A note on laminations with hyperbolic leaves

Michael Kapovich

July 28, 2017

This note is motivated by a talk by Alberto Verjovsky on solenoidal manifolds in Cuernavaca in 2017 and the preprint [8] of Sullivan and Verjovsky. The main result is Theorem 6 below describing $n$-dimensional homogeneous solenoidal manifolds with real-hyperbolic leaves. (Similar results hold for laminations whose leaves are other nonpositively curved symmetric spaces.)

Acknowledgements. This work was partially supported by the NSF grant DMS-16-04241.

An $n$-dimensional topological solenoidal lamination is a Hausdorff 2-nd countable topological space $L$ equipped with a system of homeomorphisms $\phi_\alpha : U_\alpha \subset L \to V_\alpha \times C_\alpha \subset \mathbb{R}^n \times C$, where each $C$ is a Cantor set and $U_\alpha$ is open in $L$, while $V_\alpha \times C_\alpha$ is open in $\mathbb{R}^n \times C$. In what follows, by a map of two laminations we will mean a continuous map sendings leaves to leaves. A Riemannian lamination is a lamination equipped with continuous leafwise Riemannian metric. On such a lamination we defined the leafwise Riemannian distance function. Similarly, one defines simplicial laminations, i.e. laminations which admit continuous structure of leafwise simplicial complexes. We equip such laminations with leafwise path-metrics induced by euclidean metrics on simplices where each simplex is declared to be isometric to a euclidean simplex with unit edges.

Lemma 1. Let $L, L'$ be two compact Riemannian laminations. Then for every continuous map $f : L \to L'$ of the laminations and $\epsilon > 0$ there exists a smooth map $f_\epsilon : L \to L'$ of laminations such that $d(f, f_\epsilon) < \epsilon$ in the sense that for each $x \in L$, $d(f(x), f_\epsilon(x)) < \epsilon$, where the $d$ is the leafwise Riemannian distance function.

Proof. Note first that the usual proof of the smooth approximation theorem for manifolds, which uses Whitney embedding theorem, does not apply. However, there is an alternative argument, see [11] which uses only local charts and it goes through for maps of laminations.
Lemma 2. Let $L, L'$ be compact simplicial laminations. Then for every continuous map $f : L \to L'$ of the laminations and $\epsilon > 0$ there exists a PL map $f_\epsilon : L \to L'$ of laminations such that $d(f, f_\epsilon) < \epsilon$.

Proof. Use PL approximation of continuous maps. \hfill \Box

Lemma 3. Let $L$ be a compact solenoidal Riemannian lamination. Then there exists a compact simplicial lamination $L'$ and a homotopy-equivalence $L \to L'$ which yields a quasiisometry between the leaves of the respective laminations.

Proof. Start with a finite cover $\mathcal{U}$ of $L$ which induces covers by convex regions on each leaf. By subdividing this cover further and using the fact that $L$ is solenoidal, we can assume that the nerve of this cover is locally constant in the transverse directions. Therefore, the nerve defines a simplicial lamination $L'$. The standard map-to-nerve construction obtained via a partition of unity corresponding to $\mathcal{U}$, yields the desired homotopy-equivalence. \hfill \Box

Combining Lemmata 2 and 3 we obtain:

Corollary 4. Let $L, L'$ be two compact solenoidal laminations where $L$ is simplicial and $L'$ is smooth or vice versa. Assume that $L, L'$ are homotopy equivalent. Then the respective leaves of $L, L'$ are quasiisometric.

Given a compact topological $n$-dimensional manifold $M$ with residually finite fundamental group, we consider the inverse system of finite simplicial coverings

$$M_i \to M.$$ 

The inverse limit of this system is a compact solenoidal lamination with leaves homeomorphic to the universal cover $\tilde{M}$ of $M$. We will refer to such lamination as $M_\infty$.

Lemma 5. Suppose that $L$ is a compact Riemannian lamination and $M_\infty$ is a topological lamination as above homeomorphic to $L$. Then each leaf of $L$ is quasiisometric to the fundamental group of $M$.

Proof. For simplicity, we first consider the case when $M$ admits a smooth structure. We equip $M$ with a Riemannian metric; the associated lamination $M_\infty$ then becomes Riemannian. Consider a homeomorphism $h : M_\infty \to L$. Using the approximation lemma 1, we approximate $h$ and $h^{-1}$ by smooth maps $h_\epsilon, \tilde{h}_\epsilon$. For every leaf $\mu$ of $M_\infty$ and $\lambda = h(\mu)$, we obtain that the maps $h_\epsilon, \tilde{h}_\epsilon$ are Lipschitz and satisfy

$$d(\tilde{h}_\epsilon \circ h_\epsilon, id) \leq C, d(h_\epsilon \circ \tilde{h}_\epsilon, id) \leq C.$$
for some $C < \infty$. Therefore, $h_\epsilon$ is a quasiisometry. Since the universal cover $\mu$ of $M$ is quasiisometric to $\pi_1(M)$, lemma follows.

Now consider the general case. Even though the manifold $M$ is not necessarily smoothable, it is homotopy equivalent to a finite simplicial complex $M'$. Then $M'_\infty$ has leaves quasiisometric to $\pi_1(M)$, $M'_\infty$ is homotopy equivalent to $M_\infty$ and, hence, to $L$. Now, the statement follows from Corollary 4.

**Theorem 6.** Let $L$ be a compact solenoidal $n$-dimensional Riemannian lamination with leaves isometric to $\mathbb{H}^n$ and $\text{Homeo}(L)$ acting transitively on $L$. Then there exists a closed aspherical $n$-manifold $M$ such that:

If $n \neq 4$ then $M$ is hyperbolic and if $n = 4$ then $M$ is homotopy equivalent to an $n$-dimensional hyperbolic manifold, and the lamination $M_\infty$ is homeomorphic to $L$.

**Proof.** According to [1], there exists a closed $n$-manifold $M$ such that $M_\infty$ is homeomorphic to $L$. In particular, $M$ is aspherical and its fundamental group $\pi$ is torsion-free. According to Lemma 5, $\pi$ is quasiisometric to $\mathbb{H}^n$. Applying quasiisometric rigidity for uniform lattices in $\mathbb{H}^n$, $n \geq 1$ (which is a combination of work of Tukia [9, 10], Gabai [5], Casson-Jungreis [2], Sullivan [7]) we conclude that $\pi$ is isomorphic to the fundamental group of a closed hyperbolic $n$-manifold $M'$. (See also [6] and [3, Chapter 23].) (Note that for a general finitely generated group $\pi$ quasiisometric to $\mathbb{H}^n$ we can only conclude that $\pi$ is virtually isomorphic to the fundamental group of $M'$, but, in our setting, since $\pi$ is torsion-free, we obtain an isomorphism.) We conclude that $M$ is homotopy-equivalent to $M'$. In dimensions different from 4, a closed $n$-dimensional manifold homotopy-equivalent to a hyperbolic manifold is actually homeomorphic to it: In dimension 3 it is a corollary of Perelman’s Geometrization Theorem, in dimensions $> 4$ this result is due to Farrell and Jones, [4].

**References**


