Krull Dimensions of Rings of Holomorphic Functions

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Abstract. We prove that the Krull dimension of the ring of holomorphic functions of a connected complex manifold is at least the cardinality of continuum if and only if it is $>0$.

Let $R$ be a commutative ring. Recall that the Krull dimension $\dim(R)$ of $R$ is the supremum of cardinalities lengths of chains of distinct proper prime ideals in $R$. Our main result is:

Theorem 1. Let $M$ be a connected complex manifold and $H(M)$ be the ring of holomorphic functions on $M$. Then the Krull dimension of $H(M)$ either equals 0 (if and only if $H(M) = \mathbb{C}$) or is infinite, if and only if $M$ admits a nonconstant holomorphic function $M \to \mathbb{C}$. More precisely, unless $H(M) = \mathbb{C}$, $\dim H(M) \geq \aleph$, i.e., the ring $H(M)$ contains a chain of distinct prime ideals whose length has cardinality of continuum.

Our proof of this theorem mostly follows the lines of the proof by Sasane [S], who proved that for each nonempty domain $M \subset \mathbb{C}$ the Krull dimension of $H(M)$ is infinite (he did not prove that $\dim H(M) \geq \aleph$).

Remark 2. We note that Henricksen [H] was the first to prove that the Krull dimension of the ring of entire functions on $\mathbb{C}$ has cardinality at least continuum.

In our proof we will use the Axiom of Choice in two ways: (a) to establish existence of certain maximal ideals and (b) to get existence of a nonprincipal ultrafilter $\omega$ on $\mathbb{N}$ and, hence of the ordered field $^*\mathbb{R}$ of nonstandard real (or, hyperreal) numbers. The field $^*\mathbb{R}$ contains $^*\mathbb{N}$, the nonstandard natural (or hypernatural) numbers.

The field $^*\mathbb{R}$ is a certain quotient of the countable direct product $\prod_{k \in \mathbb{N}} \mathbb{R}$; we will denote the equivalence class (in $^*\mathbb{R}$) of a sequence $(x_k)$ in $\mathbb{R}$ by $[x_k]$. Accordingly, $^*\mathbb{N}$ consists of equivalence classes $[n_k]$ of sequences of natural numbers. Roughly speaking, we will use $^*\mathbb{N}$ and a certain order relation on it to compare rates of growth of sequences of natural numbers.

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Definition 3. A commutative unital ring $R$ is ample if there exists a sequence of valuations $\nu_k$ on $R$ such that for each $\beta \in \ast \mathbb{N}$, there exists an $a = a_\beta \in R$ with the property

\[(\nu_k(a)) = \beta.\]

The main technical result of this paper is:

Theorem 4. For each ample ring $R$, $\dim(R) \geq \epsilon$. In particular, $R$ has infinite Krull dimension.

This theorem and its proof are inspired by Theorem 2.2 of [S], although some parts of the proof resemble the ones of [H].

We will verify, furthermore, that whenever $M$ is a connected complex manifold which has a nonconstant holomorphic function, the ring $H(M)$ is ample. This, combined with Theorem 4, will immediately imply Theorem 1.

Remark 5. 1. We refer the reader to Section 5.3 of [Cla] for further discussion of algebraic properties of rings of holomorphic functions and for a shorter proof of Theorem 4, which uses ultralimits and but not the hyperreal numbers.

2. Theorem 1 shows that for every Stein manifold $M$ (of positive dimension), the ring $H(M)$ has infinite Krull dimension. In particular, this applies to any noncompact connected Riemann surfaces (since every such surface is Stein, [BS]).

3. Noncompact connected complex manifolds $M$ of dimension $> 1$ can have $H(M) = \mathbb{C}$; for instance, take $M$ to be the complement to a finite subset in a compact connected complex manifold (of dimension $> 1$).

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1. Hyperreal numbers

We refer the reader to [Go] for a detailed treatment of hyperreal numbers, but we include a brief introduction below. A nonprincipal ultrafilter on $\mathbb{N}$ can be regarded as a finitely-additive probability measure on $\mathbb{N}$ which vanishes on each finite subset and takes the value 0 or 1 on each subset of $\mathbb{N}$. The existence of nonprincipal ultrafilters (the ultrafilter lemma) follows from the Axiom of Choice. Subsets of full measure are called $\omega$-large. Using $\omega$, one defines the following equivalence relation on the product

\[\prod_{k \in \mathbb{R}} \mathbb{R}.\]

Two sequences $(x_k)$ and $(y_k)$ are equivalent if $x_k = y_k$ for an $\omega$-all $k$, i.e., the set

\[\{k : x_k = y_k\}\]

is $\omega$-large. The quotient by this equivalence relation, denoted

\[\ast \mathbb{R} = \prod_{k \in \mathbb{N}} \mathbb{R}/\omega,\]
is the set of hyperreal numbers. Let $[x_k]$ be the equivalence class of the sequence $(x_k)$. The binary operations on sequences of real numbers project to binary operations on $^\ast \mathbb{R}$ making $^\ast \mathbb{R}$ a field. The total order $\leq$ on $^\ast \mathbb{R}$ is defined by $[x_k] \leq [y_k]$ if and only if $x_k \leq y_k$ for an $\omega$-all $k \in \mathbb{N}$. With this order, $^\ast \mathbb{R}$ becomes an ordered field.

The set of real numbers embeds into $^\ast \mathbb{R}$ as the set of equivalence classes of constant sequences; the image of a real number $x$ under this embedding is still denoted by $x$. We set $^\ast \mathbb{R}^+ := \{ \alpha \in ^\ast \mathbb{R} : \alpha > 0 \}$.

The projection of $\prod_{k \in \mathbb{N}} \mathbb{N} \subset \prod_{k \in \mathbb{N}} \mathbb{R}$ to $^\ast \mathbb{R}$ is denoted $^\ast \mathbb{N}$, this is the set of hypernatural numbers. We define a further equivalence relation $\sim_u$ on $^\ast \mathbb{R}$ by:

$$\alpha \sim_u \beta$$

if there exist positive real numbers $a, b$ such that $a \alpha \leq \beta \leq b \alpha$.

The equivalence class $(\alpha)$ of $\alpha \in ^\ast \mathbb{R}$ (for this equivalence relation) is a multiplicative analogue of the galaxy $\text{gal}(\alpha)$ of $\alpha$, see [Go]:

**Definition 6.** The galaxy $\text{gal}(\alpha)$ of a hyperreal number $\alpha \in ^\ast \mathbb{R}$ is the union

$$\bigcup_{n \in \mathbb{N}} [\alpha - n, \alpha + n] \subset ^\ast \mathbb{R}.$$  

In other words, $\beta \in \text{gal}(\alpha)$ if and only if there exists a real number $a$ such that $\alpha - a \leq \beta \leq \alpha + a$.

The next lemma is immediate:

**Lemma 7.** For $\alpha \in ^\ast \mathbb{R}^+$, the equivalence class $(\alpha)$ of $\alpha$ equals $\exp(\text{gal}(\log(\alpha)))$.

We let $^\ast \mathbb{R}$ denote the quotient $^\ast \mathbb{R} / \sim_u$ and $^\ast \mathbb{N}$ the projection of $^\ast \mathbb{N}$ to $^\ast \mathbb{R}$. Define the total order $\gg$ on $^\ast \mathbb{R}$ by

$$(\beta) \gg (\alpha)$$

if for every real number $c$, $c \alpha < \beta$. By abusing the notation, we will simply say that $\beta \gg \alpha$, with $\alpha, \beta \in ^\ast \mathbb{R}$.

For the reader who prefers to think in terms of sequences of (positive) real numbers, the relation $(\beta) \gg (\alpha)$ is an analogue of the relation

$$(a_n) = o((b_n)), \quad n \to \infty.$$  

**Remark 8.** The equivalence relation $\sim_u$ and the order $\gg$ are similar to the ones used by Henricksen in [H].

**Proposition 9.** The set $^\ast \mathbb{N}$ has the cardinality of continuum.

**Proof.** Note first that $^\ast \mathbb{R}$ has cardinality of continuum, hence, the cardinality of $^\ast \mathbb{N}$ is at most $\mathfrak{c}$. The proof of the proposition then reduces to two lemmata.

**Lemma 10.** The set $\text{gal}(^\ast \mathbb{R}^+)$ of galaxies $\{ \text{gal}(\alpha) : \alpha \in ^\ast \mathbb{R}^+ \}$ has the cardinality of continuum.
Proof. For each \( \alpha = [a_k] \in {^*}\mathbb{R}_+ \), the galaxy \( gal(\alpha) \) contains the hypernatural number \( \lceil \alpha \rceil = [b_k] \), where \( b_k = [a_k] \). For each hypernatural number \( \beta \in {^*}\mathbb{N} \), and natural number \( n \in \mathbb{N} \), the intersection

\[
[\beta - n, \beta + n] \cap {^*}\mathbb{N}
\]

is finite, equal \{\( \beta - n, ..., \beta + n \)\}. Therefore, \( gal(\beta) \cap {^*}\mathbb{N} = \{\beta\} + \mathbb{Z} \). It follows that the map

\[
{^*}\mathbb{N} \to gal({^*}\mathbb{R}_+), \quad \beta \mapsto gal(\beta)
\]

is a bijection modulo \( \mathbb{Z} \). Lastly, the set of hypernatural numbers \( {^*}\mathbb{N} \) has the cardinality of continuum.

Lemma 11. The map \( \lambda : {^*}\mathbb{N} \to gal({^*}\mathbb{R}_+) \), \( \lambda : \beta \mapsto gal(\log(n)) \), is surjective.

Proof. For each \( \alpha \in {^*}\mathbb{R}_+ \) let \( \beta = \lceil \exp(\alpha) \rceil \in {^*}\mathbb{N} \). Since \( \log(x+1) - \log(x) \leq 1 \) for \( x \geq 1 \), we have that

\[
\log(\beta) \in gal(\alpha).
\]

Now, we can finish the proof of the proposition. The map \( \lambda : {^*}\mathbb{N} \to gal({^*}\mathbb{R}_+) \) descends to a map \( \mu : {^*}\mathbb{N} \to gal({^*}\mathbb{R}_+) \). According to Lemma 11, the map \( \mu \) is surjective. By Lemma 10 the set \( gal({^*}\mathbb{R}_+) \) has the cardinality of continuum.

We will prove Theorem 4 in the next section by showing that for each ample ring \( R \), the ordered set \( (\mathbb{N}, \gg) \) embeds into the poset of prime ideals in \( R \) reversing the order:

\[
(\beta) \gg (\alpha) \Rightarrow P_{\beta} \subsetneq P_{\alpha}
\]

for certain prime ideals \( P_{\gamma} \subset R \) determines by \( (\gamma) \in {^*}\mathbb{N} \). Proposition 9 will then imply that the Krull dimension of \( R \) is at least \( c \).

2. Krull dimension of ample rings

Recall that a valuation on a unital ring \( R \) is a map \( \nu : R \to \mathbb{R}_+ \cup \{\infty\} \) such that:

1. \( \nu(a + b) \geq \min(\nu(a), \nu(b)) \),
2. \( \nu(ab) = \nu(a) + \nu(b) \),
3. \( \nu(a) = \infty \iff a = 0 \).
4. \( \nu(1) = 0 \).

For the following lemma, see Theorem 10.2.6 in [Coh] (see also Proposition 4.8 of [Cla] or Theorem 1 in [K]).

Lemma 12. Let \( I \) be an ideal in a commutative ring \( A \) and \( M \subset A \setminus I \) be a subset closed under multiplication. Then there exists an ideal \( J \subset A \) containing \( I \) and disjoint from \( M \), so that \( J \) is maximal with respect to this property. Furthermore, \( J \) is a prime ideal in \( A \).

Let \( R \) be an ample ring and \( \nu_k \) the corresponding sequence of valuations on \( R \). For each \( \beta \in {^*}\mathbb{N} \) we define

\[
I\beta := \{a \in R| [\nu_k(a)] \gg [\beta]\} \subset R.
\]

Lemma 13. Each \( I\alpha \) is an ideal in \( R \).
Proof. We will check that $I_\alpha$ is additive since it is clearly closed under multiplication by elements of $R$. Take $p', p'' \in I_\alpha$.

$$[\nu_k(p')] \gg \alpha, [\nu_k(p'')] \gg \alpha.$$ 

By the definition of a valuation,

$$n_k := \nu_k(p' + p'') \geq \min(\nu_k(p'), \nu_k(p'')),$$

for each $k \in \mathbb{N}$. For $m \in \mathbb{N}$, define the $\omega$-large sets

$$A' = \{k : \nu_k(p') \geq m\alpha\}, \quad A'' = \{k : \nu_k(p'') \geq m\alpha\}.$$ 

Therefore, their intersection $A = A' \cap A''$ is $\omega$-large as well, which implies that

$$\forall m \in \mathbb{N}, [n_k] \geq m\alpha \Rightarrow [n_k] \gg \alpha.$$ 

Then for each $\gamma \gg \beta$, the element $a_\gamma$ as in Definition 3, belongs to $I_\beta$. It follows that $I_\beta \neq 0$ for every $\beta$. Define the subsets

$$M_\beta := \{a \in R | \exists n \in \mathbb{N}, [\nu_k(a)] \leq n\beta \} \subset R;$$

each $M_\beta$ is closed under the multiplication. It is immediate that whenever $\alpha \leq \beta$,

$$I_\beta \subset I_\alpha, \quad M_\alpha \subset M_\beta.$$ 

It is also clear that $I_\beta \cap M_\beta = \emptyset$. At the same time, for each $\beta \gg \alpha$,

$$a_\beta \in I_\alpha \cap M_\beta.$$ 

For each $\alpha$ we let $\mathcal{J}_\alpha$ denote the set of ideals $P \subset R$ such that

$$I_\alpha \subset P, P \cap M_\alpha = \emptyset.$$ 

By Lemma 12, every maximal element $P \in \mathcal{J}_\alpha$ is a prime ideal.

Lemma 14. Every $\mathcal{J}_\alpha$ contains unique maximal element, which we will denote $P_\alpha$ in what follows.

Proof. Suppose that $P', P''$ are two maximal elements of $\mathcal{J}_\alpha$. We define the ideal $P = P' + P''$. Clearly, $P$ contains $I_\alpha$. To prove that $P$ is disjoint from $M_\alpha$, take $p' \in P', p'' \in P''$, since $p' \notin M_\alpha, p'' \notin M_\alpha$. Then the same proof as in Lemma 13 shows that $[\nu_k(p' + p'')] \gg \alpha$ which means that $p' + p'' \notin M_\alpha$. Thus, $P \in \mathcal{J}_\alpha$ and, in view of maximality of $P', P''$, we obtain

$$P' = P = P''.$$ 

For each $\beta \gg \alpha$ we define the ideal $Q_{\alpha, \beta} := I_\alpha + P_\beta$.

Lemma 15. $Q_{\alpha, \beta} \cap M_\alpha = \emptyset$.

Proof. The proof is similar to the one of the previous lemma. Let $q = c + p, c \in I_\alpha, p \in P_\beta$. Since $p \notin M_\beta, p \notin M_\alpha$ as well. Therefore,

$$[\nu_k(p)] \gg \alpha.$$ 

Since $c \in I_\alpha$,

$$[\nu_k(c)] \gg \alpha.$$ 

Hence,

$$[\nu_k(c + p)] \gg \alpha.$$
as well. Thus, \( q \notin M_{\alpha} \).

**Corollary 16.** \( Q_{\alpha\beta} \in J_\alpha \). In particular, \( Q_\alpha \subset P_\alpha \).

**Proof.** It suffices to note that \( I_\alpha \subset Q_{\alpha\beta} \) according to the definition of \( Q_{\alpha\beta} \).

**Lemma 17.** The inequality \( \beta \gg \alpha \) implies \( P_\beta \subset P_\alpha \) and this inclusion is proper.

**Proof.** By the definition of \( Q_{\alpha\beta} \) and Corollary 16, we have the inclusions

\[
P_\beta \subset Q_\alpha \subset P_\alpha.
\]

We now claim that \( P_\beta \neq Q_{\alpha\beta} = I_\alpha + P_\beta \). Recall that \( a_\alpha \in I_\alpha \subset Q_{\alpha\beta} \) and \( a_\alpha \in M_\beta \), while \( M_\beta \cap P_\beta = \emptyset \). Thus, \( a_\alpha \in Q_{\alpha\beta} \setminus P_\beta \).

According to Proposition 9, the set \( *N \) of hypernatural numbers contains a subset \( S \) of cardinality continuum such that for all \( \alpha < \beta \) in \( S \), we have \( \beta \gg \alpha \). The map

\[
\alpha \mapsto P_\alpha
\]
sends each \( \alpha \in S \) to a prime ideal in \( R \); \( \alpha < \beta \) implies that \( P_\beta \subsetneq P_\alpha \).

We conclude that the ring \( R \) contains the (descending) chain of distinct prime ideals \( P_\alpha, \alpha \in S \); the length of this chain has the cardinality of continuum. In particular, \( \dim(R) \geq c \). Theorem 4 follows.

### 3. Ampleness of rings of holomorphic functions

We will need the following classical result, see e.g. [Con, Ch. VII, Theorem 5.15]:

**Theorem 18.** Let \( D \subset \mathbb{C} \) be a domain, and let \( c_k \in D \) be a sequence which does not accumulate anywhere in \( D \) and let \( m_k \) be a sequence of natural numbers. Then there exists a holomorphic function \( g \) in \( D \) which has zeroes only at the points \( c_k \) and such that \( m_k \) is the order of zero of \( g \) at \( c_k, \ k \in \mathbb{N} \).

**Corollary 19.** If \( M \) is a connected complex manifold which admits a non-constant holomorphic function \( h : M \to \mathbb{C} \), then the ring \( H(M) \) is ample.

**Proof.** We let \( D \) denote the image of \( h \). Pick a sequence \( c_k \in D \) which converges to a point in \( \hat{\mathbb{C}} \setminus D \) and which consists of regular values of \( h \). (Here \( \hat{\mathbb{C}} \) is the Riemann sphere.) For each \( c_k \) the preimage \( C_k := h^{-1}(c_k) \) is a complex submanifold in \( M \); in each \( C_k \) pick a point \( b_k \). Define valuations

\[
\nu_k : H(M) \to \mathbb{Z}_+ \cup \{\infty\}
\]

by \( \nu_k(f) := ord_{b_k}(f) \), the total order of \( f \) at \( b_k \), cf. [Gu, Chapter C, Definition 1].

Now, given \( \beta \in *\mathbb{N}, \beta = [m_k] \), we let \( g = g_\beta \) denote a holomorphic function on \( D \) as in Theorem 18. Define \( a = a_\beta := g \circ h \in H(M) \). Then \( \nu_k(a) = m_k \), which implies that the ring \( H(M) \) is ample.

Ampleness of \( H(M) \) together with Theorem 4 imply Theorem 1.
References


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