On the Moduli Space of a Spherical Polygonal Linkage

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Abstract

We give a “wall-crossing” formula for computing the topology of the moduli space of a closed $n$-gon linkage on $S^2$. We do this by determining the Morse theory of the function $\rho_n$ on the moduli space of $n$-gon linkages which is given by the length of the last side—the length of the last side is allowed to vary, the first $(n-1)$ side-lengths are fixed. We obtain a Morse function on the $(n-2)$-torus with level sets moduli spaces of $n$-gon linkages. The critical points of $\rho_n$ are the linkages which are contained in a great circle. We give a formula for the signature of the Hessian of $D^2\rho_n$ at such a linkage in terms of the number of back-tracks and the winding number. We use our formula to determine the moduli spaces of all regular pentagonal spherical linkages.

1. Introduction

Our goal in this paper is to give a “wall-crossing” formula for determining the topology of the moduli space of a closed $n$-gon linkage on $S^2$. We will give definitions in §2. Also the definitions of the configuration space and the moduli space $M(\Lambda, X)$ of a general linkage $\Lambda$ in a constant curvature space $X$ are given in [KM3].

Let $r = (r_1, r_2, \ldots, r_n)$ be an $n$-tuple of real numbers satisfying $0 < r_i < \pi$. Let $N_r'$ be the moduli space of the free $(n-1)$-gon spherical linkage with side-lengths $r' := (r_1, \ldots, r_{n-1})$, so $N_r'$ is the quotient by $SO(3)$ of the subspace $\tilde{N}_r' \subset (S^2)^n$ defined by

$$\tilde{N}_r' = \{ u = (u_1, \ldots, u_n) \in (S^2)^n : d(u_i, u_{i+1}) = r_i, \ 1 \leq i \leq n - 1 \}.$$
Here $d$ is the spherical distance. The points $u_1, u_2, \ldots, u_n$ are called the vertices of the linkage $T \in N_r$. Clearly $N_r \cong (S^1)^{n-2}$. We will study the Morse theory of the function $ho_n : N_r \to \mathbb{R}$ given by

$$\rho_n(u) = d(u_1, u_n).$$

We will restrict to $u$’s such that $0 < \rho_n(u) < \pi$ so that $\rho_n$ is differentiable. Notice that

$$M_r := \rho_n^{-1}(r_n) \subset N_r,$$

is the moduli space of closed polygonal linkages in $S^2$ with the side-lengths $(r_1, \ldots, r_n)$.

**Definition.** We define the closed n-gon linkage $P = P(T)$ associated to a free $(n-1)$-gon linkage $T$ to be the linkage obtained by adding the length-minimizing geodesic segment\(^1\) $(u_n, u_1) = e_n \subset S^2$ joining $u_n$ to $u_1$.

Thus $r_n$ is the length of the new edge $e_n$. Hence, in terms of deformations of the closed $n$-gon $P$ in $S^2$, we can describe $N_r$ by fixing the lengths of the first $n-1$ sides and letting the length of the last side vary.

In order to state the Main Theorem we will need some definitions.

**Definition.** A linkage in $S^2$ is degenerate if it lies in a great circle $\gamma$ of $S^2$.

Suppose now that $P$ is a degenerate closed $n$-gon linkage contained in a great circle $\gamma$. We orient $\gamma$ and define $\epsilon_i \in \{\pm 1\}$ to be 1 if the orientation of the $i$-th edge of $P$ agrees with that of $\gamma$ and $-1$ otherwise. We say that the $i$-th edge of $P$ is a forward-track if $\epsilon_i = 1$ and a back-track otherwise. We let $f = f(P)$ be the number of forward-tracks and $b = b(P)$ be the number of back-tracks so $f + b = n$. Define the winding number $w = w(P)$ by

$$\sum_{i=1}^{n} \epsilon_i r_i = 2\pi w.$$

The numbers $b, f$ and $w$ depend on the orientation of $\gamma$. We will deal with this below.

We will see that the critical points of $\rho_n$ on $N_r$ are the degenerate linkages. If $T$ is a degenerate free $(n-1)$-gon linkage our goal is to give a formula for the signature of the Hessian $D^2\rho_n|_T$ in terms of $b(P), f(P)$ and $w(P)$ where $P = P(T)$ is the associated closed $n$-gon linkage (see above). Clearly we must give a rule for orienting the great circle $\gamma \supset T$.

**Definition (orienting $\gamma$).** Suppose $u = (u_1, u_2, \ldots, u_n)$ is a closed degenerate linkage contained in a great circle $\gamma$. Orient $\gamma$ so that the arc joining $u_1$ to $u_n$ is positively directed. Thus an edge $e_i$ is a back-track if it has the same direction as $e_n = (u_n, u_1)$.

We will prove the following theorem (with $b, f$ and $w$ defined using the above orientation of $\gamma$).

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\(^1\)In what follows $(a, b)$ will always denote the shortest geodesic segment connecting non-antipodal points $a, b$ in $S^2$. 

Main Theorem. Let $T \in N_r$ be a degenerate free $(n-1)$-gon linkage and $P$ be the associated degenerate closed $n$-gon linkage. Then the signature of $D^2 \rho_n|_T$ is

$$(b(P) + 2w(P) - 1, f(P) - 2w(P) - 1).$$

Remark. The analogue of the Main Theorem for polygonal linkages in the Euclidean plane was proved in Lemma 11 of [KM1].

The Main Theorem reduces the computation of the moduli spaces of spherical polygonal linkages to the combinatorics of the chambers of the polyhedron $D_n(S^2)$ (see § 2). These computations are manageable for $n = 4, 5, 6$ but become formidable for $n \geq 7$. In [G] the moduli spaces of all spherical $n$-gons for $n = 4, 5, 6$ are determined. In this paper we illustrate the wall-crossing formula by computing the moduli spaces of regular spherical pentagons.

This paper depends on the result of [KM2] that $\rho_n$ is a Morse function. This result is what underlies the deformation arguments in Lemma 5.4 and Lemma 5.6. This paper completes the partial computation of the signature of $D^2 \rho_n$ in Theorem 8.10 of that paper. In the appendix to this paper we patch up an error in [KM2] which allows us to apply the results of that paper that we need here.

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2. Preliminaries

Definition 2.1 A closed spherical $n$-gon $P = (e_1, \ldots, e_n)$ is an $n$-tuple of oriented geodesic arcs $e_j$ (in $S^2$) of lengths between 0 and $\pi$ (inclusive) such that the endpoint of $e_{i-1}$ is equal to the initial point of $e_i$, $0 \leq i \leq n$ (the indices are taken modulo $n$).

Definition 2.2 Let $P_n(S^2)$ be the space of closed $n$-gons on $S^2$ with geodesic edges.

We let $r_i$ be the length of $e_i$ in the spherical metric. The arcs $e_1, \ldots, e_n$ will be called the edges of $P$. We will use $u = (u_1, \ldots, u_n)$ to denote the set of vertices of $P$, that is, the set of initial points of the edges $e_i$. We will soon restrict ourselves to $n$-gons $P$ with the property that $0 < r_i < \pi$, $1 \leq i \leq n$. In this case $P$ is determined by its vertices $u_1, \ldots, u_n$ and we may write $P = u = (u_1, \ldots, u_n)$. 
Definition 2.3 Let \( \rho : \mathcal{P}_n(S^2) \to (\mathbb{R}_+)^n \) defined by \( \rho(u) = r = (r_1, \ldots, r_n) \) be the side length map. That is, the distances, \( d(u_i, u_{i+1}) \) in the spherical metric satisfy \( d(u_i, u_{i+1}) = r_i \) for \( 1 \leq i \leq n \) where we consider \( u_{n+1} = u_1 \).

Definition 2.4 \( D_n(S^2) = \rho(\mathcal{P}_n(S^2)) \) is the space of possible side lengths. We let \( \tilde{M}_r := \rho^{-1}(r) \) be the configuration space of closed \( n \)-gon linkages in \( S^2 \) with the side-lengths \( r \).

It is immediate that \( \tilde{M}_r \) is the set of real points of the affine variety over \( \mathbb{R} \) (i.e. \( \tilde{M}_r \) is a real algebraic set) defined by

\[
u_i \cdot u_{i+1} = \cos r_i, \quad 1 \leq i \leq n,
\]

where \( \overrightarrow{y} \cdot \overrightarrow{y} \) denotes the scalar product in \( \mathbb{R}^3 \). The group \( SO(3) \) acts on \( \tilde{M}_r \) according to

\[g(u) = (gu_1, \ldots, gu_n), \quad u \in \tilde{M}_r, \ g \in SO(3).\]

Definition 2.5 The moduli space \( M_r \) of \( n \)-gon linkages on \( S^2 \) with side lengths \( r = (r_1, \ldots, r_n) \) is defined to be the quotient space of \( \tilde{M}_r \) by \( SO(3) \).

We now prove that \( M_r \) has the structure of a real algebraic set–here we assume \( 0 < r_i < \pi, 1 \leq i \leq n \). Let \( \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3 \) denote the standard basis of \( \mathbb{R}^3 \).

Lemma 2.6 Define \( \Sigma_r \subset \tilde{M}_r \) by \( \Sigma_r = \{ u \in \tilde{M}_r : u_1 = \overrightarrow{e}_1, \ u_n = \cos r_n \overrightarrow{e}_1 + \sin r_n \overrightarrow{e}_2 \} \).

Then \( \Sigma_r \) is a cross-section to the orbits of \( SO(3) \) on \( \tilde{M}_r \).

Proof. Obvious. \( \square \)

Since the quotient map \( \tilde{M}_r \to M_r \) induces a homeomorphism from \( \Sigma_r \) to \( M_r \) and \( \Sigma_r \) is a real algebraic set, \( M_r \) is a real algebraic set by transport of structure. In what follows we identify \( M_r \) and \( \Sigma_r \). Notice that

\[\tilde{M}_r = \rho_n^{-1}(r_n), \ \rho_n : N_r \to \mathbb{R}, \ \rho_n(P) = r_n, \ \text{where} \ r = (r_1, \ldots, r_n)\]

We let \( \mathcal{Q}(S^2) \) be the quotient space of \( \mathcal{P}(S^2) \) by \( SO(3) \) and let \( \pi : \mathcal{Q}(S^2) \to (\mathbb{R}_+)^n \) be the map induced by \( \rho \). Hence for \( r \in (\mathbb{R}_+)^n \)

\[M_r = \pi^{-1}(r).\]

Our strategy is to study how the fibers of \( \pi \) vary as \( r \) varies in \( D_n(S^2) \).

We have
Lemma 2.7  (i) The Zariski tangent space $T_u(\tilde{M}_r)$ is given by

$$T_u(\tilde{M}_r) = \ker d\rho|_u$$

(ii) The Zariski tangent space $T_u(M_r)$ is given by

$$T_u(M_r) = \ker d\pi|_u$$

Corollary 2.8 The variety $\tilde{M}_r$ (resp. $M_r$) is smooth if and only if $r$ is a regular value of $\rho$ (resp. $\pi$).

From [KM2], Theorem 1.1 we deduce

Theorem 2.9 Let $P \in P_n(S^2)$ (resp. $Q_n(S^2)$). Then $P$ is a critical point of $\rho$ (resp. $\pi$) if and only if $P$ is degenerate.

3. The Results of A. Galitzer

In [G], A. Galitzer has described $D_n(S^2)$. We will need some notation to describe her results. If $I \subset \{1, 2, \ldots, n\}$ we let $\bar{I}$ denote the complement of $I$, $|I|$ be the cardinality of $I$ and $r_I = \sum_{i \in I} r_i$. Define a polyhedron $K_n \subset \mathbb{R}^n$ by the system of inequalities

$$0 \leq r_i \leq \pi, \quad 1 \leq i \leq n,$n and

$$r_I \leq r_{\bar{I}} + (|I| - 1)\pi, \quad I \subset \{1, 2, \ldots, n\}, \quad \text{with } |I| \text{ odd.}$$

Then Galitzer proves

Theorem 3.1 $K_n = D_n(S^2)$.

In addition she proves that the codimension 1 faces of $D_n(S^2)$ are given by the intersections of the hyperplanes corresponding to the above inequalities with $K_n$, i.e. the above representation of $K_n$ is irredundant.

The space $Q_n$ is difficult to work with since the mapping $\pi$ is not differentiable. To remedy this we let $P_n^0$ denote the open subset of $P_n$ corresponding to those $n$-gons such that successive vertices $u_i, u_{i+1} (i \in \mathbb{Z}/n)$ do not coincide and are not antipodal. We let $Q_n^0$ denote the quotient of $P_n^0$ by $SO(3)$. Then $Q_n^0$ is naturally a smooth manifold of dimension $2n - 3$. Indeed, $Q_n^0$ is naturally diffeomorphic to the submanifold $\Sigma \subset P_n^0$ consisting of those $n$-gons with the vertex set $u = (u_1, \ldots, u_n)$ satisfying

$$u_1 = \vec{e}_1, \quad u_n \cdot \vec{e}_3 = 0, \quad u_n \cdot \vec{e}_2 > 0.$$
Recall $\mathbf{e}_1^t$, $\mathbf{e}_2^t$, $\mathbf{e}_3^t$ is the standard basis of $\mathbb{R}^3$.

Note that $\Sigma_r = M_r \cap S$ (see Lemma 2.6) and that $\pi(Q^0_n) \supset$ interior of $K_n = K^0_n$. We will henceforth replace $\pi$ by its restriction to $Q^0_n$.

We shall see shortly (Theorem 3.3) that the set of critical values of $\pi$ inside $K^0_n$ is the union of certain hyperplane sections of $K^0_n$. We call these hyperplane sections walls of $K_n$. Connected components in $K^0_n$ of the complement of the union of the walls are called chambers. In [G], Galitzer determines the walls of $K_n$. We again summarize her results.

Let $I \subset \{1, \ldots, n\}$ be any non-empty subset. For each nonnegative integer $w$ let $H_{I,w}$ denote the hyperplane in $\mathbb{R}^n$ defined by the equation

$$r_I - r_I = 2\pi w$$

Intersection of such hyperplane with $K^0_n$ is a wall. We define minor walls as intersections of the hyperplanes

$$r_I + r_I = 2\pi w$$

with the polyhedron $K^0_n$.

We then have the following lemma of Galitzer

**Lemma 3.2** $H_{I,w} \cap K^0_n \neq \emptyset \iff |I| \geq 2w + 2$.

**Proof.** Assume $r^* \in H_{I,w} \cap K^0_n$. Since $r^* \in H_{I,w}$ we have

$$r^*_I - r^*_I = 2\pi w.$$  

But since $r^* \in K^0_n$ we also have

$$r^*_I - r^*_I < (|I| - 1)\pi.$$  

Hence $2\pi w \leq (|I| - 1)\pi$ and

$$|I| > 2w + 1.$$  

To prove the converse we first note that there exists a cross-section $s_{I,w} : H_{I,w} \cap (0,\pi)^n \to Q^0_n$ to the restriction of $\pi$ to $\pi^{-1}(H_{I,w})$ defined inductively as follows. Let $r^* \in H_{I,w} \cap (0,\pi)^n$. The vertices $u_1$ and $u_n$ are determined by the condition that the image of $s_{I,w}$ belongs to $\Sigma_r^*$ (see Lemma 2.6). Place the vertex $u_{n-1}$ on the equator so that $e_{n-1}$ is a forward track (and $d(u_{n-1}, u_n) = r^*_{n-1}$) if $n-1 \in I$ and on the other side of $u_n$ if $n-1 \in \bar{I}$. Continue inductively. The resulting degenerate linkage closes up because $r^*_I - r^*_I = 2\pi w$.

To complete the proof of the lemma it suffices to prove that if $|I| \geq 2w + 2$ then $H_{I,w} \cap (0,\pi)^n \neq \emptyset$. Choose $J = \{i_1, i_2, \ldots, i_{2w}\} \cap I$ and let $r_{i_1} = r_{i_2} = \cdots = r_{i_{2w}} = \pi - \epsilon$
where $\epsilon$ is very small. Let $K = I - J$. The hyperplane $r_K - r_I = 2w\epsilon$ comes very close to the origin. Hence it meets $(0, \pi)^n$. Choose a point in this intersection. □

The set of critical values of $\pi$ is then determined by

**Theorem 3.3** Let $r \in K_0^I$. Then $r$ is a critical value of $\pi$ if and only if $r \in H_{I, w}$ for some $I, w \geq 0$ with $|I| \geq 2w + 2$.

**Proof.** Clearly there exists a degenerate $u \in \pi^{-1}(r)$ if and only if $r$ satisfies an equation of the form $r_I - r_I = 2\pi w$. Now apply Theorem 2.9. □

**Remark 3.4** Since $\pi$ is proper it is a fibration over each chamber and the topology of the fibers does not change within a chamber.

### 4. Recuttings and Flips of Spherical $n$-gons

In this section we construct two groups acting on the space of spherical $n$-gons.

We first construct the group $\mathcal{R}$ of recuttings. Let $D^I_n(S^2) = \{ r \in D_n(S^2) : \text{all components of } r \text{ are distinct} \}$. Let $\mathcal{P}^I_n(S^2) = \rho^{-1}(D^I_n(S^2)) \cap \mathcal{P}_n(S^2)$. Let $S_n$ be the permutation group on $n$ letters. $S_n$ operates naturally on $D^I_n(S^2)$. We will construct a group $\mathcal{R}$ acting on $\mathcal{P}^I_n(S^2)$ and an epimorphism $\phi : \mathcal{R} \to S_n$ so that the projection $\rho$ is $\phi$-equivariant:

$$\rho(gP) = \phi(g)\rho(P) \quad P \in \mathcal{P}^I_n, \quad g \in \mathcal{R}.$$  

We will call elements $g \in \mathcal{R}$ recuttings. Adler [A] defined recuttings for the Euclidean plane. Here we do the recuttings for the spherical case.

We define the basic recuttings $R_i : \mathcal{P}^I_n(S^2) \to \mathcal{P}^I_n(S^2)$, $1 \leq i \leq n$ as follows. Let $u \in \mathcal{P}^I_n(S^2)$ with $u = (u_1, u_2, \ldots, u_n)$. Take any geodesic arc connecting the points $u_{i-1}$ and $u_{i+1}$, and look at its perpendicular bisector. This bisector is unique because $r_{i-1} \neq r_i$. Reflect the point $u_i$ through this perpendicular line to exchange $r_{i-1}$ and $r_i$. Leave all other vertices fixed. This is what we will call the basic recutting $R_i$ at the $i$-th vertex.

The equation for the basic recutting at the $i$-th vertex is

$$R_i(u_i) = u_i - 2 \frac{u_i \cdot (u_{i+1} - u_{i-1})}{\|u_{i+1} - u_{i-1}\|^2} (u_{i+1} - u_{i-1})$$

and

$$R_i(u_j) = u_j, \quad j \neq i.$$

Then the basic recuttings are well defined on the space $\mathcal{P}^I_n(S^2)$. We let $\mathcal{R}$ be the group generated by the basic recuttings. Since the generators act on $\mathcal{P}^I_n(S^2)$, so does $\mathcal{R}$. Notice
that the action of $\mathcal{R}$ preserves the subset of degenerate polygons and their winding numbers. Each recutting determines a permutation on the set of vertices of a closed $n$-gon which defines the homomorphism $\phi$.

We next define the basic flips $F_i$, $1 \leq i \leq n$. We define $F_i : \mathcal{P}_n^0(S^2) \to \mathcal{P}_n^0(S^2)$, $1 \leq i \leq n$, by
\[
F_i(u_1, \ldots, u_n) = (u_1, \ldots, -u_i, \ldots, u_n).
\]
We note that $F_i$ induces the map $\tilde{F}_i : D_n(S^2) \to D_n(S^2)$ given by
\[
\tilde{F}_i(r_1, \ldots, r_n) = (r_1, \ldots, \pi - r_{i-1}, \pi - r_i, \ldots, r_n).
\]

Notice that flips do not preserve the walls in $D_n(S^2)$, however the image of a wall under a flip is either a wall or a minor wall.

5. The Morse Theory of $\rho_n$

In this section we will prove the Main Theorem. We begin by discussing what we proved along these lines in [KM2]. Suppose $r^* \in K^0_n$ lies on the intersection of the walls
\[
H_{I_{1,w_1}}, H_{I_{2,w_2}}, \ldots, H_{I_{p,w_p}}
\]
Choose a degenerate linkage $u^*$ with $\pi(u^*) = r^*$. Let $\gamma$ be the great circle containing $u^*$.

**Definition 5.1** The vertical line segment $L$ through $r^*$ will be the line segment defined by
\[
r_i = r^*_i, \ 1 \leq i \leq n - 1 \text{ and } r_n^* - \delta \leq r_n \leq r_n^* + \delta.
\]

We assume that $\delta$ is chosen so that $L$ does not intersect any wall except at $r^*$. Let $X_L = \pi^{-1}(L)$.

**Lemma 5.2** $X_L$ is a smooth submanifold of $\mathcal{Q}_n$ diffeomorphic to the $(n-2)$-torus. Moreover $X_L \cong N_{r'}$, where $r' := (r^*_1, \ldots, r^*_{n-1})$ (see §1).

**Proof.** We first observe that $\rho^{-1}(L)$ is diffeomorphic to $S^2 \times (S^1)^{n-1}$. Indeed a point in $\rho^{-1}(L)$ is a closed $n$-gon where the lengths of the first $(n-1)$-sides are prescribed to be $r_1^*, r_2^*, \ldots, r_{n-1}^*$ but the length of the $n$-th side is not determined. The operation of forgetting the $n$-th side gives an isomorphism to the moduli space of the free linkage with $(n-1)$-edges. The $S^2$ factor comes from the position of the first vertex $u_1$, the circle factors come from the angles between successive edges. The quotient $\pi^{-1}(L) / SO(3)$ can be obtained by fixing the position of the first edge. Clearly $X_L \cong N_{r'}$. \qed

In [KM2], Theorem 8.10, we proved
Theorem 5.3 \( \rho_n|X_L \) is a Morse function with a finite collection of critical points \( u^*_{(1)} \cup \ldots \cup u^*_{(p)} \), all located on the critical fiber \( M_* \). Each critical point \( u^*_{(i)} \) corresponds to a degenerate \( n \)-gon linkage in \( M_* \) with \( f_i \) forward-tracks, \( b_i \) back-tracks and the winding number \( w_i \) contained in a great circle \( \gamma_i \). Then the signature of the Hessian of \( \rho_n|X_L \) at \( u^*_{(i)} \) is either \( (f_i - 2w_i - 1, b_i + 2w_i - 1) \) or \( (b_i + 2w_i - 1, f_i - 2w_i - 1) \) depending on the orientations of \( \gamma_i \), \( 1 \leq i \leq p \).

We now concentrate on a single critical point \( u^* = T^* \) of \( \rho_n \) contained in a great circle \( \gamma \) with the associated closed polygon \( P^* \) which has \( f \) forward-tracks and winding number \( w \). We orient \( \gamma \) as described in \( \S 2 \) (i.e. in the direction of rotation from \( u_1 \) to \( u_n \)). Let \( L^* \) be a vertical segment through \( \rho(u^*) \).

We begin the proof of the Main Theorem with

Lemma 5.4 There exists a vertical line segment \( L^* \subset D_n(S^2) \) and a degenerate free \((n - 1)\)-gon linkage \( T^* \) with \( \pi(T^*) = r^* \in L^* \) such that

(i) The forward-tracks of the associated closed linkage \( P(T^*) \) are the first \( f \) edges of \( T^* \).

(ii) \( w(T^*) = w(T^*), f(P(T^*)) = f \).

(iii) signature \( D^2(\rho_n|X_{L^*})|_{T^*} = signature \ D^2(\rho_n|X_{L^*})|_{T^*} \).

(iv) \( r^* \) belongs to exactly one wall in \( D_n(S^2) \) and does not belong to any minor wall.

Proof. The hyperplanes \( r_i = r_j \) intersect the hyperplane \( r_I - r_I = 2\pi w \) transversally. Hence \( H_{I, u} \cap D_n(S^2) \) is the complement of a union of hyperplane sections of \( H_{I, u} \) and hence is dense. Thus there exists \( \bar{r} \) close to \( r^* \) such that components of \( \bar{r} \) are distinct.

We let \( \bar{L} \) be the vertical segment passing through \( \bar{r} \), \( X_{\bar{L}} = \pi^{-1}(\bar{L}) \) and \( \bar{u} = s_{I, u}(\bar{r}) \) (see Lemma 3.3). We claim

signature \( D^2(\rho_n|X_{L^*})|_{\bar{u}} = signature \ D^2(\rho_n|X_{L^*})|_{\bar{u}} \)

To see this let \( B \) be the line segment in \( H_{I, u} \) joining \( \bar{r} \) to \( r^* \). For \( b \in B \), let \( L_b \) be the vertical segment through \( b \) and \( u_b = s_{I, w}(b) \). We obtain the curve \( D^2(\rho_n|X_{L_b})|_{u_b} \) which joins the two Hessians above. By Theorem 5.3 these quadratic forms are nondegenerate and the claim follows. The same argument proves that we can choose \( \bar{r} \) which belongs to exactly one wall and does not belong to any minor wall.

We now choose a permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \) which fixes \( n \) and sends \( I := \{i_1, \ldots, i_f\} \) to \( \{1, 2, \ldots, f\} \). Choose a recutting \( R \) in the subgroup of \( R \) generated by \( \{R_2, \ldots, R_{n-2}\} \) such that \( \phi(R) = \sigma \). Put \( r^\# = \sigma(\bar{r}) \) and \( u^\# = R(\bar{u}) \). The line segment
\( \tilde{L} \) through \( \tilde{r} \) is carried by \( \sigma \) to the line segment \( L^\# \) through \( r^\# \). Hence the corresponding manifold \( X_{\tilde{L}} \) is carried to \( X_{L^\#} \) by \( R \). We claim

\[
\text{signature } D^2(\rho_n | X_{L^\#})|_{\tilde{u}} = \text{signature } D^2(\rho_n | X_{L^\#})|_{u}
\]

Indeed since \( \rho_n | X_{L^\#} = \rho_n \circ R|_{X_{\tilde{L}}} \) we find that

\[
dR_{\tilde{u}} : T_{\tilde{u}}(X_{\tilde{L}}) \longrightarrow T_{u^\#}(X_{L^\#})
\]

is an isometry of the quadratic form on the right-hand side to that on the left-hand side. \( \Box \)

We can now reduce to the case \( w = 0 \).

**Lemma 5.5** There exists a flip \( F \) such that \( \tilde{T} = F(T^\#) \) satisfies

(i) \( b(\tilde{T}) = b(T^\#) + 2w(T^\#) \)

(ii) \( w(\tilde{T}) = 0 \)

(iii) \( \text{signature } D^2(\rho_n | X_{\tilde{L}})|_{\tilde{r}} = \text{signature } D^2(\rho_n | X_{L^\#})|_{T^\#} \).

**Here** \( \tilde{L} = F(L^\#) \).

**Proof.** We consider the case \( w > 0 \) (the case when \( w < 0 \) is treated similarly, just instead of flipping forward-tracks we flip back-tracks). We let \( F \) be the product of flips given by

\[
F = F_2 \circ F_1 \circ \cdots \circ F_{2w}.
\]

We note that since \( f \geq 2w + 2 \) all the edges that are flipped are forward-tracks (and they become back-tracks after flipping). Thus (i) and (ii) are clear. The statement (iii) is proved in the same fashion as (iii) in the previous lemma. \( \Box \)

We let \( K \) be the set of forward tracks of \( \tilde{T} \) (or the associated closed \( n \)-gon linkage \( \tilde{P} \)). Hence \( \tilde{r} = \pi(\tilde{P}) \) is on the wall \( H_{K,0} \).

We next deform \( \tilde{r} \) along the wall \( H_{K,0} \) to \( \hat{r} \) such that \( \hat{r}_1 + \hat{r}_2 + \cdots + \hat{r}_n < 2\pi \). The corresponding degenerate closed \( n \)-gon linkage \( s_{K,0}(\hat{r}) = \hat{u} \) will have perimeter less than \( 2\pi \). To accomplish this let \( A \subset D_n(S^2) \cap H_{K,0} \) be the line segment

\[
A = \{ \lambda \hat{r} : \epsilon < \lambda < 1 + \epsilon \}
\]

Choose \( \lambda_0 \) such that \( \sum_{i=1}^n \lambda_0 \hat{r}_i < 2\pi \). Let \( \hat{r} = \lambda_0 \hat{r} \) and \( \hat{L} \) be the vertical segment through \( \hat{r} \).

Put \( \hat{u} = s_{K,0}(\hat{r}) \).
Lemma 5.6 The signature of $D^2(\rho_n | X_L)|_{\bar{u}}$ is equal to the signature of $D^2(\rho_n | X_L)|_{\bar{u}}$.

Proof. For $a \in A$ define $L_a$ and $u_a$ as in the proof of Lemma 5.4. We obtain the curve $D^2(\rho_n | X_L)|_{u_a}$ and the proof goes as in Lemma 5.4.

Let $\hat{f}$ (resp. $\hat{b}$) be the number of forward-tracks (resp. back-tracks) of $\hat{u}$. By Lemma 5.5, $\hat{f} = f(P) - 2w(P)$ and $\hat{b} = b(P) + 2w(P)$.

We complete the proof of the Main Theorem by

Proposition 5.7 The signature of $D^2(\rho_n | X_L)|_{\bar{u}}$ is $(\hat{b} - 1, \hat{f} - 1)$.

The proposition will be a consequence of the next three lemmas. In what follows let $\hat{P} = \hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ be a degenerate closed $n$-gon linkage of perimeter less that $2\pi$. We assume that $\pi(\hat{P})$ belongs to exactly one wall and does not belong to any minor wall. Then any vertex $u_i$ is connected to $u_1$ by a unique geodesic segment $(u_1, u_i)$ which does not degenerate to a point.

Following [KK] we define local coordinates $\psi_2, \psi_3, \ldots, \psi_{n-1}$ on $X_L$ by defining $\psi_i$ to be the signed angle at $u_i$ between the oriented segment $(u_1, u_i)$ and the oriented edge $e_i$. For instance if $u_i = \overrightarrow{e_2}, u_{i+1} = -\overrightarrow{e_1}$ then $\psi_i = 0$. If $u_{i+1} = (\overrightarrow{e_1} + \overrightarrow{e_2})/\sqrt{2}$ then $\psi_i = \pi$.

We then have

Lemma 5.8 $\psi_2, \psi_3, \ldots, \psi_{n-1}$ are local coordinates near $\hat{u}$.

Proof. See [KK, §3].

Remark 5.9 In [KK] the authors study free linkages in $S^3$. Our coordinates are obtained from theirs by dropping their vector field $Y$. Thus we use an orthonormal frame $(X, Z)$ where $Z$ is the radial field.

We now have the clever observation of [KK], the reason for choosing the above coordinates.

Lemma 5.10

$$ \frac{\partial^2 \rho_n}{\partial \psi_i \partial \psi_j} |_{\bar{u}} = 0, \quad i \neq j $$

Proof. Assume $i < j$. Then by [KK, pg. 84] we find that the restriction

$$ \frac{\partial \rho_n}{\partial \psi_j} |_{\psi_k = \psi_k, \ k \neq i} $$
of the partial derivative to the curve

$$\Gamma_k := \{ \psi_k = \hat{\psi}_k, \ k \neq i \}$$

is identically zero as a function of $\psi_i$, this implies the lemma. Below we sketch a proof of vanishing of this derivative. We give the picture (Figure 1) in the Euclidean case with $\psi_j = 0$. We draw only the vertices $u_1, u_i, u_j$ and $u_n$.

![Figure 1: Vanishing of the derivative.](image)

Pick a point $u$ on the curve $\Gamma_k$. Then the points $u_1, u_j, u_n$ belong to a common geodesic circle in $S^2$. As $\psi_j$ varies the line segment $(u_j, u_n)$ rotates around $u_j$. Clearly the vertex $u_n$ moves along a (small) circle tangent at $\psi_j = 0$ to the bigger circle which is the level set of $\rho_n$ for the fixed values of $\psi_i$ and $\psi_k = \hat{\psi}_k, \ k \neq i$. Hence $\frac{\partial \rho_n}{\partial \psi_j} |_{\Gamma_k}$ is identically zero as a function of $\psi_i$. \hfill $\square$

**Lemma 5.11** (i) If $\hat{e}_i$ is a back-track then $\frac{\partial^2 \rho_n}{\partial \psi^2_i} |_{\text{g}} > 0$.

(ii) If $\hat{e}_i$ is a forward-track then $\frac{\partial^2 \rho_n}{\partial \psi^2_i} |_{\text{g}} < 0$.

**Proof.** We prove (i) and leave (ii) to the reader. We let $\hat{\psi}_i$ be a value close to $\hat{\psi}_i = \pi$ and consider the curve $\psi_j = \hat{\psi}_j, \ j \neq i$. We obtain the picture described on Figure 2 (again we have drawn the Euclidean case).

Here we have omitted all vertices except $u_1, u_i, u_{i+1}, u_{n-1}$ and $u_n$ and assumed (in the Figure 2) that $\hat{\psi}_{i+1} = 0$ and $\hat{\psi}_{n-1} = \pi$.
We set $d(u_1, u_i) = a$, $d(u_{i+1}, u_n) = b$. From the spherical “law of cosines” (see [B, Proposition 18.6.8]) we have
\[
\cos(r_n + b) = \cos a \cos r_i + \sin a \sin r_i \cos(\pi - \psi_i)
\]
Differentiating implicitly we obtain
\[
\frac{\partial^2 \rho_n}{\partial \psi_i^2} |_{\hat{u}} = \frac{\sin a \sin \hat{r}_i}{\sin(\hat{r}_n + b)}
\]
Since the perimeter of $\hat{u}$ is less than $2\pi$ we have $a < \pi$, $\hat{r}_n + b < \pi$ and (i) follows. \hfill \Box

With this, Proposition 5.7 and the Main Theorem are proved.

6. The Wall-Crossing Formula and Regular Spherical Pentagons

In this section we explain how the Main Theorem can be used to compute how the moduli spaces $M_r$ change as we cross a wall. As an illustration of our technique we compute the moduli spaces of regular spherical pentagons.

We first claim that any wall-crossing can be effected by a vertical segment. Indeed as we have seen the walls are given by $r_I - r_{\overline{I}} = 2w\pi$ with $|I| \geq 2w + 2$. Let $n_I$ be a normal vector to the above wall. Recall that the vector $\nu_n = (0, 0, \ldots, 0, 1)$ is parallel to a vertical segment through this wall. Since $\nu_n \cdot n_I \neq 0$ any vertical segment is transverse to a wall and the claim follows.
From the Main Theorem we obtain

**Theorem 6.1 (The wall-crossing formula)** Suppose we cross the wall $H_{l,w}$ at $r_n = r_n^*$ along a vertical segment $L$ with $r_n^* - \delta \leq r_n \leq r_n^* + \delta$. Then

(i) $M_{r^*+\delta}$ is obtained from $M_{r^*-\delta}$ by attaching an $(f - 2w - 1)$-handle.

(ii) $M_{r^*-\delta}$ is obtained from $M_{r^*+\delta}$ by attaching some $(b + 2w - 1)$-handle.

We now apply our formula to compute the moduli spaces of regular spherical pentagons $M_r$ with $r = (a, a, a, a, a)$. The determination of the moduli space $M_r$ for $\frac{2\pi}{5} < a < \frac{2\pi}{3}$ was first done in [G] by a different method. Assume first that $0 < a < \frac{2\pi}{5}$. Since the perimeter of $P$ is less than $2\pi$ the moduli space $M_r = M_r(S^2)$ is diffeomorphic to the corresponding Euclidean moduli space $M_r = M_r(\mathbb{R}^2)$ by [S]. Hence by [KM1, Theorem 2], $M_r$ is the genus four surface, $0 < a < \frac{2\pi}{5}$.

Now as $a$ goes from $\frac{2\pi}{5} - \delta$ to $\frac{2\pi}{5} + \delta$ we pass through the wall $r_1 + r_2 + r_3 + r_4 + r_5 = 2\pi$. We now compute what happens as we cross this wall using Theorem 6.1. Set $r_1 = r_2 = r_3 = r_4 = \frac{2\pi}{5}$ and let $r_5$ go from $\frac{2\pi}{5} - \delta$ to $\frac{2\pi}{5} + \delta$. The critical point $T \in N_r$ corresponding to the critical value $r_5 = \frac{2\pi}{5}$ is represented by the degenerate free 4-gon linkage with $P = P(T)$ obtained by dividing the equator $\gamma$ into 5 equal parts proceeding anticlockwise around the equator and taking the first four segments. Our orientation rule requires us to orient the equator so that the positive direction is clockwise hence

$$b(P) = 5, \quad f(P) = 0, \quad w(P) = -1.$$  

According to the main theorem the signature of $D^2\rho_5|_L$ is $(2, 1)$. Since $\rho_5$ increases as we cross the wall we obtain Theorem 6.1 of [G]:

$M_r$ is the genus five surface, \quad if \quad $\frac{2\pi}{5} < a < \frac{2\pi}{3}$.

The point $r = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$ lies on the intersection of five walls of the form

$$r_i + r_j + r_k + r_l - r_m = 2\pi.$$ 

There are two cases to consider, $m = 5$ and $m \neq 5$. We will analyse the first case and leave the second to the reader.

We will identify the equator of $S^2$ with the unit circle on the complex plane. Let $T$ be the degenerate free 4-gon linkage with vertices $(1, \omega, \omega^2, 1, \omega)$ where $\omega = \exp(2\pi i/3)$. By our orientation convention the unit circle has the usual (i.e. counterclockwise) orientation and

$$b(P) = 1, \quad f(P) = 4, \quad w(P) = 1.$$
Hence \( D^2 \rho_5|_T \) has signature \((2, 1)\). The equation of the wall we are considering is \( r_1 + r_2 + r_3 + r_4 - r_5 = 2\pi \). Let \( \alpha(r_1, r_2, r_3, r_4, r_5) = r_1 + r_2 + r_3 + r_4 - r_5 \). As \( a \) increases from \( \frac{2\pi}{3} - \delta \) to \( \frac{2\pi}{3} + \delta \) we pass from the half-space \( \alpha < 2\pi \) to \( \alpha > 2\pi \). Now to apply the Theorem we set \( r_1 = r_2 = r_3 = r_4 = \frac{2\pi}{3} \). To cross from \( \alpha < 2\pi \) to \( \alpha > 2\pi \) we see that \( r_5 \) must decrease from \( \frac{2\pi}{3} + \delta \) to \( \frac{2\pi}{3} - \delta \). Thus we attach the “positive” disk of according to the signature of \( D^2 \rho_5|_T \) as we pass through the critical point \( r_5 = \frac{2\pi}{3} \). Hence we attach a 2-handle. We attach 2-handles at the other 4 critical points of \( \rho_5 \) corresponding to the critical value \( r_5 = \frac{2\pi}{3} \) and we obtain

\[
M_r \approx S^2, \quad \text{if} \quad \frac{2\pi}{3} < a < \frac{4\pi}{5}.
\]

We cross no more walls of \( D_5(S^2) \) until we reach the face given by \( r_1 + r_2 + r_3 + r_4 + r_5 = 4\pi \) when \( a = \frac{4\pi}{5} \). The critical value \( r_5 = \frac{4\pi}{5} \) corresponds to the single critical point \( u = (1, \zeta^2, \zeta^4, \zeta^6, \zeta^8) \) where \( \zeta = \exp(2\pi i/5) \). We have \( u_5 = \exp(-4\pi i/5) \). Hence \( \gamma \) is oriented n the clockwise direction. We obtain

\[
b(P) = 5, \quad f(P) = 0, \quad w(P) = -2
\]

and accordingly the signature of \( D^2 \rho_5|_T \) is \((0, 3)\). Hence \( P \) is locally rigid.

We can in fact determine the moduli space \( M_r \) as follows. Apply the flips \( F_1 \) and \( F_3 \) to change \( r \) to \( r^* \) with \( r_1^* = r_2^* = r_3^* = \frac{\pi}{3}, r_4^* = \frac{4\pi}{5} \). This is a standard “Euclidean” rigid linkage and \( M_{r^*} \) = a point, as was to be expected since \( r \) is on a face.

Of course for \( a > \frac{4\pi}{5} \), \( M_r \) is empty since we are outside \( D_5(S^2) \).

7. Appendix

The statement in Section 6 of [KM2] that \( A^\bullet_{(2)}(M, adP) \) is a differential graded Lie algebra is false since the \( L^2 \)-condition is not closed under bracket. Hence our proof that \( B^\bullet(M, U; adP) \) is formal is not correct. However we can salvage all the results of [KM2] except the result that \( B^\bullet(M, U; adP) \) is formal by the following “quick fix”. First we apply the results of Section 5 of our paper [KM3] to deduce that the germ \( (M_r, [P_0]) \) is given by a single quadratic equation corresponding to the cup product:

\[
q : H^1(B^\bullet(M, U; adP)) \to H^2(B^\bullet(M, U; adP)) = \mathbb{R}.
\]

Second we claim that the results of Section 7 of [KM2] do in fact compute \( q \) above. To see this we note first that the inclusion \( B^\bullet(M, U; adP) \to A^\bullet_{(2)}(M, adP) \) is a quasi-isomorphism of complexes. Second the bracket of two elements of \( A^1_{(2)}(M, adP) \) is integrable (but not necessarily square integrable) whence the integration pairing (using the trace on \( adP \)) is well-defined on \( A^1_{(2)}(M, adP) \). By [Ga] it descends to cohomology and consequently agrees with \( q \).
Remark 7.1 Formality of $B^*(M,U;ad P)$ follows from the recent result of P. Foth [F].

References


