominant coweights. Suppose that $a \subset A$ is an alcove containing a special vertex o, $T = [o, x, y] \subset X$ is a geodesic triangle with the special vertices and the Δ-side lengths α, β, γ^*. Assume also that the broken side $Fold_{a,A}(\overline{xy})$ of the folded triangle*

$$Fold_{a,A}(T)$$

has breaks only at the special vertices of A. Then

1. *The folded triangle $T' = Fold_{o,id,\Delta}(T) \subset \Delta$ is a Littelmann triangle.*
2. *$V_\gamma \subset V_\alpha \otimes V_\beta$.*

Proof: Indeed, according to Corollary 4.21, the folded triangle T' satisfies the chain condition. Each break point x_i on the broken side of T' is

1. Either a special vertex, in which case it satisfies maximal chain condition by Remark 3.25, or

2. $Fold_{a,A}(\overline{xy})$ is geodesic at the point corresponding to x_i, so the chain at x_i can be chosen to be maximal by Proposition 4.19.

Hence Theorem 5.6 implies that T' is a Littelmann triangle. The second assertion now follows from Theorem 5.21. □

Of course, the assumption that the break points occur only at the special vertices is very restrictive. In section 7 we will to get rid of this assumption at the expense of dilation of the side-lengths.

6 Unfolding

The goal of this section is to establish an intrinsic characterization of folded triangles as the broken billiard triangles satisfying the chain condition. We first prove this characterization for Littelmann triangles and then, using this, give a general proof.

Throughout this section we assume that X is a thick locally compact Euclidean building modeled on the Coxeter complex (A, W), $\Delta \subset A$ is a Weyl chamber with the tip o. Let $g : X \to \Delta$ denote the folding $Fold_\Delta$.

Let $T \subset \Delta$ be a billiard triangle which is the composition

$$T = \overrightarrow{ox} \cup r \cup \overline{yo},$$

where $r(t) = p(t) + \alpha$, $\alpha = \overrightarrow{ox}$ and $p \in \mathcal{P}$ is a Hecke path. Thus T has the geodesic sides $\overline{ox}, \overline{oy}$ and the broken side r. We set $\gamma := \overrightarrow{oy}$ and let $\beta \in \Delta$ denote Δ-length of the path p.

6.1 Unfolding Littelmann triangles

Theorem 6.1. *Suppose that, in addition, T is a* Littelmann triangle*, i.e. $\alpha, \gamma \in L \subset P(R^\vee)$ and p is an LS path. Then T can be* unfolded *in X, i.e. there exists a geodesic triangle $\tilde{T} \subset X$ such that $g(\tilde{T}) = T$.*

Proof: Here is the idea of the proof: We know that billiard triangles in Δ satisfying the *simple chain condition* can be unfolded to geodesic triangles in X, see Corollary 4.23. Littelmann triangle T is billiard, satisfies the chain condition, but not necessarily the *simple chain condition.* Our goal is to approximate T by Littelmann polygons P_ϵ, $\lim_{\epsilon \to 0} P_\epsilon = T$, which satisfy the simple chain condition. We then unfold each P_ϵ to a geodesic quadrilateral $\tilde{T}_\epsilon \subset X$. Since X is locally compact, there is a convergent sequence $\tilde{T}_{\epsilon_j}$ whose limit is a geodesic triangle $\tilde{T}$ which folds to T. Below is the detailed argument.

Consider the geodesic path $\pi_\beta = \overline{ob} \in \mathcal{P}^+$. Since p is an LS path with the Δ-length β, according to Property 7 in section 5.2, there exists a composition $\phi \in \mathcal{F}$ of lowering operators so that

$$\phi(\pi_\beta) = p.$$

Let $c \subset \Delta$ denote an alcove which contains the germ of the segment $\overline{ob}$ at b. Pick a point u in the interior of $\overline{ob} \cap c$. Then for each $\epsilon > 0$ there exists a point $u_\epsilon \in int(c) \cap P(R^\vee) \otimes \mathbb{Q}$ such that

1. $|u - u_\epsilon| < \epsilon$.

2. The segments $\overline{ou_\epsilon}, \overline{u_\epsilon b}$ do not pass through any point of intersection of two or more walls (except for the end-points of these segments).

Observe that u_ϵ is a regular point in (A, W_{aff}), i.e. it does not belong to any wall.

In other words, the path

$$\hat{p}_\epsilon := \overline{ou_\epsilon} \cup \overline{u_\epsilon b} \in \mathcal{P}$$

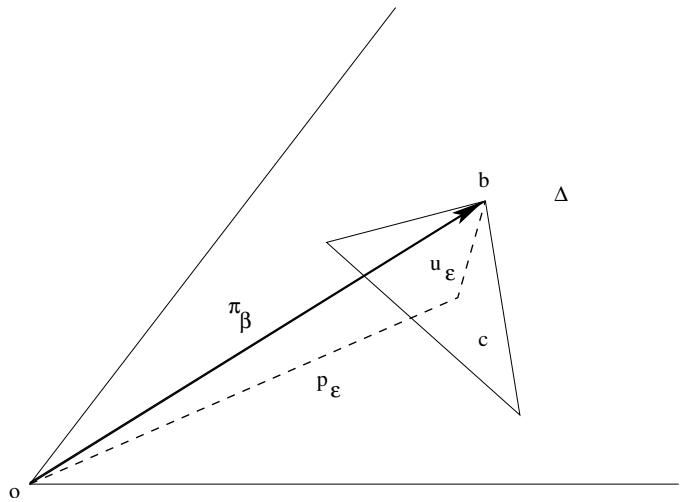

Figure 9: *Approximation.*

is *generic*. Parameterize $\hat{p}_\epsilon$ with the constant speed so that $\hat{p}_\epsilon(t_\epsilon) = u_\epsilon$. See Figure 9.

Then the path $\hat{p}_\epsilon$ belongs to $\mathcal{P}^+$; clearly it is also a generalized LS path: $\hat{p}_\epsilon \in \mathcal{LS}_2$. Moreover,

$$\lim_{\epsilon \to 0} \hat{p}_\epsilon = \pi_\beta.$$

Therefore, according to Property 6 of the root operators (see section 5.2), the operator ϕ is defined on all $\hat{p}_\epsilon$ for ϵ sufficiently small and

$$\lim_{\epsilon \to 0} \phi(\hat{p}_\epsilon) = \phi(\pi_\beta) = p.$$

Set $p_\epsilon := \phi(\hat{p}_\epsilon)$. Since $\hat{p}_\epsilon$ was generic, the path p_ϵ is generic as well. By construction, for each sufficiently small ϵ,

$$p_\epsilon(1) = p(1).$$

Observe also that the germ of the path p_ϵ at the point $p_\epsilon(t_\epsilon)$ is isomorphic (via an element of W_{aff}) to the germ of $\hat{p}_\epsilon$ at u_ϵ (since u_ϵ is regular). Similarly, p_ϵ is the composition of the path $p_\epsilon|[0, t_\epsilon]$ with the path that belongs to the W_{aff}-orbit of $\overline{u_\epsilon b}$.

For each ϵ we form a new polygon P_ϵ by replacing the broken side $r(t) = \alpha + p(t)$ (in T') with the path $\alpha + p_\epsilon(t)$. Clearly,

$$\lim_{\epsilon \to 0} P_\epsilon = T.$$

To simplify the notation we now fix $\epsilon > 0$ and let $q := p_\epsilon$.

Lemma 6.2. *For all sufficiently small ϵ, the polygon P_ϵ is contained in Δ.*

Proof: Suppose that λ is a simple root which is negative at some point of the path $\alpha + q(t)$.

Since λ is nonnegative on the limiting path $\alpha + p$, the minimum of the function $J_\lambda(t) := \lambda(q(t)), t \in [0, 1]$, converges to zero as $\epsilon \to 0$. However, as a generalized LS

path, q belongs to $\mathcal{P}_{\mathbb{Z}}$ (see Property 5 in section 5.2). Since $\alpha \in P(R^\vee)$, it follows that the minimum of $J_\lambda(t)$ is an integer. Hence it has to be equal to zero for all sufficiently small values of ϵ. Contradiction. $\square$

Since each q is a generalized LS path and $P_\epsilon \subset \Delta$, the polygon P_ϵ is a Littelmann polygon. Moreover, since q is *generic*, the polygon P_ϵ satisfies the *simple chain condition*. Thus

1. For each $t \in [0, t_\epsilon)$ either q is smooth at t or

$$(q'_-(t), q'_+(t))$$

is a chain of length 1: At $m = q(t)$ the above tangent vectors are related by a single reflection in W_m. The fixed-point set of this reflection is the unique wall passing through m.

2. The subpath $q([t_\epsilon, 1])$ in q is a geodesic segment and

$$\delta(\epsilon) := \angle(\overrightarrow{u_\epsilon o}, \overrightarrow{u_\epsilon b}) = \pi - \angle(q'_-(t(\epsilon)), q'_+(t(\epsilon))).$$

Now we are in position to apply Corollary 4.23 and unfold P_ϵ in X: For each ϵ there exists a geodesic quadrilateral $\tilde{T}_\epsilon$ (with one vertex at o) in X such that

$$g(\tilde{T}_\epsilon) = P_\epsilon.$$

Let $\tilde{z} = \tilde{z}_\epsilon$ denote the point of $\tilde{T}_\epsilon$ which maps to $z = q(t_\epsilon)$ under the folding map f. Since z is a regular point, the point $\tilde{z}$ is regular as well and the angle between the sides of $\tilde{T}_\epsilon$ at $\tilde{z}$ is the same as the angle between the sides of P_ϵ at z, i.e. equals $\delta(\epsilon)$.

Since the building X is locally compact, the sequence of quadrilaterals $\tilde{T}_\epsilon$ subconverges to a geodesic quadrilateral $T \subset X$ which is a geodesic triangle since

$$\lim_{\epsilon \to 0} \delta(\epsilon) = \pi.$$

By continuity of the folding $g : X \to \Delta$,

$$g(T) = \lim_{\epsilon \to 0} P_\epsilon = T. \quad \square$$

In the above proof we assumed that the polygon T is entirely contained in Δ. This assumption can be weakened. Let $f : X \to A$ denote the folding $Fold_{a,A}$ into the apartment A, where a is an alcove containing o. Suppose that $T \subset A$ is as above, so that $\alpha, \gamma \in P(R^\vee)$, p is a billiard path, $r = p + \alpha$. Define two subsets $J, J' \subset I = [0, 1]$:

$$J := cl(r^{-1}(int(\Delta))), \quad J' := cl(r^{-1}(int(V \setminus \Delta))).$$

Clearly, $I = J \cup J'$ and the set $J \cap J'$ is finite.

We assume that for each $t \in J$ the germ of p at t satisfies the maximal chain condition, and for each $t \in J'$ the germ of p at t is geodesic.

Theorem 6.3. *Under the above assumptions the polygon T can be unfolded to a geodesic triangle in X via the retraction f.*

Proof: Recall that unfolding of T is a local problem of behavior of the path r at the break-points (Lemma 4.8), which in our case all occur inside Δ.

We first replace T with the polygon $P = \mathbb{P}(T)$, where $\mathbb{P} = \mathbb{P}_\Delta$ is the projection of A to the Weyl chamber Δ. Then, analogously to the proof of Proposition 4.19, the new polygon P still satisfies the maximal chain condition. Therefore, according to the previous theorem, the polygon P unfolds in X via the folding map $g = Fold_\Delta : X \to \Delta$. However this means that the unfolding condition (stated in Lemma 4.8) is satisfied at each break-point of the polygon T (since the germs of r and of $\mathbb{P}(r)$ are the same). Hence the original polygon T unfolds to a geodesic triangle in X via $f : X \to A$. $\square$

Let $\underline{G}$ be a connected split semisimple algebraic group with the root system R and the cocharacter lattice L of the maximal torus $\underline{T} \subset \underline{G}$. We let $\underline{G}^\vee$ denote its Langlands' dual and set

$$G^\vee := \underline{G}^\vee(\mathbb{C}).$$

We assume that $\alpha, \beta, \gamma^* \in L$ are dominant weights of $G^\vee$ such that

$$(V_\alpha \otimes V_\beta \otimes V_{\gamma^*})^{G^\vee} \neq 0,$$

equivalently,

$$V_\gamma \subset V_\alpha \otimes V_\beta.$$

As a corollary of Theorem 6.1 we get a new proof of

Theorem 6.4 (Theorem 9.17 in [KLM3], also proven in [Ha]). *Under the above assumptions, in the thick Euclidean building X there exists a geodesic triangle with special vertices and the Δ-side-lengths α, β, γ^*. In other words,*

$$n_{\alpha,\beta}(\gamma) \neq 0 \Rightarrow m_{\alpha,\beta}(\gamma) \neq 0.$$

Proof: Since

$$V_\gamma \subset V_\alpha \otimes V_\beta,$$

according to Littelmann's Theorem 5.21, there exists a Littelmann triangle $T' \subset \Delta$, as in Theorem 6.1. Let $p \in \mathcal{P}$ denote the LS path (of the Δ-length β) representing the broken side of T'; $p = \phi(\pi_\beta)$, where $\phi \in \mathcal{F}$ is a composition of lowering operators. Thus, by Theorem 6.1, there exists a triangle $T = [o, x, y] \subset X$ such that $Fold_\Delta(T) = T'$. Therefore, by the definition of folding,

$$d_{ref}(o, x) = d_{ref}(o, x') = \alpha, \quad d_{ref}(o, y) = d_{ref}(o, y') = \gamma.$$

Since we assumed that $\alpha, \beta \in L \subset P(R^\vee)$ then x, y are special vertices of x. Since folding preserves the Δ-length,

$$d_\Delta(x, y) = \text{length}_\Delta(p) = \beta. \quad \square$$

6.2 Characterization of folded triangles

The goal of this section is to extend the results of the previous one from the case of Littelmann triangles to general broken triangles satisfying the chain condition.

Theorem 6.5. *Suppose that $p \in \mathcal{P}$ is a Hecke path, $\alpha = \overrightarrow{ou} \in \Delta$ is such that the path $q := \alpha + p$ is contained in Δ. Define the billiard triangle $T' := \overline{ou} \cup q \cup \overline{q(1)o}$. Then T' can be unfolded in X.*

Proof: The idea of the proof is that the set of unfoldable billiard paths is closed, thus it suffices to approximate p by unfoldable paths. We first prove the theorem in the case when o does not belong to the image of the path q.

According to Lemma 4.8, unfolding of a path is a purely local matter. Therefore the problem reduces to the case when q has only one break-point, $x = q(t_1)$. If p were an LS path, we would be done. In general it is not, for instance, because it might fail the *maximal chain condition*. We resolve this difficulty by *passing to a smaller Coxeter complex* and a smaller building.

Let R_x denote the root subsystem in R which is generated by the roots corresponding to the walls passing through x. This root system determines a Euclidean Coxeter complex where the stabilizer of the origin is a finite Coxeter group W'_{sph} which is conjugate to the group W_x via the translation by the vector $\overrightarrow{ox}$. Let Δ_x denote the positive Weyl chamber of (V, W'_{sph}) (the unique chamber which contains Δ). Let ξ, η and ζ denote the normalizations of the vectors $-p'_-(t_1), p'_+(t_1), \overrightarrow{xo}$.

Then, since p satisfies the chain condition, there exists an (S, W_x, Δ_x)-chain

$$(\nu_0, ..., \nu_m), \quad \nu_0 = -\xi, \quad \nu_m = \eta, \quad \nu_i = \tau_i(\nu_{i-1}), \quad 1 \le i \le m.$$

Our first observation is that although this chain may fail to be a maximal chain with respect to the unrestricted Coxeter complex (S, W_{sph}), we can assume that it is maximal with respect to the restricted Coxeter complex (S, W_x).

Next, the initial and final points of q may not belong to $P(R_x^\vee)$. Recall however that *rational points* are dense in S, see Lemma 2.4; therefore, there exists a sequence of rational points (with respect to R_x) $\xi_j \in S$ which converges to ξ. Thus, using the same reflections τ_i as before, we obtain a sequence of rational chains $(\nu_i^j), i = 0, ..., m$, where $\nu_0^j = -\xi_j, \nu_m^j = \eta_j$. We set $\zeta_j := \zeta$.

Hence for each j there exists a number $c = c_j \in \mathbb{R}_+$ so that the points

$$x_j = x + c\xi_j, \quad y_j := x + c\eta_j$$

belong to $P(R^\vee)$. We define a sequence of paths

$$q_j := \overline{x_j x} \cup \overline{x y_j} \in \tilde{\mathcal{P}}.$$

Our next goal is to choose the sequence ξ_j so that the germ of each q_j at x is contained in Δ. If x belongs to the interior of Δ then we do not need any restrictions on the sequence ξ_j. Assume therefore that x belongs to the boundary of Δ. Let F denote

the smallest face of the Coxeter complex (V, W_{sph}) which contains the point x and let H denote the intersection of all walls through the origin which contain x. It is clear that F is a convex homogeneous polyhedral cone contained in H and x belongs to the interior of F in H. If $w \in W_x$ is such that $w(-\xi) = \eta$ then w fixes H (and F) pointwise.

By Lemma 2.4, applied to the root system R_x, there exists a sequence of unit rational vectors ξ_j and positive numbers ϵ_j converging to zero so that points $x \pm \epsilon_j \xi_j$ belong to $int_H(F)$; therefore the sequence $w(x + \epsilon_j \xi_j)$ is also contained in $int_H(F)$.

Using this sequence ξ_j we define the paths q_j; clearly the germ of q_j at x is contained in $int_H(F) \subset \Delta \subset \Delta_x$.

Remark 6.6. *Note that, typically, the sequence (c_j) is unbounded and the paths q_j are not contained in Δ_x.*

We let $p_j \in \mathcal{P}$ denote the path $q_j - q_j(0)$. Then each p_j is an LS path with respect to the root system R_x: Integrality and the maximal chain condition now hold. Set

$$\lambda_j := \mathrm{length}_{\Delta_x}(p_j).$$

Remark 6.7. *Observe that,*

$$\lim_j \bar{\lambda}_j = \bar{\lambda} \in \Delta,$$

where λ is the Δ_x-length of p.

Therefore, according to Theorem 6.3 for each j the path q_j is unfoldable in a thick Euclidean building X_x modeled on the Coxeter complex

$$(A, W'_{aff}), \quad \text{where} \quad W'_{aff} = V \ltimes W_x.$$

This means that there exists a geodesic path $\tilde{q}_j$ in X_x, whose Δ_x-length is λ_j, and which projects to q_j under the folding $X_x \to A$.

Let $z_j \in \tilde{q}_j$ be the points which correspond to the point x under the folding map

$$\tilde{q}_j \to q_j.$$

Thus the "broken triangle" $[\xi_j, \zeta, \eta_j]$ in S_x unfolds in $\Sigma_{z_j}(X_x)$ into a triangle $[\tilde{\xi}_j, \tilde{\zeta}_j, \tilde{\eta}_j]$ such that

$$d_{ref}(\tilde{\xi}_j, \tilde{\zeta}_j) = d_{ref}(\xi_j, \zeta_j),$$
$$d_{ref}(\tilde{\eta}_j, \tilde{\zeta}_j) = d_{ref}(\eta_j, \zeta_j),$$
$$d(\tilde{\eta}_j, \tilde{\zeta}_j) = \pi.$$

Observe that the metric distance from o to z_j is uniformly bounded. Since X_x is locally compact, the sequence of buildings $\Sigma_{z_j}(X_x)$ subconverges to the link of a vertex $u \in X_x \subset X$.

Remark 6.8. *The spherical buildings $\Sigma_{z_j}(X_x)$, $\Sigma_u(X_x)$ have to be modeled on the same spherical Coxeter complex (S, W_x), since the structure group can only increase in the limit and the structure group at z_j was already maximal possible, i.e. W_x.*

Accordingly, the triangles $[\tilde{\xi}_j, \tilde{\zeta}_j, \tilde{\eta}_j]$ subconverge to a triangle $[\tilde{\xi}, \tilde{\zeta}, \tilde{\eta}]$ whose refined side-lengths are

$$d_{ref}(\xi, \zeta), d_{ref}(\zeta, \eta), \pi.$$

This shows that the triangle $[\xi, \zeta, \eta]$ can be unfolded in a building which is modeled on (S, W_x). We now apply the *locality* lemma 4.8 to conclude that the path q can be unfolded in X to a geodesic path. Thus the broken triangle T' unfolds to a geodesic triangle as well.

If o belongs to the image of q we argue as follows. The path q, as before, has only one break point, which in this case occurs at the origin:

$$q = \overline{zo} \cup \overline{oy}.$$

There exists an element $w \in W_{sph}$ which sends the vector $\eta = -\xi$ to ξ, where η is the normalization of the vector $\overrightarrow{oy}$. Then consider the geodesic path

$$\tilde{q} := \overline{zo} \cup w(\overline{oy}).$$

It is clear that $g(\tilde{q}) = \mathbb{P}(\tilde{q}) = q$. □

By combining Theorem 6.5 and Corollary 4.21 we obtain the following

Theorem 6.9 (Characterization of folded triangles). *A polygon $P \subset \Delta$ of the form*

$$\overline{ox} \cup (p + \alpha) \cup \overline{yo}, \text{ where } \alpha = \overrightarrow{ox}, \gamma = \overrightarrow{oy} \in \Delta,$$

can be unfolded to a geodesic triangle in X if and only if p is a Hecke path, i.e. a billiard path which satisfies the chain condition.

7 Proof of the saturation theorem

We first prove Theorem 1.6 formulated in the Introduction. Part 1 of Theorem was proven in [KLM3], so we prove Part 2. Let X be a (thick) Euclidean building of rank r modeled on a discrete Coxeter complex (A, W); the building X is the Bruhat-Tits building associated with the group $G = \underline{G}(\mathbb{K})$. Then the assumption that $m_{\alpha,\beta}(\gamma) \neq 0$ is equivalent to the assumption that there exists a geodesic triangle $T = [\tilde{x}, \tilde{y}, \tilde{z}] \subset X$, where $\tilde{x}, \tilde{y}, \tilde{z}$ are special vertices of X and whose Δ-side-lengths are $\alpha, \beta, \gamma^* \in \Delta \cap P(R^\vee)$.

Recall that there exists an apartment $\tilde{A} \subset X$ which contains the segment $\overline{\tilde{x}\tilde{y}}$. We let $\tilde{W}$ denote the affine Weyl group operating on $\tilde{A}$. Our first step is to replace the geodesic triangle T with a geodesic polygon

$$\tilde{P} := [\tilde{z}, \tilde{x} = \tilde{x}_1, ..., \tilde{x}_n, \tilde{x}_{n+1} = \tilde{y}]$$

as follows. We now treat the point $\tilde{x}$ as the origin in the affine space $\tilde{A}$. Let $\tilde{\Delta} \subset \tilde{A}$ be a Weyl chamber in $(\tilde{A}, \tilde{W})$, so that $\tilde{\Delta}$ has its the tip at $\tilde{x}$ and $\overline{\tilde{x}\tilde{y}} \subset \tilde{\Delta}$.

Consider the vectors $\varpi_1, ..., \varpi_r \in \tilde{\Delta}$ which are the fundamental coweights of our root system. Then the vector $\overrightarrow{\tilde{x}\tilde{y}}$ is the integer linear combination

$$\overrightarrow{\tilde{x}\tilde{y}} = \sum_{i=1}^{r} n_i \varpi_i, n_i \in \mathbb{N} \cup \{0\}.$$

Accordingly, we define a path $\tilde{p}$ in $\tilde{\Delta}$ with the initial vertex $\tilde{x}$ and the final vertex $\tilde{y}$ as the concatenation

$$\tilde{p} = \pi_{\underline{\lambda}} = \pi_{\lambda_1} ... * \pi_{\lambda_r} = \tilde{p}_1 \cup .. \cup \tilde{p}_r,$$

where $\lambda_i := n_i \varpi_i$, $\underline{\lambda} = (\lambda_1, ..., \lambda_r)$. Observe that the path $\tilde{p}$ satisfies the assumptions of Theorem 4.16. Moreover, $\tilde{p}$ is a generalized Hecke path.

Next, let $A \subset X$ be an apartment containing $\overline{\tilde{z}\tilde{x}}$, $a \subset A$ be an alcove containing $\tilde{z}$; consider the retraction $f = Fold_{a,A} : X \to A$. This retraction transforms $\tilde{P}$ to a polygon $\hat{P} := f(\tilde{P}) \subset A$ which has geodesic sides $f(\overline{\tilde{z}\tilde{x}})$, $f(\overline{\tilde{y}\tilde{z}})$. Note that the break-points in

$$\hat{p} := f(\tilde{p}) = \hat{p}_1 \cup ... \cup \hat{p}_r,$$

are the images of the vertices $\tilde{x}_i$ of $\tilde{P}$ which are break-points $\tilde{p}$, but in addition we possibly have break-points within the segments $f(\tilde{p}_i)$. The latter can occur only at the values of t for which the geodesic segments of $\tilde{p}_i$ intersect transversally the walls of $(\tilde{A}, \tilde{W})$. Since $\tilde{x}$ is a special vertex and the edges of each $\tilde{p}_i$ are parallel to multiples of ϖ_i, it follows that each segment $\tilde{p}_i$ is contained in the 1-skeleton of X. Thus the break-points of $f(\tilde{p}_i)$ are automatically vertices of $\tilde{A}$. We subdivide the path $\tilde{p}$ so that all break-points of $\hat{p}$ are the images of the vertices of $\tilde{P}$.

Let $k = k_R$ be the saturation constant of the root system R. Then, according to Lemma 2.9, for each vertex $v \in A$, the point $kv \in A$ is a special vertex of A. Therefore, applying a dilation $h \in Dil(A, W)$ with the conformal factor k to the polygon $\hat{P}$, we obtain a new polygon $k \cdot \hat{P} = h(\hat{P})$, whose vertices are all special vertices of A. Thus we can identify the Weyl chamber Δ with a chamber in A whose tip o is at the vertex $h(\tilde{z})$ and which contains the geodesic segment $h(\overline{\tilde{z}\tilde{x}})$.

Lastly, let $P = \mathbb{P}(k\hat{P}) = g(\tilde{P})$ denote the projection of the polygon $k\hat{P}$ to the Weyl chamber Δ, where $g = Fold_{z,h,\Delta}$. We set $x := g(\tilde{x}), y := g(\tilde{y}), x_i := g(\tilde{x}_i)$, $p := g(\tilde{p})$, etc.

Proposition 7.1. *$P = \overline{ox} \cup p \cup \overline{yo}$ is a Littelmann polygon such that*

$$\underline{\text{length}}_{\Delta}(p) = k \cdot \underline{\text{length}}_{\Delta}(\tilde{p}).$$

Proof: We have to show that the path p satisfies the chain condition with maximal chains at each vertex.

The chain condition at each vertex of the path p follows immediately from Theorem 4.16.

The maximality condition is immediate for the break-points which occur at the special vertices of A, in particular, for all break-points which are images of the break-points of $\hat{p}$. The remaining break-points are the ones which occur at the points $\mathbb{P}(\hat{x}_i)$, where $\hat{x}_i$ are smooth points of $k\hat{p}$ at which this path transversally intersects the walls of A passing through o. However at these points the maximality condition follows from Proposition 4.19.

The second assertion of the proposition was proven in Lemma 4.5. □

Remark 7.2. *Observe that in the case when β is the sum of* minuscule fundamental coweights*, the multiplication by k in the above proof is unnecessary since all the vertices of the polygon $\hat{P}$ (and hence $P = \mathbb{P}(\hat{P})$) are already special. Thus, in this case, the polygon*

$$P = Fold_{\Delta}(\tilde{P})$$

is a Littelmann polygon.

The above proposition shows that the polygon P is a Littelmann polygon in Δ, which has two geodesic sides having the Δ-lengths $k\alpha, k\gamma^*$ and the concatenation of the remaining sides equal to a generalized LS path of the Δ-length $k\beta$. Therefore, according to Littelmann's theorem (see Theorem 5.21),

$$V_{k\gamma} \subset V_{k\alpha} \otimes V_{k\beta}. \quad \square$$

This concludes the proof of Theorem 1.6. □

Corollary 7.3. *Suppose that $\alpha, \beta, \gamma \in L \subset P(R^\vee)$ are dominant weights for the complex semisimple Lie group $\underline{G}^\vee(\mathbb{C})$, such that $\alpha + \beta + \gamma \in Q(R^\vee)$ and that there exists $N \in \mathbb{N}$ so that*

$$V_{N\gamma} \subset V_{N\alpha} \otimes V_{N\beta}.$$

Then for the saturation constant $k = k_R^2$ we have

$$V_{k\gamma} \subset V_{k\alpha} \otimes V_{k\beta}.$$

Proof: Let X be the Euclidean (Bruhat-Tits) building associated to the group $\underline{G}(\mathbb{K})$. Then the assumption that

$$V_{N\gamma} \subset V_{N\alpha} \otimes V_{N\beta}$$

implies that $(N\alpha, N\beta, N\gamma^*)$ belongs to $D_3(X)$ (see [KLM3, Theorem 9.17], or [KLM3, Theorem 10.3], or Theorem 6.4). Since $D_3(X)$ is a homogeneous cone and $N > 0$, $(\alpha, \beta, \gamma^*) \in D_3(X)$ as well. Moreover, according to Theorem 2.18, since $\alpha, \beta, \gamma \in P(R^\vee)$ and $\alpha + \beta + \gamma \in Q(R^\vee)$, there exists a triangle $T \subset X$ with the Δ-side-lengths α, β, γ^*, whose vertices are also vertices of X. Now the assertion follows from Part 3 of theorem 1.6. □

Using Remark 7.2 we also obtain:

Theorem 7.4. *Let X be a building as above. Suppose that $T = [x, y, z]$ is a geodesic triangle in X with the Δ-side-lengths $(\alpha, \beta, \gamma^*)$, which are dominant weights of $G^\vee$ and so that one (equivalently, all) vertices of T are special and at least one of the weights α, β, γ is the sum of minuscule weights. Then*

$$V_\gamma \subset V_\alpha \otimes V_\beta.$$

This theorem was originally proven by Tom Haines in the case when all the weights α, β, γ are sums of minuscules.

Conjecture 7.5. *(T. Haines) Suppose that α, β, γ are sums of minuscule weights. Then, in the above theorem, the assumption that one vertex of T is special can be replaced by $\alpha + \beta + \gamma \in Q(R^\vee)$.*

Note that (among irreducible root systems) the root systems G_2, F_4, E_8 have no minuscule weights, B_n, C_n, E_7 have exactly one minuscule weight and the root systems A_n, D_n, E_6 have more than 1 minuscule weights. For the root system A_n Haines conjecture follows from the saturation theorem. For D_n and E_6 it would follow from the affirmative answer to Question 1.3.

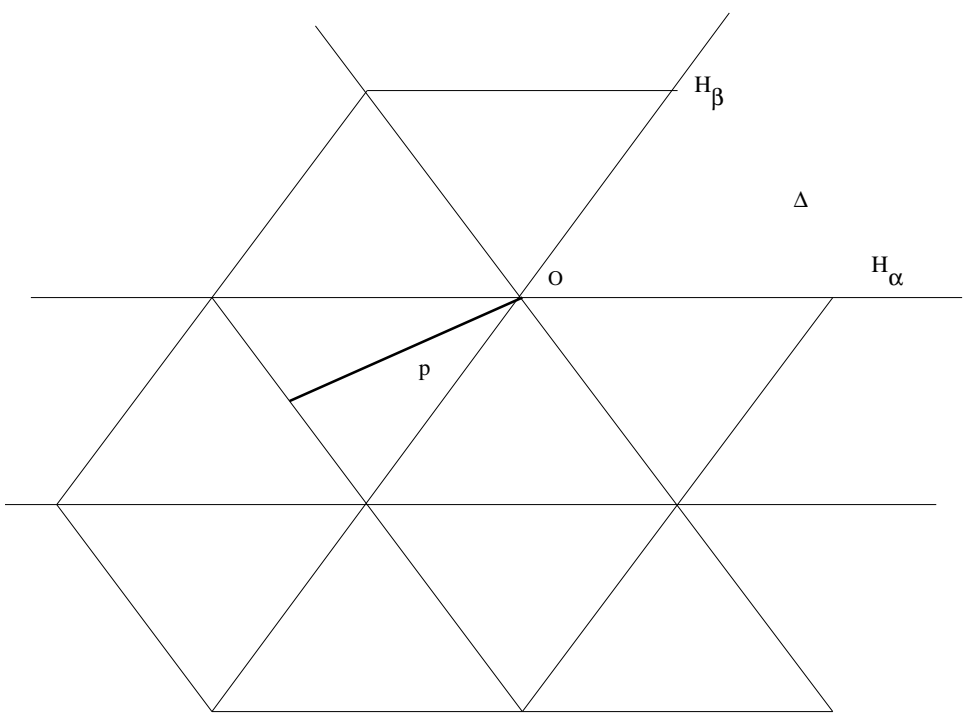

Figure 10: *A Hecke path which does not satisfy the integrality condition.*

Proposition 7.6. *Suppose that the root system R has exactly one minuscule coweight λ. Then the above conjecture holds for R.*

Proof: Let (A, W) denote the Euclidean Coxeter complex corresponding to the root system R and let X be a thick Euclidean building modeled on (A, W). Given

$(\alpha, \beta, \gamma^*) \in D_3(X)$ such that $\alpha, \beta, \gamma \in P(R^\vee), \alpha + \beta + \gamma \in Q(R^\vee)$ we have to construct a geodesic triangle $T = [o, x, y] \subset X$ with special vertices and the Δ-side lengths $(\alpha, \beta, \gamma^*)$. Clearly, it suffices to treat the case when the root system R is irreducible and spans V. Therefore R has type B_n, C_n, or E_7. In particular, the index of connection i of R equals 2 and $-1 \in W_{sph}$. In particular, $\gamma = \gamma^*$. Let λ denote the unique minuscule coweight of R and let Λ denote the span in λ in V.

Observe that λ does not belong to the coroot lattice $Q(R^\vee)$ and thus, since $i = 2$,

$$\mathbb{N} \cdot \lambda \cap Q(R^\vee) = 2\mathbb{N} \cdot \lambda.$$

Suppose now that $\alpha = a\lambda, \beta = b\lambda, \gamma = b\lambda$, where $a, b, c \in \mathbb{N} \cup \{0\}$ and

$$(\alpha, \beta, \gamma) \in D_3(X), \quad \alpha + \beta + \gamma \in Q(R^\vee).$$

Thus $a + b + c$ is an even number and the triple (a, b, c) satisfies the ordinary metric triangle inequalities.

Let $(A', W') = (\mathbb{R}, 2\mathbb{Z} \ltimes \mathbb{Z}/2)$ denote the rank 1 Coxeter complex; its vertex set equals $\mathbb{Z}$. The positive Weyl chamber in (A', W') is $\mathbb{R}_+$ and we can identify Δ'-distances with the usual metric distances. Let X' denote a thick building which is modeled on (A', W') (i.e. a simplicial tree with edges of unit length and thickness ≥ 3). Then the above properties of a, b, c imply that X' contains a triangle $T' = [o', x', y']$ with the metric side-lengths a, b, c. If this triangle is contained in a single apartment $A' \subset X'$, we send T' to a geodesic triangle $T \subset X$ via the isometry $A' \to \Lambda \to A \to X$. If not, we obtain a folded (Hecke) triangle $P = f'(T') = [o', x', u', f'(y')] \subset \Delta'$. Note that the unit tangent directions $\xi', \eta' \in S_{u'}(A')$ to the segments $\overline{u'x'}, \overline{u'f(y')}$ are antipodal. Now embed the apartment A' into $A \subset X$ via the isometry ι that sends A' to Λ, o' to o, 1 to λ (the latter is a special vertex). Then the point $u := \iota(u')$ is also a special vertex in A. We claim that the resulting broken triangle $[o, x, u, y] \subset A$ is a Hecke triangle in A. Indeed, the directions ξ, η at Σ_u which are images of ξ', η' under ι are antipodal and $\eta \in \Delta$. Therefore, since u is a special vertex and $-1 \in W_{sph}$, according to Lemma 3.12 $\xi \geq \eta$. Thus $\overline{xu} \cup \overline{uy}$ is a Hecke path. It follows that the broken triangle $[o, x, u, y]$ is a Hecke triangle and hence it unfolds to a geodesic triangle T in X. The triangle T has special vertices and Δ-side lengths (α, β, γ).

Below is an alternative to the above argument. Let $\Sigma \subset \Delta^3$ denote the collection of triples of dominant weights τ, η, μ such that

$$(V_\tau \otimes V_\eta \otimes V_\mu)^{G^\vee} \neq 0,$$

where $G^\vee$ is assumed to be simply-connected. This set is an additive semigroup, see for instance the Appendix in [KLM3]. It suffices to prove that $(\alpha, \beta, \gamma) \in \Sigma$.

Set $t := \frac{1}{2}(a + b - c)$. (The number t is the metric length of the "leg" of the geodesic triangle $T' \subset X'$ in the above argument, the leg which contains the vertex x'.) Since $a + b + c$ is even, the number t is an integer. Set

$$a_1 := a - t, b_1 := b - t, c_1 := c, \quad \alpha_1 := a_1\lambda, \beta_1 := b_1\beta, \gamma_1 := c_1\gamma.$$

Then $c_1 = a_1 + b_1$ and the metric triangle inequalities for a, b, c imply that $t \geq 0, a_1 \geq 0, b_1 \geq 0$. Thus $\alpha_1, \beta_1, \gamma_1$ are still dominant weights of $G^\vee$ and they satisfy

$$\gamma_1 = \alpha_1 + \beta_1.$$

Then, since $-1 \in W_{sph}$ and $\gamma_1 = \gamma_1^*$, we have: $(\alpha_1, \beta_1, \gamma_1) \in \Sigma$. Moreover, $(t\lambda, t\lambda, 0)$ also clearly belongs to Σ and we have

$$(\alpha, \beta, \gamma) = (\alpha_1, \beta_1, \gamma_1) + (t\lambda, t\lambda, 0) = (\alpha, \beta, \gamma). \quad \square$$

Example 7.7. *There exists a Hecke path $p \in \mathcal{P}$ such that $p(1) \in P(R^\vee)$, however for the saturation constant $k = k_R$, the path $k \cdot p$ is not an LS path.*

Proof: Our example is for the root system A_2, in which case $k = 1$. We will give an example of a Hecke path $p \in \mathcal{P}$ such that $p(1) \in P(R^\vee)$ but p does not belong to $\mathcal{P}_\mathbb{Z}$. Since, according to Theorem 5.5, each LS path belong to $\mathcal{P}_\mathbb{Z}$, it proves that p is not an LS path.

The Hecke path p in question has Δ-length $\varpi_1 + \varpi_2$, where ϖ_1, ϖ_2 are the fundamental coweights; the break-point of p occurs at the point $-(\varpi_1 + \varpi_2)/2$ where the path p backtracks back to the origin. Thus, for the simple roots α and β, the minimum of the functions $\alpha(p(t)), \beta(p(t))$ equals $-1/2$. See Figure 10. $\square$

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