GEOMETRY OF QUASI-PLANES

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ABSTRACT. In this paper we discuss metric cell complexes satisfying a coarse form of 2-dimensional Poincare duality. We prove that such spaces are either Gromov-hyperbolic or have polynomial growth. As an application we prove that 2-dimensional Poincare duality groups over commutative rings are commensurable with surface groups.

1. INTRODUCTION

This paper is the second in a sequence of three papers which deal with spaces satisfying coarse Poincare duality [13] in conjunction with a coarse generalization of the Seifert fibered space conjecture [12]. In the present paper we study geometry of *quasi-planes*, i.e. simply-connected metric cell complexes satisfying *coarse Poincare duality* (over a commutative ring $\mathcal{R}$) in dimension 2. We refer the reader to section 3 for the precise definition, at the moment we only note that quasi-planes are metric spaces satisfying a coarse version of the Jordan separation theorem (for curves in $\mathbb{R}^2$). We introduce a *surrounding function* for quasi-planes and prove

Theorem 1.1. *Suppose that Z is a quasi-homogeneous quasi-plane. Then we have the following dichotomy:*

1. Either the surrounding function is super-linear, in which case Z is Gromov-hyperbolic, with topological circle as the ideal boundary. In this case each cobounded quasi-action $G \curvearrowright Z$ is quasi-isometrically conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.

2. Or the surrounding sunction is at most linear, in which case Z has polynomial growth and, in case Z is quasi-isometric to a finitely generated group Q, the group Q is virtually abelian of rank 2.

We note that in the case when Z is the zero-skeleton of a 2-dimensional triangulated planar surface S, such that S is quasi-isometric to a finitely generated group G, the above theorem was established by G. Mess in [15]. His paper was an inspiration for our work.

As an application we get the following characterization of 2-dimensional Poincare duality groups over commutative rings, originally established in [14] using somewhat different methods:

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Theorem 1.2. *Suppose that $\mathcal{R}$ is a commutative ring with a unit, G is a 2-dimensional Poincare duality group over $\mathcal{R}$. Then G is commensurable to a surface group.*

This theorem extends earlier results of Eckmann, Müller and Linnel [8, 7], in the case $\mathcal{R} = \mathbb{Z}$, and results of Bowditch [3], in the case when $\mathcal{R}$ is a field. Theorem 1.2 was conjectured by Dicks and Dunwoody in [5]. The key to the proof of this theorem is a construction of an action of G on a quasi-plane Z (over $\mathcal{R}$), which then allows us to apply Theorem 1.1 to conclude that either G is virtually nilpotent or G acts discretely cocompactly isometrically on the hyperbolic plane.

In order to prove Theorems 1.1 and 1.2 we establish a local-to-global characterization of Gromov-hyperbolic spaces. As an application of this characterization we also get a theorem of independent interest:

Theorem 1.3. *Suppose that G is a finitely presented group such that* some[1] *asymptotic cone of G is a metric tree. Then G is Gromov-hyperbolic.*

This contrasts with the fact that there are examples [17], [6], of finitely generated groups G for which *some* asymptotic cone of G is a metric tree but G is *not Gromov-hyperbolic.*

In the forthcoming paper [12] we will apply our results to study *coarse fibrations* of manifolds by lines. In particular, we prove a coarse analogue of the Seifert fibered space conjecture and give an alternative proof of G. Mess' part of the proof of the original Seifert fibered space conjecture.

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2. Preliminaries

2.1. **Definitions and notation.** We let $\mathbb{Z}_+ := \{m \in \mathbb{Z} \mid m \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. We let $d_H(\cdot,\cdot)$ denote the Hausdorff distance between subsets of a metric space; the usual (infimal) distance will be denoted $d(\cdot,\cdot)$. All maps between cell complexes will be continuous unless otherwise specified. Given a map f we let $\mathrm{Im}(f)$ denote the image of f.

We will be using singular (co)homology with $\mathbb{Z}$ coefficients unless we indicate otherwise. For each negative integer k we set $H_k(\cdot) = 0$, $H_c^k(\cdot) = 0$, etc.

Let r_i, R_i be two sequences of positive real numbers. We will use the notation

$$R_i \gtrsim r_i$$

[1] It is easy to see that if *all* asymptotic cones of a finitely-generated group G are metric trees, then G is Gromov-hyperbolic.

if there exist a pair of constants A, B (independent of i) such that for all but finitely many $i \in \mathbb{N}$ we have:

$$R_i \geq Ar_i + B.$$

We will use the notation $R_i \simeq r_i$ if $R_i \gtrsim r_i$ and $r_i \gtrsim R_i$.

A subset $S \subset Z$ of a metric space is called *δ-dense* if each point $z \in Z$ is within distance $\leq \delta$ from S. A subset in Z which is δ-dense from some $\delta < \infty$, is called a *net* in Z.

A metric space Z has *bounded geometry* if there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each metric ball $B(x, r) \subset Z$, one has:

$$Vol(B(x, r)) := |B(x, r)| \leq \phi(r),$$

where $|S|$ denotes the cardinality of a set S.

For a subset $D \subset Z$ define the *c-frontier* $\partial_c D$ as

$$\partial_c D := \{x \in Z \setminus D : d(x, D) \leq c\}.$$

For $c = 1$ we set $\partial := \partial_1$.

2.2. **Maps and actions.** Let Y be a subset of a metric space X. A map $f : Y \to X$ has *bounded displacement* if there is $C \in \mathbb{R}$ such that $d(f, i_Y) \leq C$ where $i_Y : Y \to X$ is the inclusion.

A map $f : X \to X'$ between metric spaces is *uniformly proper* if there are constants L, A, and a continuous strictly increasing *distortion function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \eta(t) = \infty$ such that

$$\eta(d(x_1, x_2)) \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A \tag{2.1}$$

for all x_1, $x_2 \in X$. A family of maps $\{f_i\}_{i\in\mathcal{J}}$ is *uniformly proper* if one can choose L, A, and η as above so that (2.1) holds for all the maps in the family.

A map $f : X \to X'$ between metric spaces X and X' is an (L, A)*-quasi-isometry* if for all $x_1, x_2 \in X$ we have

$$\frac{1}{L}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$

and $\mathrm{Im}(f)$ is A-dense in X'. We let $\widehat{\mathrm{QI}}(X, X')$ denote the collection of all quasi-isometries from X to X'. Two quasi-isometries $f_1, f_2 : X \to X'$ are *equivalent* if $d(f_1, f_2) < \infty$; we let $\mathrm{QI}(X, X')$ denote the set of equivalent classes of quasi-isometries, and use $\mathrm{QI}(X)$ (resp. $\widehat{\mathrm{QI}}(X)$) in place of $\mathrm{QI}(X, X)$ (resp. $\widehat{\mathrm{QI}}(X, X)$). Composition of quasi-isometries induces a group structure on $\mathrm{QI}(X)$.

A *quasi-action* of a group G on a metric space X, denoted $G \overset{\rho}{\curvearrowright} X$, is a map $\rho : G \to \widehat{\mathrm{QI}}(X)$ such that for suitable constants L, A,

1. $\rho(g)$ is an (L, A) quasi-isometry for all $g \in G$,

2. $d(\rho(e), \mathrm{id}_X) < A$, and
3. $d(\rho(g_1 g_2), \rho(g_1)\rho(g_2)) < A$ for all $g_1,\ g_2 \in G$.

We will usually write $g(x)$ rather than $\rho(g)(x)$, suppressing the name of the quasi-action when it is understood.

A quasi-action $G \overset{\rho}{\curvearrowright} X$ is called *cobounded* if for some (for every) point $x \in X$ the *quasi-orbit* $G \cdot x = \{g(x) : g \in G\}$ is a net in X. We say that a metric space X is *quasi-homogeneous* if there are constants L, A such that for all $x,\ x' \in X$ there is an (L, A) quasi-isometry $f : X \to X'$ with $f(x) = x'$. Note that if there exists a cobounded quasi-action $G \curvearrowright X$ then X is quasi-homogeneous.

2.3. **Gromov hyperbolicity.** Let Z be a geodesic metric space. A geodesic triangle $\Delta \subset Z$ is called *R-thin* if every side of Δ is contained in the R-neighborhoodof the union of two other sides. An *R-fat triangle* is a geodesic triangle which is not R-thin. A geodesic metric space Z is called *δ-hyperbolic in the sense of Rips* (Rips was the first to introduce this definition) if each geodesic triangle in Z is δ-thin.

Let X be a metric space (which is no longer required to be geodesic). For each base-point $p \in X$ define a number $\delta_p \in [0, \infty]$ as follows. For each $x \in X$ set $|x|_p := d(x, p)$ and

$$(x, y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x, y)).$$

Then

$$\delta_p := \inf_{\delta \in [0,\infty]} \{\delta | \forall x, y, z \in X, (x, y)_p \geq \min((x, z)_p, (y, z)_p) - \delta\}.$$

We say that X is *δ-hyperbolic in the sense of Gromov*, if $\infty > \delta \geq \delta_p$ for some $p \in X$. We note that if X a geodesic metric space which is δ-hyperbolic in Gromov's sense then X is 4δ-hyperbolic in the sense of Rips and vice-versa (see [10, 6.3C]).

A metric space Z is *Gromov-hyperbolic* if it is δ-hyperbolic for some $\delta < \infty$.

2.4. **Growth of spaces.** A bounded geometry metric space Z has *polynomial growth* if there is a constant $c \in \mathbb{R}_+$ such that for each ball $B(x, r) \subset X$ one has

$$r^c \gtrsim Vol(B(x, r)), r \in \mathbb{N}.$$

The optimal constant c is called the *degree of the polynomial growth.* A space Z has *superpolynomial growth* if it does not have polynomial growth for any c.

Similarly, a bounded geometry metric space Z has *exponential growth* if there is a constant $a > 0$ such that for each $x \in Z$ one has:

$$Vol(B(x, R)) \gtrsim e^{aR}, r \in \mathbb{N}.$$

A space has *subexponential growth* if

$$e^{aR} \gtrsim Vol(B(x, R)), r \in \mathbb{N},$$

for all $a > 0$.

A metric space Z is *doubling* if there is a constant $N \in \mathbb{N}$ such that each ball $B(x, 2R) \subset Z$ (where $R \geq 1$) is contained in the union of $\leq N$ balls of radius R.

Lemma 2.2. *Suppose that Z is a bounded geometry doubling metric space. Then Z has polynomial growth.*

Proof. For each $r \geq 1$ choose $n \in \mathbb{Z}$ such that $n - 1 \leq \log_2(r) \leq n$. Hence for each metric ball $B(x, r) \subset X$ we have:

$$B(x, r) \subset \bigcup_{i=1}^{N} B(x_i, \frac{r}{2}) \subset \bigcup_{j}^{N^2} B(x_j, \frac{r}{4}) ... \subset \bigcup_{\ell=1}^{N^{n-1}} B(x_\ell, r2^{-(n-1)}).$$

Let $M := \sup_{x \in Z} Vol(B(x, 2))$ and $c := \frac{1}{\log_N(2)}$. Then

$$Vol(B(x, r)) \leq N^{n-1} M \leq N^{c \log_N(r)} M = r^c M. \quad \square$$

Theorem 2.3. *(M. Gromov, [9]) Suppose that G is a finitely-generated group of polynomial growth. Then G is virtually nilpotent.*

An improvement of this theorem was established by Wilkie and Van der Dries [18]:

Theorem 2.4. *Suppose that G is a finitely-generated group and there is a sequence of metric balls $B(x_i, R_i) \subset G$ and constant $c \geq 0$ such that $R_i^c \gtrsim Vol(B(x_i, R_i))$. Then G is virtually nilpotent and has polynomial growth of degree $\leq c$.*

2.5. **Metric cell complexes.** We will be working with CW complexes endowed with an extra structure. Let X be a CW complex, and $X^{(m)}$ denote its m-skeleton, $m \in \mathbb{Z}_+$. *We will not assume that X is finite-dimensional,* this degree of generality will be important for the applications of our results presented in [12]. Recall that a subcomplex Y of X is a closed subset which is a union of open cells, such that the boundary of each open cell $\sigma \subset Y$ is contained in Y.

A *control map* for X is a function $p : X \to X^{(0)}$ such that

1. $p\big|_{X^{(0)}} = \mathrm{id}_{X^{(0)}}$,
2. p is constant on open cells in X,
3. $p(x)$ belongs to the smallest subcomplex containing x, for all $x \in X$.

A *morphism* $(X, p) \to (X', p')$ is a skeleton preserving continuous map so that for each $i \in \mathbb{N}$ the diameter of $p'(\sigma)$ is uniformly bounded (σ are i-cells in $X^{(i)}$).

A *bounded geometry metric cell complex* is a CW complex X equipped with a control map p, whose 1-skeleton $X^{(1)}$ is connected and equipped with a path metric with respect to which all edges have the same length, subject to the condition that there exists a function $D(m)$ so that every closed cell $\sigma \subset X^{(m)}$ intersects at most $D(m)$ closed cells in $X^{(m)}$.

Remark 2.5. Note that for such a complex, the metric space $X^{(0)}$ has bounded geometry in the sense of Section 2.1.

To simplify the terminology, we will refer to bounded geometry metric cell complexes as simply *metric cell complexes*: the bounded geometry will be assumed by default.

We say that a metric cell complex X is *Gromov-hyperbolic* if its zero-skeleton $X^{(0)}$ is Gromov-hyperbolic.

Let X be a metric cell complex. If $V \subset X^{(0)}$ and $R \in \mathbb{Z}_+$, we denote the closed metric R-neighborhood of V in the 0-skeleton by

$$N_R^{(0)}(V) := \{x \in X^{(0)} \mid d(x, V) \leq R\}.$$

Given $m \in \mathbb{Z}_+$, $R \in \mathbb{Z}_+$, and a subcomplex $Y \subset X$, we define the *R-neighborhood of Y in the m-skeleton*, $N_R^{(m)}(Y)$, to be the largest subcomplex of $X^{(m)}$ whose 0-skeleton is $N_R^{(0)}(Y^{(0)})$. Given $x \in X^{(0)}$ we let $B^{(m)}(x, r) := N_R^{(m)}(\{x\})$, be the r-ball in $X^{(m)}$ with the center at x and radius r. Note that

1. $N_0^{(m)}(Y) \supset Y \cap X^{(m)}$,

2. If Y, $Y' \subset X$ are subcomplexes and $R + R' < d(Y^{(0)}, Y'^{(0)})$, then for all $m \in \mathbb{Z}_+$ we have $N_R^{(m)}(Y) \cap N_{R'}^{(m)}(Y') = \emptyset$.

If X is an m-dimensional metric cell complex then we will use the abbreviation $N_R(V) := N_R^{(m)}(V)$.

We will only use the notation $B(x, r)$ to denote closed metric balls in the 1-skeleton $X^{(1)}$, i.e. $B(x, r) := \{y \in X^{(1)} \mid d(x, y) \leq r\}$, where $x \in X^{(1)}$. If Y is a subset of X, we define its diameter to be $\operatorname{diam}(Y) := \operatorname{diam}(p(Y))$. Similarly, we define the distance between two functions $f, f' : S \to X$ to be the quantity $d(p \circ f, p \circ f')$.

A metric cell complex X is called *uniformly k-acyclic* if there is a function $\psi_k(R) = \psi(R)$ such that for each subcomplex $K \subset X^{(k)}$ of diameter $\leq R$ the maps

$$\tilde{H}_i(K) \to \tilde{H}_i(N_{\psi(R)}^{(k+1)} K), i = 0, ..., k,$$

are trivial. A complex X is called *uniformly acyclic* if it is uniformly k-acyclic for each $k = 0, 1, ...$ One defines uniform contractibility in the similar fashion using homotopy groups instead of the homology groups.

An important class of metric cell complexes is given by *Rips complexes* of bounded geometry metric spaces. Let Z be such a metric space and $D \in \mathbb{R}_+$. The *D-Rips complex* $\mathrm{Rips}_D(Z)$ is defined to be the simplicial complex whose vertex set is Z, where distinct points $x_0, ..., x_n \in Z$ span an n-simplex in $\mathrm{Rips}_D(Z)$ iff $d(x_i, x_j) \leq D$ for all $0 \leq i, j \leq n$. Thus we get a direct system of Rips complexes $\mathrm{Rips}_D(Z)$ with the inclusion morphisms $\mathrm{Rips}_D(Z) \to \mathrm{Rips}_{D'}(Z)$ for $D \leq D'$.

We metrize R-Rips complexes by taking the largest metric for which all simplicial embeddings $\sigma \to \mathrm{Rips}_R(Z)$ are 1-Lipschitz, where σ is a regular Euclidean simplex with side length R. We define the control map $c : \mathrm{Rips}_R(Z) \to Z$ by sending each simplex to one of its vertices.

If Z has bounded geometry, so does $\mathrm{Rips}_R(Z)$ for each $R < \infty$. If Z is δ-hyperbolic then for each $R \geq 10\delta$, the complex $\mathrm{Rips}_R(Z)$ is uniformly contractible, see [10, Lemma 17.A].

Proposition 2.6. *Suppose that $Y^{(2)}$ is simply-connected. Then $\mathrm{Rips}_d(Y^{(0)})$ is 1-connected provided that d is sufficiently large.*

Proof. Let d_i denote the supremum of the diameters of the sets $p(\sigma^i)$, where the supremum is taken over all i-cells $\sigma^i \subset Y^{(i)}$. Since Y is connected, it is clear that $\mathrm{Rips}_d(Y^{(0)})$ is connected for each $d \geq d_1$.

Let $d \geq d_1$. Consider a loop $\gamma : S^1 \to \mathrm{Rips}_d^{(1)}(Y^{(0)})$. After homotoping γ if necessary, we may assume that it is a simplicial map with respect to some triangulation $\mathcal{T}$ of S^1. Define a map $\gamma_1 : S^1 \to Y^{(1)}$ as follows. For each vertex v of $\mathcal{T}$, let $\gamma_1(v) \in Y^{(0)} \equiv \mathrm{Rips}_d^{(0)}(Y^{(0)})$ be equal to $\gamma(v)$. For each edge $e = [v_1 v_2]$ of $\mathcal{T}$, let $\gamma_1\big|_e$ be a geodesic in $Y^{(1)}$ between $\gamma_1(v_1)$ and $\gamma_1(v_2)$. There is a natural map $Y^{(1)} \xrightarrow{i_1} \mathrm{Rips}_d^{(1)}(Y^{(0)})$ which takes each $v \in Y^{(0)}$ to the corresponding vertex of $\mathrm{Rips}_d^{(0)}(Y^{(0)})$ and maps each edge of $Y^{(1)}$ at constant speed to the corresponding edge of $\mathrm{Rips}_d^{(1)}(Y^{(0)})$. Let $\gamma_2 := i_1 \circ \gamma_1$.

If $d \geq d_2$ then i_1 can be extended to a map $Y^{(2)} \xrightarrow{i_2} \mathrm{Rips}_d^{(2)}(Y^{(0)})$. This implies that γ_2 is null-homotopic in $\mathrm{Rips}_d^{(2)}(Y^{(0)})$.

On the other hand, we claim that γ_2 is homotopic to γ in $\mathrm{Rips}_d^{(2)}(Y^{(0)})$. To see this, for each edge $e = [v_1 v_2]$ of $\mathcal{T}$, let $y_0 = \gamma(v), y_1, ..., y_m = \gamma(w)$ be the vertices of $Y^{(1)}$ on $\gamma_1(e)$ so that $\gamma_2(e)$ is the concatenation of the edges

$$[y_0 y_1], ..., [y_{m-1} y_m] \subset \mathrm{Rips}_d^{(2)}(Y^{(0)}).$$

Since $\gamma_1(e)$ is a geodesic between y_0, y_m and $d_{Y^{(1)}}(y_0, y_m) \leq d$, we get:

$$d_{Y^{(1)}}(y_0, y_i) \leq d, i = 1, ..., m-1.$$

Hence each triple of vertices y_0, y_i, y_m spans a 2-simplex Δ_i in $\mathrm{Rips}_d^{(2)}(Y^{(0)})$. Together these simplices define a homotopy between $\gamma(e)$ and $\gamma_2(e)$ (rel. the end-points). Thus the loops γ and γ_2 are homotopic. $\square$

2.6. **Finiteness properties of groups.** We recall that a group G is said to have type F_n ($n = 1, 2, ..., \infty$) if there exists an $(n-1)$-connected n-dimensional cell complex X and a discrete free action $G \curvearrowright X$ such that $X^{(i)}/G$ is compact for each $i < \infty$.

This notion of finiteness has the following homological generalization (see [4]): A group G is said to be of type FP_n (over a commutative ring $\mathcal{R}$) if there exists a partial resolution of $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules:

$$P_n \to \ldots \to P_0 \to \mathcal{R} \to 0.$$

The group G is of type FP if there exists a finite resolution

$$0 \to P_n \to \ldots \to P_0 \to \mathcal{R} \to 0.$$

of $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules.

Proposition 2.7. *Suppose that there exists an $(n-1)$-connected n-dimensional cell complex Y and a discrete action $G \curvearrowright Y$ such that $Y^{(i)}/G$ is compact for each $i < \infty$. Then G is of type F_n.*

Proof. We note that if the action $G \curvearrowright Y$ were free, then this action would satisfy the properties stated in the definition of a group of type F_n and there would be nothing to prove. Our goal is to modify Y to make the action free. We do this by induction on skeleta. Let $i = 1$. Since $G \curvearrowright Y^{(1)}$ is cocompact and Y is connected, we conclude that G is finitely-generated. Hence we take $X^{(1)}$ to be a Cayley graph of G. Suppose that $2 \le i \le n$ and an $(i-1)$-connected complex $X^{(i)}$ together with a free discrete cocompact action $G \curvearrowright X^{(i)}$ was constructed. We convert X into a metric cell complex by taking a G-equivariant control map $p : X \to X^{(0)}$. Let $x_0 \in X^{(0)}$ be a base-point.

Lemma 2.8. *There are finitely many spherical i-cycles $\sigma_1, \ldots, \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_i(X^{(i)})$, in the sense that the normal closure of the cycles $\{\hat{\sigma}_j : j = 1, \ldots, k\}$ is $\pi_i(X^{(i)})$, where each $\hat{\sigma}_j$ is obtained from σ_j by attaching a "tail" from x_0.*

Proof. Without loss of generality we can assume that $X^{(i)}$ and Y are simplicial complexes. Let $f : X^{(i)} \to Y^{(i)}$ be a G-equivariant continuous map. Consider the embedding

$$X^{(i)} \hookrightarrow Cone(f)$$

where $Cone(f)$ is the mapping cone. Since $Cone(f)$ is i-connected, according to [11, Proof of Lemma 5.8], there are finitely many spherical i-cycles $\sigma_1, \ldots, \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_i(X^{(i)})$. Here is the brief outline of the construction of σ_j's:

Let $\tau_\alpha : S^i \to Y^{(i)}, \alpha \in \mathbb{N}$, denote the attaching maps of the $(i+1)$-cells in Y, these maps are just simplicial homeomorphic embeddings from the boundary S^i of the standard $(i+1)$-simplex into $Y^{(i)}$. Starting with a G-equivariant projection $Y^{(0)} \to X^{(0)}$ one inductively constructs a (non-equivariant) map $\bar{f} : Y^{(i)} \to X^{(i)}$ so that $f \circ \bar{f} : Y^{(i)} \to Y^{(i+1)}$ is homotopic to the identity inclusion $id : Y^{(i)} \hookrightarrow Y^{(i+1)}$ via a homotopy H whose tracks have "uniformly bounded complexity": The compositions

$$H \circ (\tau_\alpha \times id) : S^i \times I \to Y^{(i+1)}$$

are simplicial maps with a uniform upper bound on the number of simplices in a triangulation of $S^i \times I$. We let σ_α denote the composition $g_\alpha \circ \bar{f} \circ \tau_\alpha$ where $g_\alpha \in G, \alpha = 1, ..., k$ are chosen so that the image of σ_α intersects a fixed fundamental domain for the action $G \curvearrowright X^{(i)}$. $\square$

We now equivariantly attach $(i+1)$-cells along G-orbits of the cycles σ_j: For each j and $g \in G$ we attach an $(i+1)$-cell along $g(\sigma_j)$. Note that if σ_j is stabilized by a subgroup of order $m = m(j)$ in G, then we attach m copies of the $(i+1)$-dimensional cell along σ_j. We let $X^{(i+1)}$ denote the resulting complex and we extend the G-action to $X^{(i+1)}$ in obvious fashion. It is clear that $G \curvearrowright X^{(i+1)}$ is free, discrete and cocompact. $\square$

Corollary 2.9. *Suppose that G is Gromov-hyperbolic. Then G is of type F_∞.*

Proof. Use the action of G on its contractible Rips complex Y. $\square$

2.7. Gromov's coarse version of the Cartan-Hadamard theorem.

Theorem 2.10. *(Cf. [10], [2, Theorem 8.1.2]) There are constants d_0, C_1, C_2, and C_3 with the following property. Let X be a metric space of bounded geometry. Assume that for some δ, and $d \geq \max(C_1\delta, d_0)$, every ball of radius $C_2 d$ in X is δ-hyperbolic, and $\mathrm{Rips}_d(X)$ is 1-connected. Then X is $C_3 d$-hyperbolic.*

One can give a direct proof of this theorem modelled on the proof of the Cartan-Hadamard theorem. Instead of doing this, we will use 6.8M and 6.8N from [10].

Remark 2.11. We note that the δ-hyperbolicity in the statements of 6.8M and 6.8N is to be taken in Gromov's sense.

Taking $A_0' = 500d^2$ in [10, 6.8M] we get:

Theorem 2.12 (6.8M, adapted version)**.** *Suppose that X is a metric space of bounded geometry, such that for some $d \geq 0$ every simplicial circle S' in $P_d^1(X)$ with*

$$500d^2 \leq A(S') \leq 64(500d^2)$$

satisfies

$$L(S') \geq d\sqrt{(4000)(64)(500)} \tag{2.13}$$

and $P_d^2(X)$ is 1-connected. Then $P_d^1(X)$ is $(400)\sqrt{500}d$-hyperbolic in the sense of Rips (see [10, 6.8.J]) and X is $(400)\sqrt{500}d$-hyperbolic in the sense of Gromov.

Here $P_d^m(X)$ denotes the m-skeleton of the Rips complex $\mathrm{Rips}_d^{(m)}(X)$ (endowed with a metric for which each simplex is path-isometric to a regular Euclidean simplex of side length d). The quantities $L(S')$ and $A(S')$ are the length of S' and the minimal area of a null-homotopy of S', where length and area are computed using the metric on $P_d^2(X)$ rather than the combinatorial length and area.

Theorem 6.8N from [10] states

Theorem 2.14 (6.8N)**.** *If X is δ-hyperbolic and $d \geq 8\delta$, then every simplicial circle $S' \subset P^1_d(X)$ satisfies $L(S') \geq \frac{d}{4\sqrt{3}} A(S')$.*

Proof of Theorem 2.10. Choose d_0 such that

$$\frac{500 d_0^2}{4\sqrt{3}} \geq \sqrt{(4000)(64)(500)}, \tag{2.15}$$

and set $C_1 := 32$, $C_2 := 64 \cdot 500$. Let $S' \subset P^1_d(X)$ be a simplicial circle with

$$500 d^2 \leq A(S') \leq 64(500 d^2) \tag{2.16}$$

and let $f : D \to P^2_d(X)$ be a least area simplicial 2-disk filling S'. There are at most (64)(500) triangles in the triangulated 2-disk D which are mapped isomorphically by f, by (2.16). Therefore, if we look at $\mathrm{Im}(f) \subset P^2_d(X)$, and let $W \subset \mathrm{Im}(f)$ be the closure of the 2-simplices contained in $\mathrm{Im}(f)$, then connected components W_i of W have diameter $\leq (64)(500)d$. This means that we can decompose D along disjoint arcs as the union of disks $D_i, i = 1, ..., k+1$ and regions E_j, so that each $f(E_j)$ is 1-dimensional and the diameter of each "minimal 2-disk" $f(D_i)$ is at most $(64)(500)d$.

By assumption, every ball of radius $C_2 d = 64 \cdot 500 d$ is δ-hyperbolic and $d \geq C_1\delta = 32\delta$, so by applying Theorem 2.14 to $f(\partial D_1), \dots f(\partial D_{k+1})$ and adding up the results, we obtain

$$L(S') \geq \frac{d}{4\sqrt{3}} A(S') \geq \frac{d}{4\sqrt{3}} 500 d^2 \geq \frac{d}{4\sqrt{3}} 500 d_0^2 \geq d\sqrt{(4000)(64)(500)}$$

where the last inequality comes from (2.15). By Theorem 2.12 we conclude that X is $C_3 d$-hyperbolic (in Gromov's sense) where $C_3 := (400)(\sqrt{500})$. ☐

As a corollary we get:

Corollary 2.17. *There exist a constant $0 < c < \infty$ such that for each 1-connected 2-dimensional metric cell complex Y, there exists a constant $\rho = \rho(Y)$ with the property:*

If for some $R \geq \rho$, each ball $B_R(y) \subset Y$ is cR-hyperbolic (in the sense of Rips), then Y is Gromov-hyperbolic.

Proof. First of all, since Y is 1-connected, there exists a constant $D = D(Y)$ such that $\mathrm{Rips}^{(2)}_d(Y^{(0)})$ is 1-connected for each $d \geq D$, see Proposition 2.6.

Let C_1, C_2 and d_0 be the constants from Theorem 2.10. Choose ρ so that $\rho/C_2 \geq \max(D, d_0)$. Let $c := \frac{1}{4C_1C_2}$.

Set $d := R/C_2$ and $\delta := cR$. Then $\mathrm{Rips}^{(2)}_d(Y^{(0)})$ is 1-connected. If each ball $B_R(y) \subset Y$ is δ-hyperbolic (in the sense of Rips) then for each $x \in Y^{(0)}$, the ball

$B_R(x) \subset Y^{(0)}$ is 4δ-hyperbolic (in Gromov's sense). Since (by our choice of the constant c)

$$d = \frac{R}{C_2} \geq 4C_1C_2cR$$

(in fact, the equality holds), and

$$d \geq \frac{\rho}{C_2} \geq \max(D, d_0),$$

Theorem 2.10 implies that Y is Gromov-hyperbolic. ☐

Corollary 2.18. *Suppose that G is a finitely-presented group such that* some *asymptotic cone of G is a tree. Then G is Gromov-hyperbolic; in particular,* every *asymptotic cone of G is a tree. Thus, under the hypothesis of the corollary, all asymptotic cones of G are isometric.*

Proof. Let ω be a nonprincipal ultrafilter on $\mathbb{N}$, let R_j be a sequence of positive real numbers such that $\omega\text{-}\lim R_j = \infty$. Let Y be a (simply-connected) Cayley complex for G, $y_j \in Y$ is a sequence of vertices. We give Y structure of a metric cell complex so that G acts on Y isometrically. By our assumption, $\omega\text{-}\lim \frac{1}{R_j}(Y, y_j)$ is a tree for some choice of $\omega, (R_j)$ (and the base-points y_j). Thus each geodesic triangle in an R_j-ball $B(y_j, R_j)$ is δ_j-thin, where

$$\omega\text{-}\lim \frac{\delta_j}{R_j} = 0.$$

Hence the same is true for each ball $B(y, R_j)$, $y \in Y^{(0)}$. For sufficiently large j, $R_j \geq \rho = \rho(Y)$ and $\frac{\delta_j}{R_j} < c$, where ρ, c are the constants from the previous corollary. Hence, by Corollary 2.17, the complex Y is Gromov-hyperbolic and therefore G is too. ☐

We note that the above corollary is false for finitely-generated groups, see [6], [17].

Question 2.19. Suppose that G is a finitely-presented group. Is it true that all asymptotic cones of G are homeomorphic?

3. Coarse Poincare duality

Let $\check{H}^*_c(\cdot)$ denote the compactly supported Čech cohomology with coefficients in a commutative ring $\mathcal{R}$. The (relative) homology and cohomology in this section are also taken with coefficients in $\mathcal{R}$. We first recall the usual Poincare duality:

Theorem 3.1. *Suppose that X is a metric cell complex homeomorphic to an n-dimensional manifold. Then for each closed subset $W \subset X$ and $k \in \mathbb{Z}$ there is an isomorphism*

$$P_{W,k} : \check{H}^k_c(W) \to H_{n-k}(X, X \setminus W)$$

which is local in the following sense: $\mathrm{Supp}(P_{W,k}(\tau)) \subset N_{D_X}(\mathrm{Supp}(\tau))$ *for each* $\tau \in Z^k_c(W)$. *The constant* D_X *does not depend on* W *and* τ. *The family* $\{P_{W,k}\}$ *is compatible with homomorphisms induced by inclusions.*

The *coarse Poincare duality* is a coarse analogue of the above property; we remind the reader that by convention (see section 2) all our metric cell complexes have bounded geometry:

Definition 3.2. Let X be a uniformly acyclic metric cell complex. We say that X satisfies *coarse n-dimensional Poincare duality* if the following holds. For each $m \in \mathbb{N}$ there is a constant[2] $D = D_m \geq 0$ so that if $k \in \mathbb{Z}$ and $m \geq 1 + \max(k, n-k)$, then the metric cell complex complex $Y := X^{(m)}$ satisfies:

There is a system of homomorphisms $\{P_K\}$ defined for subcomplexes $K \subset Y$:

$$P_K : H^k_c(N_D(K)) \to H_{n-k}(Y, Y \setminus K),$$

which are compatible with homomorphisms induced by inclusions, and which determine an *approximate isomorphism*[3] in the sense that the homomorphisms α and β in the following commutative diagram are zero:

$$\begin{array}{ccccccc}
ker(P_{N_D(K)}) & \to & H^k_c(N_{2D}(K)) & \xrightarrow{P_{N_D(K)}} & H_{n-k}(Y, V_D(K)) & \to & coker(P_{N_D(K)}) \\
\alpha\downarrow & & \downarrow & & \downarrow & & \beta\downarrow \\
ker(P_K) & \to & H^k_c(N_D(K)) & \xrightarrow{P_K} & H_{n-k}(Y, V_0(K)) & \to & coker(P_K).
\end{array}$$

Here and in what follows $V_\rho(K) := Y - N_\rho(K)$, all neighborhoods $N_\rho = N^{(m)}_\rho$ are taken within the m-dimensional complex Y.

The homomorphisms P_K are local in the following sense: if $[\sigma] \in H^k_c(N_D(K))$ is represented by a cocycle $\sigma \in Z^k_c(N_D(K))$, then $P_K(\sigma)$ can be represented by a relative cycle τ supported in $N_D(\mathrm{Supp}(\sigma))$.

We say that a metric cell complex X satisfies *coarse n-dimensional Poincare duality in dimension j*, if the constant $D = D_m$ and the family of operators $\{P_K\}$, as above, exists for $k = j$, and $m = 1 + \max(j, n-j)$ (we do not require X to be uniformly acyclic).

Definition 3.3. A bounded geometry metric space Z satisfies coarse n-dimensional Poincare duality (over $\mathcal{R}$) if there exists a metric cell complex Y whose 0-skeleton is quasi-isometric to Z, so that Y satisfies coarse n-dimensional Poincare duality.

Definition 3.4. A 1-connected metric cell complex Y satisfying 2-dimensional coarse Poincare duality in dimension 1 will be called a *quasi-plane* (over $\mathcal{R}$).

---

[2]If X is finite-dimensional, as in Section 3, we can of course take the constant D independent on m and replace $X^{(m)}$ with X, etc.

[3]Vanishing of α means the *approximate monomorphism* and vanishing of β means the *approximate epimorphism.*

Lemma 3.5. *(See* [13]*.) Suppose that $n \geq 2$ and X is a metric cell complex which satisfies coarse n-dimensional Poincare duality over $\mathcal{R}$ in dimension* 1*. Then $X^{(1)}$ is 1-ended.*

We note that in [11] we have proven a number of coarse versions of Jordan separation theorem for complexes satisfying coarse Poincare duality. In particular:

Proposition 3.6. *Suppose that Z is a (finite-dimensional) metric cell complex satisfying n-dimensional Poincare duality, M is a bounded geometry uniformly acyclic cell complex which is homeomorphic to an n-manifold. Then for each uniform proper map $f : M \to Z$, the image $f(M^{(0)})$ is a net in $Z^{(0)}$.*

We next note that (unlike the usual Poincare duality) coarse Poincare duality is a quasi-isometry invariant property:

Proposition 3.7. *(See* [13]*.) Suppose X and X' are (uniformly acyclic) cell complexes with quasi-isometric* 0*-skeleta. Then X satisfies coarse n-dimensional Poincare duality iff X' does.*

Remark 3.8. The setting in [11] and [13] is that of (finite-dimensional) *metric simplicial complexes*. To adopt the results of [11] and [13] to the discussion in the present paper, one has to (inductively) replace each k-skeleton of a metric cell complex X with an appropriate metric simplicial complex $C^{(k)}$. Since in the formulation of the coarse Poincare duality for metric cell complexes, the constants D_m are allowed to depend on the dimension, this does not cause problems.

The following therem is a special case of a more general result proven in [13] for Gromov-hyperbolic metric cell complexes satisfying coarse Poincare duality:

Theorem 3.9. *(See* [13]*.) 1. Suppose that X is a Gromov-hyperbolic quasi-plane over $\mathcal{R}$. Then $\partial_\infty X$ is homeomorphic to the circle S^1 and $X^{(1)}$ is quasi-isometric to $\mathbb{H}^2$.*

2. Suppose in addition that $G \curvearrowright X$ is a cobounded quasi-action. Then this quasi-action is quasi-isometrically conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.

4. The surrounding function

Let Y be a metric cell complex whose 1-skeleton $Y^{(1)}$ is 1-ended, connected, and equipped with a path metric d where edges have length 1. We suppose further that $Y^{(0)}$ is *quasi-homogeneous*, i.e. there are constants $L_0, A_0 \in \mathbb{R}$ such that for all $y, y' \in Y^{(0)}$, there is an (L_0, A_0)-quasi-isometry $Y^{(0)} \to Y^{(0)}$ which maps y to y'.

Definition 4.1. A subgraph $\Gamma \subset Y^{(1)}$ *surrounds* a subset $\Sigma \subset Y^{(1)}$ if $\Gamma \cap \Sigma = \emptyset$, and Γ separates Σ from infinity, i.e. any proper path $\mathbb{R}_+ \to Y^{(1)}$ starting at Σ intersects Γ.

If $\Gamma \subset Y^{(1)}$ is a subgraph, we let $|\Gamma|$ denote the cardinality of the vertex set of Γ; we will refer to $|\Gamma|$ as the *size* of Γ.

Definition 4.2. Given $y \in Y^{(0)}$, $R \in \mathbb{R}_+$, let $\mathrm{Sur}(y, R)$ be the minimum of sizes of connected subgraphs $\Gamma \subset Y^{(1)}$ which surround $B(y, R) \subset Y^{(1)}$. We will refer to a graph Γ which realizes the minimum as a *smallest* graph which surrounds $B(y, R)$.

Lemma 4.3. *The function* $\mathrm{Sur}(\cdot, \cdot)$ *is "quasi-invariant": There are constants* C *and* R_0 *such that* $\mathrm{Sur}(y', \frac{R}{C}) \le C\,\mathrm{Sur}(y, R)$ *for all* $y, y' \in Y$, $R > R_0$.

Proof. Pick $y, y' \in Y^{(0)}$. There are (L, A)-quasi-isometries $f : Y^{(0)} \to Y^{(0)}$ and $f' : Y^{(0)} \to Y^{(0)}$ such that $f(y) = y'$, $f'(y') = y$, and $d(f' \circ f, \mathrm{id}_{Y^{(0)}}) < A$, $d(f \circ f', \mathrm{id}_{Y^{(0)}}) < A$, where L, A are independent of y, y'.

Choose a smallest connected subgraph $\Gamma \subset Y^{(0)}$ which surrounds $B(y, R)$. Then $d(y', f(\Gamma^{(0)})) \ge \frac{1}{L}R - A$. If $x' \in Y^{(0)}$ and $Ld(y', x') + A \le R$, then $f'(x') \in B(y, R)$, and consequently if $\rho : \mathbb{R}_+ \to Y^{(1)}$ is any proper path with $\rho(0) = x'$, then by the uniform connectedness of $Y^{(0)}$, the set $f'((\mathrm{Im}(\rho) \cap Y^{(0)})$ must pass within distance $D_1 = D_1(L, A, Y)$ of Γ, which means that ρ must pass within distance $D_2 = D_2(L, A, Y)$ of $f(\Gamma^{(0)})$. Applying the uniform connectedness of $Y^{(1)}$ again, we can find a connected subgraph $\Gamma' \subset Y^{(1)}$ such that

$$N_{D_2}(f(\Gamma^{(0)})) \subset \Gamma' \subset N_{D_3}(f(\Gamma'))$$

for $D_3 = D_3(L, A, Y)$. If R' satisfies $R' + D_3 < \frac{1}{L}R - A$, then $B(y', R') \cap \Gamma' = \emptyset$, and hence Γ' surrounds $B(y', R)$. Since $Y^{(1)}$ has bounded geometry, there is a constant $c = c(Y)$ such that $|\Gamma'| \le c|f(\Gamma^{(0)})| \le c|\Gamma|$. Taking $C = \max(c, 2L)$, and R_0 sufficiently large, the lemma follows. $\square$

Observe that the quantity $\mathrm{Sur}(y, R)$ is finite since the space $Y^{(1)}$ is 1-ended. Note also that, since Y is quasi-homogeneous, there are constants L, A such that for each point $y \in Y^{(0)}$ there exists a 1-Lipschitz (L, A) quasi-geodesic γ through y (with $L \ge 1, A \ge 0$).

Lemma 4.4. *1. For each* $R \ge R_0 := 2AL^2$ *and each connected subgraph* Γ *which surrounds* $B(y, R)$ *we have:* $\mathrm{diam}(\Gamma) \ge \frac{1}{L^2}R$.

2. There exists $C_0 > 0$ *such that for all* $R \ge 1$, $\mathrm{Sur}(y, R) \ge C_0 R$.

3. Suppose that Γ *is a graph which surrounds a ball* $B(y, R)$, $R \ge R_0$. *Then for* $r < R/L^2$ *and each* $z \in \Gamma$ *the size of* $\Gamma \cap B(z, r)$ *is at least* r.

4. There is a constant $C_1 = C_1 \ge 1$ *such that any connected subgraph* Γ *which surrounds an* R*-ball* $B(y, R)$ *must lie in* $B(y, C_1|\Gamma|)$.

Set $C_2 = \max(2C_1, 1/C_0)$.

5. If Γ *surrounds* $B(y, R) \subset Y^{(1)}$, *then the connected component of* y *in* $Y^{(1)} \setminus \Gamma$ *is contained in* $B(y, C_2|\Gamma|)$.

Proof. 1. Let $\Gamma \subset Y^{(1)}$ be a connected graph which surrounds the ball $B(y, R)$. Consider an (L, A) quasi-geodesic $\gamma : \mathbb{R} \to Y^{(1)}$ as above, $\gamma(0) = y$. Since Γ surrounds $B(y, R)$ there are two points $y_\pm \in \Gamma$ such that $\gamma(T_\pm) = y_\pm$, with $T_- < 0 < T_+$. Thus

$$\mathrm{diam}(\Gamma) \geq d(y_-, y_+) \geq \frac{1}{L}(T_+ - T_-) - A \geq \frac{1}{L}(\frac{2}{L}R - 2A) - A \geq \frac{1}{L^2}R,$$

for $R \geq 2AL^2$. This proves (1).

Note also that there are $t_- \in [T_-, 0], t_+ \in [0, T_+]$ such that $\gamma(t_\pm) = x_\pm$ are within distance R from y. Since Γ is connected we have:

$$d(y_+, y_-) \leq |\Gamma|.$$

On the other hand, since γ is a (L, A) quasi-geodesic,

$$R \lesssim d(x_+, x_-) \lesssim |T_+ - T_-| \sim d(y_+, y_-).$$

Hence $R \lesssim |\Gamma| \leq \mathrm{Sur}(y, R)$.

3. Recall that, according to (2), $\mathrm{diam}(\Gamma) \geq \frac{1}{L^2}R$. Hence for each $r < R/L^2$ and $z \in \Gamma$, the metric sphere $S(z, r)$ has nonempty intersection with Γ. Thus, since Γ is connected, the intersection $\Gamma \cap B(z, r)$ contains at least r points, vertices of a path in $\Gamma \cap B(z, r)$ connecting z to $S(z, r)$.

4. Set $C_1 := 2L^2(A + 1)$. Let γ be a quasi-geodesic as in (1). Let's estimate the distance $d(y_\pm, y)$. We have:

$$d(y_\pm, y) \leq L|T_\pm| + A, \quad |T_+ - T_-| \leq L(d(y_+, y_-) + A).$$

Hence

$$d(y_\pm, y) \leq L(L(d(y_+, y_-) + A)) + A = L^2 d(y_+, y_-) + (L^2 + 1)A.$$

However, since Γ is connected, $d(y_+, y_-) \leq |\Gamma|$, and therefore

$$d(y_\pm, y) \leq L^2|\Gamma| + (L^2 + 1)A.$$

If $z \in \Gamma$, then $d(z, y_+) \leq |\Gamma|$, which implies that

$$d(z, y) \leq 2L^2|\Gamma| + (L^2 + 1)A \leq 2L^2(A + 1)|\Gamma|,$$

because $|\Gamma| \geq 1$. Thus $\Gamma \subset B(C_1|\Gamma|, y)$.

5. Consider a point $z \in Y^{(1)} \setminus \Gamma$ which lies in the same component of $Y^{(1)} \setminus \Gamma$ as y. Suppose that $z \notin N_R(\Gamma)$. Then Γ surrounds both $B(y, R)$, $B(z, R)$ and hence, by (4),

$$\Gamma \subset B(C_1|\Gamma|, y), \quad \Gamma \subset B(C_1|\Gamma|, z).$$

By the triangle inequality we conclude that $d(y, z) \leq 2C_1|\Gamma|$. If $d(z, \Gamma) \leq R$ then (by (2)) $d(z, y) \leq R \leq |\Gamma|/C_0$. Therefore $z \in B(y, C_2|\Gamma|)$ in this case as well. □

Lemma 4.5. *Suppose there is a constant $C \geq 2$ such that* $\mathrm{Sur}(y, R) < CR$ *for all $y \in Y^{(0)}$, and all $R \geq 1$. Then $Y^{(0)}$ is* doubling *(see section 2), and hence has polynomial growth.*

Proof. In the proof we will be using the constants C_i and R_0 from the previous lemma.

Without loss of generality we may assume that $C, C_2 \in \mathbb{N}$. Recall that (according to Lemma 4.4, Part 5), if Γ surrounds $B(y,R) \subset Y^{(1)}$ and has size $\le CR$, then the connected component of y in $Y^{(1)} \setminus \Gamma$ is contained in $B(y, C_2CR)$.

Pick $y \in Y^{(0)}$, $R \ge 1$. Choose a connected graph $\Gamma_1 \subset Y^{(1)}$ with size at most CR which surrounds $B(y,R)$, and set $\mathcal{L}_1 = \{\Gamma_1\}$. Let $\mathcal{N}_1$ be an $\frac{R}{2L^2}$-separated $\frac{R}{2}$-net in Γ_1. Then, by Lemma 4.4 (Part 3), the cardinality of $\mathcal{N}_1$ is at most

$$\frac{|\Gamma_1|}{R/(4L^2)} \le c := 4L^2C.$$

Let $\mathcal{L}_2$ be a collection of connected subgraphs (each having size at most CR) of $Y^{(1)}$ surrounding the R-balls centered at points in $\mathcal{N}_1$. Proceed inductively in this fashion, building up k layers of surrounding connected subgraphs of $Y^{(1)}$. The union $V_k := \mathcal{N}_0 \cup \ldots \cup \mathcal{N}_k$ has cardinality at most

$$c^{k+1} = (4L^2C)^{k+1}.$$

We claim that the collection of C_2CR-balls centered at points in V_k covers $B(y, \frac{kR}{2})$. To see this, consider a path σ of length at most $\frac{kR}{2}$ starting at y. We inductively break σ into a concatenation of at most k subpaths of length at least $\frac{R}{2}$ as follows. Let σ_1 be the initial segment of σ until it arrives at Γ_1. The path σ_1 terminates within distance $\frac{R}{2}$ of a point $y_1 \in \mathcal{N}_1$. Let σ_2 be the initial segment of $\sigma \setminus \sigma_1$ until it arrives at the connected subgraph $\Gamma_2 \in \mathcal{L}_2$ surrounding $B(y_1, R)$. Et cetera. At each step, the segment σ_i has length at least $\frac{R}{2}$, and (by Lemma 4.4, Part 5) all of them are contained in $\cup_{q \in V_k} B(q, C_2CR)$.

Thus

$$B(y, \frac{kR}{2}) \subset \bigcup_{i=1}^{c^{k+1}} B(y_i, C_2CR),$$

for each $R \ge R_0$, and each $y \in Y^{(0)}$. Choosing k such that $[\frac{k}{2}] = 2C_2C$, and setting $\rho = C_2CR$ we see that

$$B(y, 2\rho) \subset \bigcup_{i=1}^{N} B(y_i, \rho),$$

where $N = c^{4C_2C+2}$ is independent of ρ. Hence $Y^{(0)}$ is doubling and thus has polynomial growth. $\square$

Below is an alternative argument for the polynomial growth of $Y^{(0)}$ in the case when it is quasi-isometric to a (finitely generated) group. Let $V(y,r)$ denote the number of points in $B(y,r) \cap Y^{(0)}$ for $y \in Y^{(0)}$.

Proposition 4.6. *Suppose there is a constant C and a sequence $R_j \in \mathbb{R}_+$ diverging to ∞, such that $\mathrm{Sur}(y, R_j) < CR_j$ for all $y \in Y^{(0)}$, and each j. Assume in adition that $Y^{(1)}$ is the Cayley graph of a finitely generated group Q. Then Q is virtually nilpotent and has at most quadratic growth.*

Proof. We recall (N. Varopoulos, [19]) that if there are constants $C_0, a \in \mathbb{R}$ so that

$$Vol(y,r) \geq C_0 r^a, \quad \forall r \geq 1, \quad \forall y \in Y^{(0)}, \tag{4.7}$$

then there is $C_1 \in \mathbb{R}$ so that

$$|\partial D| \geq C_1 |D|^{(a-1)/a}$$

for all finite subsets $D \subset Y^{(0)}$. (Here $\partial D = \partial_1 D$ is the "boundary" of D, see section 2.) Let $\mathrm{Inrad}(D)$ denote the radius of the largest metric ball $B(z,r) \subset Y^{(1)}$ such that in $B(z,r) \cap Y^{(0)}$ is contained in D. Hence

$$|\partial D| \geq C' |D|^{(a-1)/a} \geq C_1 [C_0 \cdot \mathrm{Inrad}(D)^a]^{(a-1)/a} = C_2\, \mathrm{Inrad}(D)^{a-1}.$$

On the other hand, the hypothesis of the proposition implies that for each $y \in Y^{(0)}$ each ball $B(y, R_j) \subset Y^{(1)}$ is surrounded by a connected graph $\Gamma_{R_j} \subset Y^{(1)}$ of the size $\leq CR$ ($R \in \mathbb{N}$). Let D_R denote the vertex set of the component of y in $Y^{(1)} \setminus \Gamma_R$. Then

$$|\partial D_{R_j}| \leq C_3 |D_{R_j}| \leq C_4 R_j \leq C_4\, \mathrm{Inrad}(D_{R_j})$$

for all $j \in \mathbb{N}$. Thus (4.7) cannot hold in $Y^{(0)}$ for any $a > 2$. Thus for each $a > 2$ there are sequences $y_j \in Y^{(0)}, r_j \in \mathbb{N}$ so that

$$Vol(y_j, r_j) \leq C_0 r_j^a.$$

Hence, by the improvement of Gromov's theorem on groups of polynomial growth due to Van den Dries and Wilkie [18], the group Q is virtually nilpotent and moreover its growth $\leq a$ for each $a > 2$. Thus Q has at most quadratic growth. $\square$

We now strengthen our assumptions on Y. We assume in addition that $Y^{(2)}$ is 1-acyclic and satisfies 2-dimensional coarse Poincare duality in dimension 1, i.e. the statement of Definition 3.2 holds for $n = 2$ and $k = 1 = n - k = m - 1$.

Lemma 4.8. *There are constants D and R_1 such that when $R \geq R_1$, the D-neighborhood of each R-fat geodesic triangle Δ in $Y^{(1)}$ will surround a ball of radius $\frac{R}{10}$.*

Proof. Let $\Delta \subset Y^{(1)}$ be an R-fat geodesic triangle, and let x be a point on one of the sides γ of Δ whose distance to the remaining two sides exceeds R. Let $\alpha \in C^1(\gamma)$ be a 1-cocycle supported in $B^{(1)}(x,1) \cap \gamma$, representing the fundamental class of the side γ relative to its boundary. Construct a map $f : Y^{(2)} \to \gamma$ by letting $f\big|_{Y^{(0)}} \to \gamma$ be a nearest point map, and extending f to $Y^{(2)}$ using the uniform contractibility of γ. For all $y \in Y^{(2)}$ we will have

$$d(p(f(y)), p(y)) \leq d(p(y), \gamma^{(0)}) + C_1 \tag{4.9}$$

where $p : Y \to Y^{(0)}$ is the control map, and C_1 is independent of γ. Using $f\big|_{N^{(2)}_{\frac{R}{3}}(\gamma)}$, pullback α to a 1-cocycle

$$\hat{\alpha} := \left(f\Big|_{N^{(2)}_{\frac{R}{3}}(\gamma)}\right)^*(\alpha) \in Z^1(N^{(2)}_{\frac{R}{3}}(\gamma)).$$

Extending $\hat{\alpha}$ by zero, we get a cochain α' in $C^1(N^{(2)}_{\frac{R}{3}}(\Delta))$.

Recall that there is a constant C_2 depending only on $Y^{(2)}$ such that if σ is a 2-cell of $Y^{(2)}$ and τ is a 1-cell appearing in the boundary of σ, then $d(p(\tau), p(\sigma)) < C_2$. Using this and (4.9), it follows that if $R \geq C_3$ for $C_3 = C_3(Y^{(2)})$, σ is a 2-cell in $N^{(2)}_{\frac{R}{3}}(\Delta)$, τ is a 1-cell in the boundary of σ, and $\alpha'(\tau) \neq 0$, then the whole boundary of σ lies outside the $\frac{R}{3}$-neighborhoods of the other two sides of Δ; therefore the boundary of σ lies in $N^{(2)}_{\frac{R}{3}}(\gamma)$, which means that σ itself lies in $N^{(2)}_{\frac{R}{3}}(\gamma)$, and so $\alpha'(\partial\sigma) = \hat{\alpha}(\partial\sigma) = 0$. Thus α' is a 1-cocycle when $R \geq C_3$, which we henceforth assume. The restriction of α' to Δ is nontrivial in H^1 because of excision and the nontriviality of α.

Applying our coarse Poincaré duality assumption we get that α' is "dual" to an element

$$c := P_{N^{(2)}_{\frac{R}{3}}(\Delta)}(\alpha') \in H_1(Y^{(2)}, Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-D}(\Delta))$$

which maps nontrivially to $H_1(Y^{(2)}, Y^{(2)} \setminus N^{(2)}_D(\Delta))$ provided $\frac{R}{3} \geq 2D$, where D is the constant in the statement of coarse Poincaré duality. Since Y is 1-acyclic, this means that the 0-chain ∂c is a linear combination $\partial c = \sum a_i y_i$ with nonzero coefficients, where the y_i's lie outside $N^{(2)}_{\frac{R}{3}-D}(\Delta)$, and some pair y_k, y_l of the support points cannot be joined by a curve in $Y^{(2)} \setminus N^{(2)}_D(\Delta)$.

Set $y'_k := p(y_k)$, $y'_l := p(y_l)$. Then $y'_k, y'_l \in Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-r_1}(\Delta)$ for $r_1 = r_1(Y^{(2)})$, provided $\frac{R}{3} \geq r_1$. Also, the points y_k and y'_k (resp. y_l and y'_l) lie in the same component of $Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-r_2}(\Delta)$ for $r_2 = r_2(Y^{(2)}) > r_1$, provided $\frac{R}{3} \geq r_2$. Therefore $N^{(1)}_D(\Delta)$ separates one of the balls $B(y'_k, \frac{R}{10})$, $B(y'_l, \frac{R}{10})$ from infinity when R is sufficiently large. $\square$

Proposition 4.10. *Suppose in addition that $\pi_1(Y)$ is trivial. Then there are constants R_2 and C_h such that if $\mathrm{Sur}(p, r) > C_h r$ for some $p \in Y$, $r \geq R_2$, then Y is Gromov hyperbolic.*

Proof. We will verify hyperbolicity by invoking a corollary of Gromov's coarse Cartan-Hadamard theorem, see Corollary 2.17. Let $\rho = \rho(Y^{(2)})$ and c be constants from Corollary 2.17. According to Lemma 4.3, there exists $C > 0$ so that

$$\mathrm{Sur}(y, r) \leq C\,\mathrm{Sur}(y', Cr) \tag{4.11}$$

for all $y' \in Y^{(0)}$ provided $r > R_0 = r_0/C$, which we will henceforth assume. Pick a number

$$C_h \geq \frac{40mC^2}{c}.$$

We assume that $r \geq \frac{4mC\rho}{C_h}$. Suppose $\mathrm{Sur}(y, r) \geq C_h r$ for some $y \in Y^{(0)}$. Then

$$C_h r \leq \mathrm{Sur}(y, r) \leq C\,\mathrm{Sur}(y', Cr) \tag{4.12}$$

for each $y' \in Y^{(0)}$. Consider a geodesic triangle $\Delta \subset Y^{(1)}$ which is $10Cr$-fat. By Lemma 4.8, there exists $r_1 > 0$ so that if $r \geq r_1/C$, which we will assume from now on, then $N_D^{(1)}(\Delta)$ (the D-neighborhood of Δ in $Y^{(1)}$) surrounds some Cr-ball $B(y', Cr)$. Therefore

$$\frac{C_h r}{C} \leq \mathrm{Sur}(y', Cr) \leq m\,\mathrm{length}(\Delta),$$

where m is the maximum of cardinality of $(D + 1)$-balls in $Y^{(0)}$. Therefore, any triangle contained in a ball of radius $R = \frac{C_h r}{4mC}$ is $\delta := 10Cr$-thin. According to our choice of C_h we have:

$$\frac{\delta}{R} = \frac{10Cr}{C_h r/(4mC)} = \frac{40mC^2}{C_h} = \frac{40mC^2}{C_h} \leq c.$$

Our assumption that $r \geq \frac{4mC\rho}{C_h}$ implies that $R \geq \rho$. Hence (since Y is 1-connected) we are in position to apply Corollary 2.17 and conclude that Y is Gromov-hyperbolic. $\square$

Corollary 4.13. *Suppose that Y is a quasi-plane so that $Y^{(0)}$ is quasi-homogeneous. Then either:*

1. *The surrounding function of Y is superlinear, i.e. for all C_h there is a constant R_2 such that $\mathrm{Sur}(y, R) > C_h R$ for some $y \in Y^{(0)}$ and all $R \geq R_2$. In this case $Y^{(1)}$ is Gromov-hyperbolic.*
2. *Or the surrounding function of Y is sublinear, i.e. there is a constant C such that $\mathrm{Sur}(y, R) < CR$ for all $y \in Y^{(0)}$, $R \geq 1$. In this case $Y^{(1)}$ is doubling, and has polynomial growth.*

5. $PD(2)$ groups are virtually surface groups

In this section, we show how to apply the surrounding function to prove that $PD(2)$ groups over arbitrary commutative rings $\mathcal{R}$ are virtually surface groups. This was proven by Eckmann, Linnell and Müller [7, 8] in the case of $\mathcal{R} = \mathbb{Z}$ and, more recently, by Brian Bowditch for field coefficients, [3]. Our main result is

Theorem 5.1. *Suppose that G is a 2-dimensional Poincare duality group over a commutative ring $\mathcal{R}$ with a unit. Then G is virtually a surface group.*

5.1. $PD(n)$ **groups over a ring.** Poincare duality groups of dimension n over rings are generalizations of the fundamental groups of closed aspherical n-manifolds. Thus we begin our discussion with the motivating example of Poincare duality (over $\mathbb{Z}$) for manifolds. If M is an oriented triangulated (connected) n-manifold, Poincare duality induces a chain homotopy equivalence $C_c^*(M) \to C_{n-*}(M)$ between the complex of compactly supported simplicial cochains and the simplicial chain complex. If a group G acts freely simplicially on M, then Poincare duality is G-equivariant, up to twisting by the orientation module. More precisely, let $D := H_c^n(M)$ equipped with the left G-action $g \cdot \alpha := (g^{-1})^* \alpha$. Then Poincare duality defines a G-equivariant map $C_c^*(M) \to D \otimes_{\mathbb{Z}} C_{n-*}(M)$ where G acts on $D \otimes_{\mathbb{Z}} C_{n-*}(M)$ by the diagonal action. The task of this subsection is to establish a similar statement for Poincare duality groups in general.

Let $\mathcal{R}$ be a commutative ring with unit. Recall [1], that an n-dimensional Poincare duality group over $\mathcal{R}$ (for short, $PD(n)$ group over $\mathcal{R}$), is an FP-group over $\mathcal{R}$ (see section 2.6) such that $H^i(G, \mathcal{R}G)$ is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module when $i = n$ and is trivial otherwise. This implies that the cohomological dimension of G over $\mathcal{R}$ is equal to n [4, p. 202], and so we may choose a resolution

$$0 \to A_n \to \ldots \to A_0 \to \mathcal{R} \to 0 \tag{5.2}$$

of the trivial $\mathcal{R}G$-module $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules, [4, p. 199]. Applying the functor $\mathrm{Hom}_{\mathcal{R}G}(\cdot, \mathcal{R}G)$ to the resolution A_*, we obtain cochain complex

$$0 \leftarrow B^n \leftarrow \ldots \leftarrow B^0 \leftarrow 0$$

where $B_i := \mathrm{Hom}_{\mathcal{R}G}(A_i, \mathcal{R}G)$. The complex B_* computes $H^*(G, \mathcal{R}G)$ (recall that $H^n(G, \mathcal{R}G) \cong \mathcal{R}$ and $H^i(G, \mathcal{R}G) = 0$ if $i \neq r$), and hence we can extend B^* to a "dual" resolution

$$0 \leftarrow D \leftarrow B^n \leftarrow \ldots \leftarrow B^0 \leftarrow 0 \tag{5.3}$$

where, following [4, p. 219], we set $D := H^n(G, \mathcal{R}G)$. The $\mathcal{R}G$-bimodule structure on $\mathcal{R}G$ induces a right $\mathcal{R}G$-module structure on the resolution (5.3). We will find it more convenient to convert the right G-action to a left G-action by taking inverses of group elements; in particular, we will view D as a left $\mathcal{R}G$-module.

For each left $\mathcal{R}G$-module M, define a new left $\mathcal{R}G$-module $\tau(M)$, by "twisting" with the G-action on D, i.e. $\tau(M)$ is the $\mathcal{R}$-module $D \otimes_{\mathcal{R}} M$ equipped with the diagonal G-action $g \cdot (d \otimes m) = gd \otimes gm$. Consider the chain complex of left $\mathcal{R}G$-modules obtained by applying the functor τ to the resolution (5.2):

$$0 \to \tau(A_n) \to \ldots \to \tau(A_0) \to \tau(\mathcal{R}) \to 0. \tag{5.4}$$

Lemma 5.5. *1. The $\mathcal{R}G$-chain complex (5.4) is a resolution of $\tau(\mathcal{R})$.*

2. The left G-modules $\tau(\mathcal{R})$ and D are isomorphic.

3. $\tau(A_i)$ is a finitely generated projective $\mathcal{R}G$-module for each i.

Proof. The module D is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module, and hence tensoring with D over $\mathcal{R}$ does nothing to $\mathcal{R}$-module structure. So clearly (5.4) is still a resolution by $\mathcal{R}$-modules, and therefore a resolution by $\mathcal{R}G$-modules. This proves 1.

For any $\mathcal{R}G$-module M, the $\mathcal{R}$-module $M \otimes_{\mathcal{R}} \mathcal{R}$ with the diagonal G-action is $\mathcal{R}G$-isomorphic to M, which proves 2.

Before proving 3, we first make a few observations about modules isomorphic to $\mathcal{R}$ and the operation τ.

Let E be an $\mathcal{R}$-module isomorphic to $\mathcal{R}$, and let $\mathcal{R}^\times$ denote the group of units in $\mathcal{R}$. Then the map $\mathcal{R}^\times \to \mathrm{Aut}_{\mathcal{R}}(E)$ which assigns to $r \in \mathcal{R}^\times$ the automorphism $E \to E$ given by $e \mapsto re$, is an isomorphism. Hence if E is an $\mathcal{R}G$-module whose underlying $\mathcal{R}$-module is isomorphic to $\mathcal{R}$, then we obtain a character $\chi : G \to \mathcal{R}^\times$ from the equation

$$g \cdot e = \chi(g)e,$$

$g \in G$, $e \in E$. When we tensor E over $\mathcal{R}$ with an $\mathcal{R}G$-module M and let G act diagonally, then up to $\mathcal{R}G$-isomorphism, we may think of the resulting $\mathcal{R}G$-module $M' = E \otimes_{\mathcal{R}} M$ as having the same underlying $\mathcal{R}$-module as M, where the G-action is twisted by the character χ:

$$(g, m) \mapsto \chi(g)gm,$$

for $g \in G$, $m \in M$. Keeping the above remarks in mind, it is then clear that when we apply τ to the group ring $\mathcal{R}G$, we get back an $\mathcal{R}G$-module isomorphic to $\mathcal{R}G$: To get the isomorphism $\phi : \mathcal{R}G \to \tau(\mathcal{R}G)$ just set $\phi(\delta_g) := \chi(g)\delta_g$ for all $g \in G$. Direct sums are respected by τ, so τ transforms free RG-modules into free $\mathcal{R}G$-modules, and hence projective modules into projective modules. Obviously τ preserves finite generation. Combining these statements yields assertion 3. $\square$

The lemma implies that the resolution (5.3) and the resolution (5.4) both define resolutions of D by left $\mathcal{R}G$-modules. Therefore the two resolutions are $\mathcal{R}G$-chain homotopy equivalent, i.e. there are $\mathcal{R}G$-chain mappings

$$P : B^* \to \tau(A_{n-*}), \quad \bar{P} : \tau(A_*) \to B^{n-*} \tag{5.6}$$

and $\mathcal{R}G$-chain homotopy operators

$$\bar{P} \circ P \overset{H}{\sim} \mathrm{id}_{B^*}, \quad P \circ \bar{P} \overset{\bar{H}}{\sim} \mathrm{id}_{\tau(A_*)} \,. \tag{5.7}$$

5.2. $PD(2)$ **groups and quasiplanes.** The goal of this section is to show that for each $PD(2)$-group G over $\mathcal{R}$ there is a 2-dimensional metric cell complex Y on which G acts freely properly discontinuously and cocompactly so that Y satisfies coarse Poincare duality in dimensions 0 and 1.

Lemma 5.8. *Suppose that G is an FP_2 group over $\mathcal{R}$. Then there exists a 2-dimensional cell complex Y which is 1-acyclic over $\mathcal{R}$ and a free cocompact cellular action $G \curvearrowright Y$.*

Proof. First we note that since G is FP_1 group over $\mathcal{R}$ and the ring $\mathcal{R}$ is nontrivial, the group G is finitely generated (see [4]).

Considier a Cayley graph Γ of G associated with a finite generating set for G. We let $Y^{(1)} := \Gamma$ be the 1-skeleton of Y. We then have a natural monomorphism of $\mathcal{R}G$-modules:

$$\iota : Z_1(\Gamma) \otimes \mathcal{R} = H_1(\Gamma) \otimes \mathcal{R} \to Z_1(\Gamma, \mathcal{R}) = H_1(\Gamma \otimes \mathcal{R}).$$

Next observe that Γ is homotopy-equivalent to the bouquet of circles B (of course the homotopy-equivalence h is not G-invariant), therefore we have the commutative diagram:

$$\begin{array}{ccc} Z_1(\Gamma, \mathbb{Z}) \otimes \mathcal{R} & \xrightarrow{\iota} & Z_1(\Gamma, \mathcal{R}) \\ h_* \otimes id \downarrow & & h_* \downarrow \\ Z_1(B, \mathbb{Z}) \otimes \mathcal{R} & \xrightarrow{j} & Z_1(B, \mathcal{R}). \end{array}$$

It is clear however that j is onto, hence ι is onto as well. The $\mathcal{R}G$-module $H_1(\Gamma, \mathcal{R})$ is finitely generated, see [4, Proposition 4.3, Chapter VIII]; let $c_1, \ldots, c_k : S^1 \to \Gamma$ denote loops whose images in $H_1(\mathcal{G}, \mathcal{R})$ (under the homomorphism ι) generate $H_1(\Gamma, \mathcal{R})$ as the $\mathcal{R}G$-module. For each $g \in G$ we attach, a 2-disk $D^2_{g,i}$ to Γ along $g \circ c_i$. It is clear that G acts freely on the resulting 2-complex Y and that $H_1(Y, \mathcal{R}) = 0$. $\square$

In what follows we will also need to analyze the case of $PD(2)$-groups which act cocompactly on an $(n-1)$-acyclic (over $\mathcal{R}$) n-dimensional complex Y, $n \geq 2$. Henceforth we assume that Y is such a complex. All (co)homology groups through this section will be with coefficients in $\mathcal{R}$.

It will be convenient to metrize the 1-skeleton of Y so that the edges have unit length, thus Y has a natural structure of a metric cell complex with a control map $Y \to Y^{(0)}$. All (co)homology in the following computations will be taken with coefficients in $\mathcal{R}$.

The cellular chain complex of Y determines a partial resolution of $\mathcal{R}$ by $\mathcal{R}G$-modules:

$$C_n(Y) \to \ldots \to C_1(Y) \to C_0(Y) \to \mathcal{R} \to 0. \tag{5.9}$$

The fundamental lemma of homological algebra, applied to the resolution (5.2) and the partial resolution (5.9), provides chain mappings

$$(f_i : A_i \to C_i(Y))_{0\le i\le n}, \text{ and } (\bar{f}_i : C_i(Y) \to A_i)_{0\le i\le n}$$

and homotopy operators

$$(\bar{f}_i \circ f_i \overset{K_i}{\sim} \mathrm{id}_{A_i})_{0\le i\le n}, \quad (f_i \circ \bar{f}_i \overset{\bar{K}_i}{\sim} \mathrm{id}_{C_i(Y)})_{0\le i<n}.$$

Using f_*, $\bar{f}_*$, K_*, and $\bar{K}_*$, we can transfer the operators P, $\bar{P}$, H, $\bar{H}$ (see (5.6), (5.7)) to $C_*(Y)$ and $C_c^*(Y)$, as chain mappings

$$(P_i : C_c^i(Y) \to \tau(C_{2-i}(Y))_{0\le i\le n}, \quad (\bar{P}_i : \tau(C_{2-i}(Y)) \to C_c^i(Y))_{0\le i\le n}$$

and chain homotopies

$$(H_i : C_c^i(Y) \to C_c^{i-1}(Y))_{0<i\le n}, \quad (\bar{H}_i : \tau(C_i(Y)) \to \tau(C_{i+1}(Y)))_{0\le i<n}.$$

Here we have used the natural isomorphisms of $\mathcal{R}G$-modules

$$\mathrm{Hom}_{\mathcal{R}G}(C_i(Y), \mathcal{R}G) \simeq C_c^i(Y),$$

where as with (5.4), right G-actions are converted to left G-actions. The key fact for us is that $(H_i)_{0<i\le n}$ gives a chain homotopy $\bar{P}_i \circ P_i \sim \mathrm{id}_{C_c^i(Y)}$, and $(\bar{H})_{0\le i<n}$ gives a chain homotopy

$$P_i \circ \bar{P}_i \sim \mathrm{id}_{\tau(C_i(Y))}.$$

As there are only finitely many G-orbits of cells in Y, it follows that if σ is an oriented i-cell in Y and $\hat{\sigma} \in C_c^i(Y)$ is the associated element, then

$$P_i(\hat{\sigma}) \in \tau(C_{2-i}(Y)) = D \otimes_{\mathcal{R}} C_{2-i}(Y)$$

can be expressed as a sum $\sum_j d_j \otimes \beta_j$ where the β_j's are oriented $(2-i)$-cells contained in a D-neighborhood of σ, where D is a universal constant. Similar statements apply to the operators $\bar{P}_*$, H_*, and $\bar{H}_*$. We therefore deduce that when T is one of these operators and $c \in \mathrm{Domain}(T)$, then $T(c)$ is supported in the D_0-neighborhood of the support of c, where D_0 is independent of c. We now forget about the G-action on Y; henceforth we will identify $\tau(C_*(Y))$ with $C_*(Y)$ and $\tau(C_c^*(Y))$ with $C_c^*(Y)$.

Lemma 5.10. *Y satisfies coarse 2-dimensional Poincare duality over $\mathcal{R}$ in dimensions $0 \le i < n$.*

Proof. This lemma is a special case of the *Coarse Poincare Duality Theorem* 6.7 proven in [11] (for the integer coefficients). For the reader's convenience we will outline a proof.

When $K \subset Y$ is a (nonempty) subcomplex we will consider the direct system of tubular neighborhoods $\{N_R(K)\}_{R\ge 0}$ of K and the inverse system of the closures of their complements

$$\{V_R := \overline{Y - N_R(K)}\}_{R\ge 0}.$$

We get four inverse and four direct systems of (co)homology groups with coeffients in $\mathcal{R}$:

$$\{H_c^i(N_R(K))\},\ \{H_i(Y,V_R)\},\ \{H_c^i(Y,N_R(K))\},\ \{H_i(V_R)\}$$
$$\{H_c^i(V_R)\},\ \{H_1(Y,N_R(K))\},\ \{H_c^i(Y,V_R)\},\ \{H_i(N_R(K))\}$$

with the usual restriction and projection homomorphisms. Note that by excision, we have isomorphisms

$$H_i(Y,V_R) \simeq H_i(N_R(K),\partial N_R(K)),\ \text{etc.}$$

Extension by zero defines a group homomorphism $C_c^i(N_{R+D_0}(K)) \overset{ext}{\subset} C_c^i(Y)$. When we compose this with

$$C_c^i(Y) \overset{P}{\to} C_{2-i}(Y) \overset{proj}{\to} C_{2-i}(Y,V_R)$$

we get a well-defined induced homomorphism

$$P_{R+D_0}: H_c^1(N_{R+D_0}(K)) \to H_1(Y,V_R).$$

We get, in a similar fashion, homomorphisms

$$H_c^i(N_{R+D_0}(K)) \xrightarrow{P_{R+D_0}} H_{2-i}(Y,V_R) \xrightarrow{\bar{P}_R} H_c^i(N_{R-D_0}(K)) \tag{5.11}$$

$$H_c^i(V_R) \xrightarrow{P_R} H_{2-i}(Y,N_{R+D_0}(K)) \xrightarrow{\bar{P}_{R+D}} H_c^i(Y_{R+2D_0}) \tag{5.12}$$

$$H_c^i(Y,N_{R+D_0}(K)) \xrightarrow{P_{R+D_0}} H_{2-i}(V_R) \xrightarrow{\bar{P}_R} H_c^i(Y,N_{R-D_0}(K)) \tag{5.13}$$

$$H_c^{2-i}(Y,V_R) \xrightarrow{P_R} H_{2-i}(N_{R+D_0}(K)) \xrightarrow{\bar{P}_{R+D_0}} H_c^{2-i}(Y,V_{R+2D_0}) \tag{5.14}$$

Note that the homomorphisms in (5.11), (5.12), (5.13) and (5.14) inherit the bounded displacement property of P and $\bar{P}$.

We now check that the maps P_R satisfy the approximate monomorphism property stated in Definition 3.2, we leave verification of the rest of the assertions to the reader. Let

$$\xi \in Z_c^i(N_{R+2D_0}(K))$$

be a cocycle representing an element $[\xi] \in Ker(P_{R+2D_0})$, and let $\xi_1 \in C_c^i(Y)$ be the extension of ξ by zero. Then we have

$$P(\xi_1) = \partial\eta + \zeta$$

where $\eta \in C_{2-i}(Y)$ and $\zeta \in C_{2-i}(\overline{Y - N_{R+D}(K)})$. Applying $\bar{P}$ and the chain homotopy H, we get

$$\delta H(\xi_1) + H\delta(\xi_1) = \bar{P}\circ P(\xi_1) - \xi_1 = \bar{P}(\partial\eta + \zeta) - \xi_1$$

so

$$\xi_1 = \delta\bar{P}(\eta) + \bar{P}(\zeta) - \delta H(\xi_1) - H\delta(\xi_1).$$

The second and fourth terms on the right hand side vanish upon projection to $H_c^i(N_R(K))$, so $[\xi] \in Ker(H_c^i(N_{R+2D_0}(K)) \to H_c^i(N_R(K))$. $\square$

Remark 5.15. The 2-complex Y is 1-acyclic and satisfies coarse 2-dimensional Poincare duality in dimension 1. Hence Y is 1-ended by Lemma 3.5.

5.3. **Coarse geometry of quasi-planes.** *Proof of Theorem 5.1.* We prove the theorem by analyzing the coarse geometry of the complex Y along the lines of the sections 4 and 3. The difference however is that (unlike in sections 4 and 3) we do not know that Y is simply-connected; on the other hand, in our present situation Y is quasi-isometric to the finitely-generated group G.

Case 1. G is Gromov-hyperbolic. Then G admits a free discrete cocompact action on a contractible metric complex Y' (see Corollary 2.9). Although the complex Y' is not finite-dimensional we get a resultion

$$... \to C_n(Y', \mathcal{R}) \to ... \to C_0(Y', \mathcal{R}) \to \mathcal{R} \to 0$$

by free $\mathcal{R}G$-modules. Thus we get a chain-homotopy equivalence between the above resultion and

$$0 \to \tau(A_2) \to \tau(A_1) \to \tau(A_0) \to \tau(\mathcal{R}) \to 0.$$

Using this chain-homotopy-equivalence we can transfer the Poincare duality operators to operators

$$\mathrm{Hom}_{\mathcal{R}G}(C_i(Y'), \mathcal{R}G) \xrightarrow{P'} C_{2-i}(Y', \mathcal{R}G), C_i(Y', \mathcal{R}G) \xrightarrow{\bar{P}'} \mathrm{Hom}_{\mathcal{R}G}(C_{2-i}(Y'), \mathcal{R}G)$$

which are homotopy-inverses of each other. We note that the complex

$$\mathrm{Hom}_{\mathcal{R}G}(C_*(Y'), \mathcal{R}G)$$

is nothing but $C_c^*(Y', \mathcal{R})$ (after ignoring the $\mathcal{R}G$-module structure). The complex Y' is quasi-isometric to the finite-dimensional contractible Rips complex X of G. Hence, by choosing continuous quasi-isometries we get maps

$$Y' \xrightarrow{f} X \xrightarrow{\bar{f}} Y'$$

which are uniform proper homotopy-inverses. Applying these maps to $P', \bar{P}'$ we get duality operators

$$P : C_c^i(X, \mathcal{R}) \to C_{2-i}(X, \mathcal{R}), \bar{P} : C_i(X, \mathcal{R}) \to C_c^{2-i}(X, \mathcal{R})$$

so that the compositions $P \circ \bar{P}$ and $\bar{P} \circ P$ are chain-homotopic to the identity by homotopies with uniformly bounded tracks.

Applying Lemma 5.10 to X we conclude that X is a quasi-plane over $\mathcal{R}$. Thus we are in position to apply Theorem 3.9: The complex X is quasi-isometric to $\mathbb{H}^2$ and the discrete cocompact action $G \curvearrowright X$ is quasi-isometrically conjugate to a discrete cocompact and isometric action of G on $\mathbb{H}^2$. Therefore G is virtually a surface group. This concludes the proof in the Gromov-hyperbolic case.

Case 2. G is not Gromov-hyperbolic. Then, according to Lemma 3.8 in [16], there exist constants $L \geq 1, c > 0$, a sequence $R_j \in \mathbb{R}_+$ diverging to ∞, such that for each $R = R_j$, there exists an L-bilipshitz embedding

$$f = f_R : S_R \to Y^{(1)}.$$

Here S_R is the circle of radius R in $\mathbb{R}^2$. Let S'_R denote $f(S_R)$.

Lemma 5.16. *For each $R = R_j \gg 1$, the graph $N^{(1)}_{D_0}(S'_R)$ surounds a ball of radius $\frac{1}{2}LR$ in $Y^{(1)}$.*

Proof. Our proof is analogous to the proof of Lemma 4.8 and of the coarse Jordan separation theorem [11, Corollary 7.8]. We recall that Y satisfies coarse 2-dimensional Poincare duality in the dimension 1, let D_0 be the constant which appears in the definition of the coarse 2-dimensional Poincare duality.

Take $1 < r < \frac{1}{2}(\sqrt{3}LR - 1)$. Define a retraction $\bar{f} : N_r(S'_R) \to S_R$ as follows: For each vertex y in $N_r(S'_R)$ we let $\bar{f}(y) \in S_R$ be a point x such that $f(x) \in S'_R$ is a nearest point to y. Extend f linearly to the 1-skeleton of $N_r(S'_R)$. The inequality $r < \frac{1}{2}(\sqrt{3}LR - 1)$ ensures that the map $\bar{f}$ extends to the 2-skeleton of $N_r(S'_R)$. The same inequality ensures that $\bar{f} \circ f : S_R \to S_R$ is homotopic to the indentity.

Let $\alpha \in C^1(S_R, \mathcal{R})$ be a cocycle whose support set has unit diameter and which represents the generator of $H^1(S_R, \mathcal{R})$. Then the homotopy $\bar{f} \circ f \simeq id$ implies that $\alpha' := \bar{f}^*(\alpha)$ is not null-cohomologous in $N_r(S'_R)$. We now apply the coarse Poincare duality to the cocycle α' and we get a dual relative 1-cycle

$$\sigma := P(\alpha') \in H_1(Y, Y \setminus N_{r-D_0}(S'_R))$$

which maps nontrivially to $H_1(Y, Y \setminus N_{D_0}(S'_R))$ provided that $r \geq 2D_0$. Since Y is 1-acyclic over $\mathcal{R}$, this means that the 0-chain $\partial\sigma$ is a linear combination $\partial\sigma = \sum a_i y_i$ with nonzero coefficients $a_i \in \mathbb{R}$, where the y_i's lie outside $N_{r-D_0}(S'_R)$, and $\sum a_i y_i$ represents a nontrivial class in $\tilde{H}_0(Y \setminus N_{D_0}(S'_R), \mathcal{R})$. Thus there is a pair of points y_k, y_l in the support of $\sum a_i y_i$, which cannot be joined by a curve in $Y \setminus N_{D_0}(S'_R)$.

It follows that the connected graph $N^{(1)}_{D_0}(S'_R)$ separates one of the balls $B(y_k, r - 2D_0)$, $B(y_l, r - 2D_0)$ from infinity. □

Combining the above lemma with Proposition 4.6 we get:

Corollary 5.17. *The group G is virtually nilpotent and has at most quadratic growth.*

The above corollary together with the fact that G is one-ended implies that G contains a nilpotent subgroup Γ of finite index so that Γ fits into a short exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z} \to 1;$$

since Γ is nilpotent, this sequence splits and hence $\Gamma \cong \mathbb{Z}^2$. This concludes the proof of Theorem 5.1. □

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