INTEGER NORMS ARE POLYHEDRAL

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The main purpose of this note is to correct the erroneous proof of polyhedrality of integer norms given in section 2.1 (chapter "Thurston Norm") of my book "Hyperbolic Manifolds and Discrete Groups." About 15 years ago I wrote a rather intricate correction of the proof (using, among other things, the thick-thin decomposition of the locally symmetric space of $SL(n,\mathbb{Z})$). The proof given below is due to Roman Vershynin, it is much simpler and is totally elementary.

Definition 1. *A norm* $||\cdot||$ *on* $\mathbb{R}^n$ *is called* integer *if it takes integer values on integer vectors. A norm on* $\mathbb{R}^n$ *is called* polyhedral *if its unit ball is a polyhedron.*

Theorem 2. *Every integer norm on* $\mathbb{R}^n$ *is polyhedral.*

Proof. Let $||x||_2$ denote the standard Euclidean norm on $V=\mathbb{R}^n$; we will use the same notation for the dual Euclidean norm on V^*. Let $f(x)=||x||$ be a norm on V; let B denote the closed unit ball of this norm. Thus, there exists $M>0$ such that

$$M^{-1}||x||_2 \le ||x|| \le M||x||_2.$$

We will use the notation $||\cdot||^*$ for the dual norm of $||\cdot||$ on V^* and $\langle\cdot,\cdot\rangle$ for the pairing between V and V^*: For $\alpha\in V^*$,

$$||\alpha||^* = \max_{v\in B}\langle \alpha, v\rangle.$$

Every point $x\in\partial B$ admits a (not necessarily unique) supporting affine hyperplane H_x, which is parallel to the kernel of a nonzero linear functional $\alpha_x\in V^*$ such that

$$\langle\alpha_x, b\rangle \le ||\alpha_x||^*, \forall b\in B$$

with the equality attained at $b=x$.

Thus, if $S\subset\partial B$ is a dense subset then for every $v\in V$

$$||v|| = \sup_{x\in S}\frac{\langle\alpha_x, v\rangle}{||\alpha_x||^*}. \tag{1}$$

If the norm-function f is differentiable at $x\in\partial B$ then

$$\alpha_x = df_x.$$

Since convex functions are differentiable a.e. in V and their set of points of differentiability is conical (i.e. with every $x\in V$ it contains the ray $\mathbb{R}_+\cdot x$), the set $S\subset\partial B$ of differentiability points of f is dense in ∂B.

Lemma 3. *For every* $x\in S$, $||\alpha_x||_2\le M$.

Proof. For every nonzero vector $h\in V$ with unit Euclidean norm,

$$\frac{||x+h||-||x||}{||h||_2} \le \frac{||h||}{||h||_2} \le M.$$

Applying this inequality to the vectors $th, t \in \mathbb{R}^{\times}$, where $||h||_2 = 1$, and taking the limit as $t \to 0+$, we obtain:

$$\alpha_x(h) = df_x(h) \le M. \quad \square$$

Lemma 4. *Suppose now that* $||\cdot||$ *is an integer norm on* $\mathbb{R}^n$*. Then for every* $x \in S$*,* α_x *is an integer vector.*

Proof. Fix $x \in S$. By the definition of the derivative,

$$||x + h|| = ||x|| + \langle \alpha_x, h\rangle + o(||h||_2),$$

as $||h||_2 \to 0$.

Since the closed Euclidean balls of radius $\sqrt{n}$ centered at integer points cover the entire $\mathbb{R}^n$, for every $N \in \mathbb{N}$ there exists a vector $y \in \mathbb{R}^n$ such that

$$Nx + y \in \mathbb{Z}^n, ||y||_2 \le \sqrt{n}.$$

Therefore, for the vectors $h = \frac{1}{N} y$, we have

$$||x + \frac{1}{N} y|| = ||x|| + \frac{1}{N} < \alpha_x, y\rangle + o(\frac{1}{N}),$$

as $N \to \infty$. Similarly, for any given integer vector $z \in \mathbb{Z}^n$,

$$||x + \frac{1}{N}(y + z)|| = ||x|| + \frac{1}{N}\langle \alpha_x, y + z\rangle + o(\frac{1}{N}),$$

as $N \to \infty$. Hence,

$$||x + \frac{1}{N}(y + z)|| - ||x + \frac{1}{N} y|| = \frac{1}{N}\langle \alpha_x, z\rangle + o(\frac{1}{N}),$$

and, thus,

$$||Nx + (y + z)|| - ||Nx + y|| = \langle \alpha_x, z\rangle + o(1),$$

as $N \to \infty$. Observe that, by the choice of y and z and since $||\cdot||$ is an integer norm, the left-hand side of this equation is integer. Hence, taking the limit as $N \to \infty$, we see that left-hand side converges to an integer, while the right-hand side converges to $\langle \alpha_x, z\rangle$. Thus, $\langle \alpha_x, z\rangle \in \mathbb{Z}$ for each integer vector $z \in \mathbb{Z}^n$, i.e. α_x is an integer linear functional. $\square$

We now can conclude the proof of the theorem. The linear functionals $\alpha_x, x \in S$, are all integer with the norm bounded by M. Hence, they form a finite set A and for every $v \in V$,

$$||v|| = \sup_{x \in S} \frac{\langle \alpha_x, v\rangle}{||\alpha_x||^*} = \max_{\alpha \in A} \frac{\langle \alpha, v\rangle}{||\alpha||^*}.$$

Hence, the unit ball B of the norm $||\cdot||$ is given by finitely many linear inequalities

$$\frac{\langle \alpha, v\rangle}{||\alpha||^*} \le 1, \alpha \in A,$$

i.e. is a polyhedron. $\square$