PROBLEMS ON BOUNDARIES OF GROUPS AND KLEINIAN GROUPS

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1. Background

Ideal boundaries of hyperbolic spaces. Suppose that X is a hyperbolic metric space. Pick a base-point $o \in X$. This defines the *Gromov product* $(x, y)_o \in \mathbb{R}_+$ for points $x, y \in X$. The ideal boundary $\partial_\infty X$ of X is the collection of equivalence classes $[x_i]$ of sequences (x_i) in X where $(x_i) \sim (y_i)$ if and only if

$$\lim_{i\to\infty} (x_i, y_i)_o = \infty.$$

The topology on $\partial_\infty X$ is defined as follows. Let $\xi \in \partial_\infty X$. Define r-neighborhood of ξ to be

$$U(\xi, r) := \{\eta \in \partial_\infty X : \exists (x_i), (y_i) \text{ with } \xi = [x_i], \eta = [y_i], \liminf_{i,j\to\infty} (x_i, y_j)_o \geq r\}.$$

Then the basis of topology at ξ consists of $\{U(\xi, r), r \geq 0\}$. We will refer to the resulting ideal boundary $\partial_\infty X$ as the *Gromov–boundary* of X. One can check that the topology on $\partial_\infty X$ is independent of the choice of the base-point. Moreover, if $f : X \to Y$ is a quasi-isometry then it induces a homeomorphism $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$. The Gromov product extends to a continuous function

$$(\xi, \eta)_o : \partial_\infty X \times \partial_\infty X \to [0, \infty].$$

The geodesic boundary of X admits a family of *visual metrics* d^a_∞ defined as follows. Pick a positive parameter a. Given points $\xi, \eta \in \partial_\infty X$ consider various *chains* $c = (\xi_1, ..., \xi_m)$ (where m varies) so that $\xi_1 = \xi, \xi_m = \eta$. Given such a chain, define

$$d_c(\xi, \eta) := \sum_{i=1}^{m-1} e^{-a(\xi_i,\xi_{i+1})_o},$$

where $e^{-\infty} := 0$. Finally,

$$d^a_\infty(x, y) := \inf_c d_c(\xi, \eta)$$

where the infimum is taken over all chains connecting ξ and η. Taking different values of a results in Hölder–equivalent metrics. Each quasi-isometry $X \to Y$ yields a quasi-symmetric homeomorphism (see section 8 for the definition)

$$(\partial_\infty X, d^a_\infty) \to (\partial_\infty Y, d^a_\infty).$$

Conversely, each quasi-symmetric homeomorphism as above extends to a quasi-isometry $X \to Y$, see [51].

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Then the *ideal boundary* of a Gromov–hyperbolic group G is defined as

$$\partial_\infty \Gamma_G,$$

where Γ_G is a Cayley graph of G. Hence Γ_G is well-defined up to a quasi-symmetric homeomorphism.

Ideal boundaries of $CAT(0)$ spaces. Consider a $CAT(0)$ space X. Two geodesic rays $\alpha, \beta : \mathbb{R}_+ \to X$ are said to be *equivalent* if there exists a constant $C \in \mathbb{R}$ such that

$$d(\alpha(t), \beta(t)) \le C, \forall t \in \mathbb{R}_+.$$

The *geodesic boundary* $\partial_\infty X$ of X is defined to be the set of equivalence classes $[\alpha]$ of geodesic rays α in X. Fix a base-point $o \in X$. If X is locally compact (which we will assume from now on), then there exists a unique a representative α in each equivalence class $[\alpha]$ so that $\alpha(0) = o$. With this convention the *visual topology* on $\partial_\infty X$ is defined as the compact-open topology on the space of maps $\mathbb{R}_+ \to X$. One can check that this topology is independent of the choice of the base-point and that isometries $X \to Y$ induce homeomorphisms $\partial_\infty X \to \partial_\infty Y$.

Example. If $X = \mathbb{R}^n$ then $\partial_\infty X$ is homeomorphic to S^{n-1}.

If X is a $CAT(-1)$ space then it is also Gromov-hyperbolic. Then the two ideal boundaries of X (one defined via sequences and the other defined via geodesic rays) are canonically homeomorphic to each other. More specifically, each geodesic ray α defines sequences $x_i = \alpha(t_i)$, for $t_i \in \mathbb{R}_+$ diverging to infinity. The equivalence class of such (x_i) is independent of (t_i) and one gets a homeomorphism from the $CAT(0)$-boundary to the Gromov-boundary.

In general, quasi-isometries of $CAT(0)$ spaces to not extend to the ideal boundaries in any sense. Moreover, Bruce Kleiner and Chris Croke constructed examples [22] of pairs of $CAT(0)$ spaces X, X' which admit geometric (i.e. isometric, discrete, cocompact) actions by the same group G so that $\partial_\infty X, \partial_\infty X'$ are not homeomorphic.

Therefore, given a $CAT(0)$–group G one can talk only of the *collection of $CAT(0)$ boundaries* of G, i.e. the set

$$\{\partial_\infty X : \exists G \curvearrowright X\}$$

where the actions $G \curvearrowright X$ are geometric.

2. Topology of boundaries of hyperbolic groups

Problem 1 (Misha Kapovich)**.** What spaces can arise as boundaries of hyperbolic groups? As a sub-problem: For which k do k-dimensional stable Menger spaces appear as boundaries?

Example 1 (Damian Osajda)**.** Let X be a thick right-angled hyperbolic building of rank $n+1$, i.e. with apartments isometric to $\mathbb{H}^{n+1}$. Then the ideal boundary of X is a stable Menger space $M_{n,k}$. However $n+1$-dimensional right-angled hyperbolic reflection groups exist only for $n \le 3$.

Problem 2. Can one remove the "right-angled" assumption in Osajda result?

Background: The Menger space $M_{k,n}$ is obtained by iteratively subdividing an n-cube into 3^n subcubes and removing those that do not touch the k-skeleton, see [4] for a detailed discussion of the topology of these spaces. Below are few properties of $M_{k,n}$:

- $M_{k,n}$ has topological dimension k.
- $M_{k,n}$ is stable when $n \geq 2k+1$ (that is, replacing n by a larger value does not change $M_{k,n}$).
- Any k-dimensional compact metric space embeds in some stable $M_{k,n}$.

Problem 3 (Panos Papasoglu)**.** What 2-dimensional spaces arise as boundaries of hyperbolic groups? Can restrict to cases with no virtual splitting, no local cut points or cut arcs, and no Cantor set that separates.

Background: 2-dimensional Pontryagin surfaces and 2-dimensional Menger spaces $M_{2,5}$ appear as boundaries of hyperbolic Coxeter groups, see [27]. According to work of Misha Kapovich and Bruce Kleiner [37]: if $\partial_\infty G$ is 1-dimensional, connected and has no local cut points, then $\partial_\infty G$ is homeomorphic to a Sierpinski carpet ($M_{1,2}$) or the Menger space $M_{1,3}$.

Problem 4 (Mike Davis)**.** Are there torsion-free hyperbolic groups G with

$$cd_{\mathbb{Q}}(G)/cd_{\mathbb{Z}}(G) < 2/3 \quad ?$$

Background: Here cd_R is the cohomological dimension over a ring R. Mladen Bestvina and Geoff Mess [6] have shown that:

a. For torsion-free hyperbolic groups $cd_R(G) = cd_R(\partial_\infty G) + 1$.
b. There are hyperbolic groups G such that $cd_{\mathbb{Z}}(G) = 3$ and $cd_{\mathbb{Q}}(G) = 2$.

Problem 5 (Nadia Benakli)**.** What can be said about boundaries arising from *strict hyperbolization* constructions of Charney and Davis, [18]?

Problem 6 (Ilia Kapovich)**.** Is there an example of a group G which is hyperbolic relative to some parabolic subgroups that are nilpotent of class ≥ 3 whose Bowditch boundary is homeomorphic to some n-sphere?

Remark 1 (Tadeusz Januszkiewicz). Strict hyperbolization of piecewise linear manifolds gives many examples of hyperbolic groups G with $\partial_\infty G$ homeomorphic to S^n.

Problem 7 (Misha Kapovich)**.** Suppose that Z is a compact metrizable topological space, $G \curvearrowright Z$ is a convergence action which is topologically transitive, i.e. each G–orbit is dense in Z. Is there a Gromov-hyperbolic space X with the ideal boundary Z so that the action $G \curvearrowright Z$ extends to a uniformly quasi-isometric quasi-action $G \curvearrowright X$?

Background: Suppose that Z is a topological space, $Z^{(3)}$ is the set of triples of distinct points in Z. The space $Z^{(3)}$ has a natural topology induced from Z^3. A topological group action $G \curvearrowright Z$ is called a *convergence action* if the induced action $G \curvearrowright Z^{(3)}$ is properly discontinuous. A convergence action $G \curvearrowright Z$ is called *uniform* if $Z^{(3)}/G$ is compact. Examples of convergence group actions are given by uniformly quasi-Moebius actions $G \curvearrowright Z$, e.g. are induced on $Z = \partial_\infty X$ by uniformly quasi-isometric quasi-actions $G \curvearrowright X$. Brian Bowditch [12] proved that each uniform convergence action $G \curvearrowright Z$ is equivalent to the action of a hyperbolic group on its ideal boundary.

Problem 8 (Tadeusz Januszkiewicz)**.** Find topological restrictions on the ideal boundaries of $CAT(-1)$ cubical complexes.

Background. A $CAT(-1)$ cubical complex is a $CAT(-1)$ complex X where every n-cell is a combinatorial cube, isometric to a polytope in $\mathbb{H}^n$, so that the isometry preserves the combinatorial structure. For instance, such a complex can cover closed hyperbolic 3-manifold. It was proven by Januszkiewicz and Świątkowski [35] that $\partial_\infty X$ cannot be homeomorphic to S^4. Moreover, $\partial_\infty X$ cannot contain an essential k-sphere for $k \geq 4$.

3. Boundaries of Coxeter groups

Let G be a finitely-generated Coxeter group with Coxeter presentation $\langle S|R\rangle$. This presentation determines a *Davis-Vinberg complex* X (see [24]), whose dimension equals *rank* of the maximal finite special subgroup of G with respect to the above presentation. The complex X admits a natural piecewise-Euclidean $CAT(0)$ metric. The group G acts on X properly discontinuously and cocompactly. Hence, X has visual boundary ∂X, which we can regard as a boundary of G. Topology of ∂X was studied in [27, 28]. For instance, [27] constructs examples of hyperbolic Coxeter groups whose boundaries are both orientable and non-orientable Pontryagin surfaces and 2-dimensional Menger compacta. Recall that a Pontryagin surface is obtained as follows. Let K be a connected, compact (without boundary) triangulated surface. Define $P(K)$ by replacing each closed 2-simplex σ in K with a copy K_σ of the closure of $K \setminus \sigma$. We get the map

$$P(K) \to K$$

by sending each K_σ to σ. Set $P_n := P(P_{n-1})$. Then the corresponding Pontryagin surface P_∞ based on P_0 is inverse limit of the sequence

$$....P_n \to P_{n-1} \to ... \to P_1 \to P_0.$$

It turns out that P_∞ can have only three distinct topological types:

1. If $P_0 \cong S^2$, then $P_\infty \cong S^2$.

2. If P_0 is oriented but has genus ≥ 1, then P_∞ is oriented (i.e. $H^2(P_\infty, \mathbb{Z}) \cong \mathbb{Z}$) but not homeomorphic to S^2.

3. If P_0 is not oriented then P_∞ is unoriented. In this case, the rational homological dimension of P_∞ equals 1.

Problem 9 (Alexander Dranishnikov)**.** Is it true that isomorphic Coxeter groups have homeomorphic boundaries?

Remark 2. It appears that the answer is positive provided that all labels are powers of 2. REFERENCE?

Problem 10 (Alexander Dranishnikov)**.** Does there exist a Coxeter group G_n with n-dimensional boundary ∂G_n, so that the rational homological dimension of ∂G_n equals 1?

Problem 11 (Alexander Dranishnikov)**.** Under which conditions on the Coxeter diagram of G, the boundary of a Coxeter group is n-connected and locally n-connected?

Partial results in this direction are obtained in [26]. The main motivation for this problem comes from the problem of realizing Menger spaces as boundaries of Coxeter groups.

Problem 12 (Misha Kapovich)**.** Can exotic homology manifolds as in [14] appear as ideal boundaries of Coxeter groups?

4. Universality phenomena

The term *universality* loosely describes the following situation:

There is a class $\mathcal{C}$ of groups (spaces) of different nature, whose ideal boundaries are all homeomorphic.

Usually such results come from topological rigidity results for certain families of compacta.

Examples of universality phenomena.

1. Consider the class of all 2-dimensional hyperbolic groups which are 1-ended, do not split over virtually cyclic groups, are not commensurable to surface groups, are not relative $PD(3)$ groups. Then the ideal boundaries of all groups in this class are homeomorphic to the Menger curve. See [37].

2. The boundaries of the right angled rank $n+1$ hyperbolic buildings in Example 1 are all homeomorphic (since they are all homeomorphic to the stable Menger space $M_{n,k}$).

3. Let N be a closed n-manifold, Δ be its triangulation. Then Δ determines a right-angled Coxeter graph $Cox(N,\Delta)$ and $n+1$-dimensional David-Vinberg complex $C(N,\Delta)$. We assume, in addition, that Δ is a flag-complex, satisfying the *no-square condition* (which guarantees hyperbolicity of the resulting Coxeter group).

Suppose Δ_1, Δ_2 are two such triangulations of N, which admit a common subdivision. Let $C_i := C(N, \Delta_i)$. Then (H. Fischer [32]):

$$\partial_\infty C_1 = \partial_\infty C_2.$$

Note that

$$\partial_\infty C(N,\Delta) = \partial_\infty C(N\#N, \Delta\#\Delta).$$

In particular, the boundaries which appear in case $n = 2$ are of three types: S^2, oriented Pontryagin surface, non-orientable Pontryagin surface.

Problem 13 (Tadeusz Januszkiewicz)**.** Find more universality phenomena.

Problem 14 (Misha Kapovich)**.** Is it true that $\partial_\infty C(N,\Delta)$ is a topological invariant of N?

Problem 15 (Misha Kapovich, Tadeusz Januszkiewicz)**.** Suppose that (N_1,Δ_1) and (N_2,Δ_2) are closed 3-manifolds equipped with flag-triangulations, so that

$$\partial_\infty C(N_1,\Delta_1) = \partial_\infty C(N_2,\Delta_2).$$

Does it follow that every prime connected sum summand of N_i appears as a connected sum summand of N_{i+1}, $i = 1, 2$? What can be said in higher dimensions?

5. Markov compacta

The notion of Markov compactum is a generalization of the boundary of a group. Let $\mathcal{K} := \{K_i, \phi_i^{i+1}, i \geq 0\}$ be an inverse system of finite simplicial complexes

$$\phi_i^{i+1} : K_{i+1} \to K_i.$$

For a simplex $\sigma \in K_j$ let $\phi^{-1}(\sigma)$ denote the inverse subsystem K_σ formed by the subcomplexes (*building blocks*)

$$K_{i+1,\sigma} := (\phi_i^{i+1})^{-1}(K_{i,\sigma}), j \geq i, \quad K_{j,\sigma} := \sigma.$$

The inverse system $\mathcal{K}$ is called *Markov* if it contains only finitely many isomorphism classes of inverse subsystems K_σ. A *Markov compactum* is a compactum obtained as the inverse limit of a Markov inverse system.

Thus $\mathcal{K}$ is obtained from K_0 by inductively replacing simplices σ in K_i with the building blocks $K_{i+1,\sigma}$ using only finitely many "replacement rules". For instance, the Pontryagin surfaces are Markov compacta. Markov compacta appear naturally as boundaries of hyperbolic and Coxeter groups.

For every compactum Z either

$$\dim Z^n = n \dim(Z)$$

or

$$\dim Z^n = (n-1)\dim(Z) + 1.$$

In the latter case, Z is called a Boltyansky compactum.

Problem 16 (Alexander Dranishnikov)**.** Let Z be a compactum which is a Z-boundary of a group G. Then Z is never a Boltyansky compactum.

In the special case when Z is an Markov compactum, so that all building blocks

$$K_\sigma \to \sigma$$

are isomorphic, it was proven in [29] that Z cannot be a Boltyansky compactum.

6. Boundaries of $CAT(0)$ spaces

Problem 17 (Kim Ruane)**.** Examples of Kleiner and Croke [22], [23] of non-unique boundaries are badly non-locally-connected. Is that essential in having the "flexibility" to have many boundaries? That is, does local connectedness imply uniqueness of the boundary (in the 1-ended case) for CAT(0) groups?

Background: Suppose that X, Y are Gromov-hyperbolic spaces and $f : X \to Y$ is a quasi-isometry. Then f extends naturally to a homeomorphism $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$. In particular, the ideal boundaries of X and Y are not homeomorphic. The situation for the $CAT(0)$ spaces is quite different.

Definition 1. A group action $G \curvearrowright X$ on a metric space X is called *geometric* if it is isometric, properly discontinuous and cocompact.

For a CAT(0) group G acting geometrically on spaces X_i, there is an induced action of G on the boundary $\partial_\infty X_i$. For G-spaces X_1 and X_2, the boundaries may be (a) non-homeomorphic, or (b) homeomorphic, but not G-equivariantly.

The Croke-Kleiner examples are torus complexes which are "combinatorially" the same but where the angle α between the principal circles varies. [23], [22] showed that these complexes K_α, which all have the same fundamental group (a right-angled Artin group, in particular), have universal covers whose boundaries are not homeomorphic when $\alpha = \pi/2$ and $\alpha \neq \pi/2$. Julia Wilson showed that any two distinct values of α give non-homeomorphic boundaries.

Problem 18 (Dani Wise)**.** Suppose that G is a $CAT(0)$ group which does not split over a small subgroup. Does it follow that $\partial_\infty G$ is unique?

Problem 19 (Dani Wise)**.** Is the boundary well-defined for groups acting geometrically on $CAT(0)$-cube complexes? More precisely, suppose that X_1, X_2 are cube complexes which admit geometric actions of a group G. Does it follow that $\partial_\infty X_1 = \partial_\infty X_2$?

Problem 20 (Ross Geoghegan)**.** What topological invariants distinguish boundaries? In particular, what topological properties of boundaries are quasi-isometry invariants? Does something coarser than the topology stay invariant?

Remark 3. All boundaries for a given group are shape equivalent, so cannot be distinguished by their Čech cohomology. See [30] for the definition of shape equivalence.

It was shown by Eric Swenson [57] that for a proper cocompact $CAT(0)$ space X, the ideal boundary $\partial_\infty X$ has finite topological dimension. It was shown by Ross Geoghegan and Pedro Ontaneda [33] that the topological dimension of $\partial_\infty X$ is a quasi-isometry invariant of X.

Here and below a space X is called *cocompact* if $Isom(X)$ acts cocompactly on X.

A useful class of maps is called *cell-like*: inverse images of points are compact metrizable and each is shape equivalent to a point. (For a finite-dimensional compact subset Y of $\mathbb{R}^n$ (or of any ANR) "shape equivalent to a point" is equivalent to saying "Y can be contracted to a point in any of its neighborhoods.")

Remark 4. Cell-like maps are simple homotopy equivalences.

Problem 21 (Ross Geoghegan)**.** If G acts geometrically on two CAT(0) spaces, are the resulting boundaries cell-like equivalent? (That is, does there exist a space Z with cell-like maps to each of the two spaces?)

Remark 5. Ric Ancel, Craig Guilbault, and Julia Wilson have some examples when the answer is positive: they showed that the complexes K_α (see Croke-Kleiner examples above) are all cell-like equivalent.

Suppose that $G \curvearrowright X_i$, $i = 1, 2$ are isometric cocompact properly discontinuous actions of G on two CAT(0) spaces.

Problem 22 (Thomas Delzant)**.** Is there a convex core for the diagonal action of G on $X_1 \times X_2$? (A special case is surface groups G with X_1 and X_2 corresponding to different hyperbolic structures.) If there is a convex core, can Z (the space with cell-like maps to X_1 and X_2) be taken to be the boundary of the core?

Remark 6 (Bruce Kleiner). Convex sets are actually rare (see Remark 31), so maybe there is a different problem with better prospects.

Danny Calegari: One can try to define a new ideal boundary for $CAT(0)$ spaces (which is different from the visual boundary $\partial_\infty X$) by looking at the space of all quasi-geodesics in X. For example, in $\mathbb{R}^2$, consider all (equivalence classes of) K-quasi geodesics, with the compact-open topology. Varying K gives a filtration of the space of all quasi geodesics. Can one do interesting analysis on such a space?

Problem 23 (Danny Calegari)**.** Define a topology on the set of quasi geodesics in a (proper geodesic, or coarsely homogeneous, or cocompact) $CAT(0)$ space which

(1) has a description as an increasing union of compact metrizable spaces
(2) has an inclusion of its visual boundary $\partial_\infty X$ into it
(3) is quasi-isometry invariant
(4) has reasonable measure classes which are quasipreserved

According to a theorem by Brian Bowditch and Gadde Swarup [13, 56], if G is a 1-ended hyperbolic group then $\partial_\infty G$ has no cut points.

For G a CAT(0) group, a theorem of Eric Swenson says that if $c \in \partial_\infty G$ is a cut point, then there is an infinite-torsion subgroup of G fixing c.

Problem 24 (Conjecture: Eric Swenson)**.** Any $CAT(0)$ group has no infinite-torsion subgroups.

A *Euclidean retract* is a compact space that embeds into some $\mathbb{R}^n$ as a retract. A compact metrizable space Z is a *Z-set* in $\tilde{X}$ if it is "homotopically negligible" (for every open $U \subset \tilde{X}$, the inclusion $U \setminus Z$ in U is a homotopy equivalence). A *Z-structure* on a group G is a pair $(\tilde{X}, Z)$ such that

- $\tilde{X}$ is a Euclidean retract,
- Z is a Z-set in $\tilde{X}$,
- $X := \tilde{X} \setminus Z$ admits a covering space action of G with X/G compact,
- the set of translates of any compact set $K \subset X$ is a null sequence in $\tilde{X}$ (that is, for each $\epsilon > 0$ there are only finitely many translates with diam $> \epsilon$).

Finally, Z is a boundary of G (or Z-structure boundary) if there exists a Z-structure $(\tilde{X}, Z)$ on G.

The above notion boundary of G was generalized by T. Farrell and J. Lafont as follows:

An EZ-boundary of a group G is a boundary $Z = \partial_{EZ} G$ so that the action of G on X extends to topological action of G on Z.

Problem 25 (Misha Kapovich)**.** Let G be a hyperbolic group and $\partial_{EZ} G$ be its EZ boundary. Is it true that $\partial_{EZ} G$ is equivariantly homeomorphic to the Gromov boundary of G?

Problem 26 (Mladen Bestvina)**.** Can there be two different boundaries in the sense of Z-structures for a group G that are not cell-like equivalent?

Remark 7. Note that this problem is even open for $\mathbb{Z}^n$. For CAT(0) spaces, the visual boundaries are Z-structure boundaries, so Problem 21 is a special case.

Problem 27 (Bruce Kleiner)**.** Is the property of splitting over a 2-ended subgroup an invariant of Bestvina boundaries?

Some necessary conditions are known for compact, metrizable spaces X to be the boundary of some proper cocompact $CAT(0)$ space:

(1) X should have 1,2, or infinitely many components
(2) X is finite dimensional (Theorem of Swenson)
(3) X has nontrivial top Čech cohomology (Geoghegan-Ontaneda)

In the case when X admits a cocompact free(?) action by a discrete subgroup of isometries, one necessary condition is due to Bestvina: the dimension of every nonempty open set $U \subset X$ is equal to the dimension of X.

Problem 28 (Ross Geoghegan)**.** Extend these lists, or give a complete classification.

Problem 29 (Kevin Whyte)**.** Does every CAT(0) group have finite asymptotic dimension?

7. Asymptotic topology

Problems below are mostly motivated by the following rigidity results of Panos Papasoglu, [50]:

Theorem 1. Suppose that G is a finitely-presented 1-ended group. Then:

1. The JSJ decomposition of G is invariant under quasi-isometries.
2. A quasiline coarsely separates Cayley graph of G iff G splits over virtually-$\mathbb{Z}$ or G is virtually a surface group.
3. No quasi-ray coarsely separates the Cayley graph of G.

Problem 30 (Panos Papasoglu)**.** Do these results hold for general finitely generated groups?

Problem 31 (Panos Papasoglu)**.** Are splittings over $\mathbb{Z}^2$ (or $\mathbb{Z}^n$) invariant under quasi-isometry? The analogous problem also makes sense for the JSJ decompositions.

Problem 32 (Panos Papasoglu)**.** Suppose G is finitely generated and there is a sequence of quasicircles that separate its Cayley graph. Is G virtually a surface group?

Problem 33 (Conjecture of Panos Papasoglu)**.** If G is finitely generated with asymptotic dimension $\geq n$, and X is a subset of the Cayley graph with asymptotic dimension $\leq n-2$ that coarsely separates the Cayley graph, then G splits over some subgroup $H \leq G$ with asymptotic dimension $\leq n-1$.

A *homogeneous continuum* is a locally connected compact metric space whose group of homeomorphisms acts transitively. Papasoglu showed that every simply connected homogeneous continuum has the property that no simple arc separates it.

Problem 34 (Panos Papasoglu)**.** Do all homogeneous continua (with dimension greater than 2) have this property?

8. Analytical aspects of boundaries of groups

We begin with the basic definitions of the quasiconformal analysis. For a quadruple of points x, y, z, w in a metric space X, $[x, y, z, w]$ denotes their *cross-ratio*, i.e.

$$[x, y, z, w] = \frac{d(x, y)}{d(y, z)} \frac{d(z, w)}{d(w, x)}.$$

- *Quasiconformal* (analytic definition): A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is *quasiconformal* iff
  (1) $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$ and
  (2) There exists $K = K_f(x) < \infty$ so that $||Df(x)||^n \le K J_f(x)$ a.e. (here J_f is the Jacobian of f).

  The essential supremum $K(f)$ of $K_f(x)$ on $\mathbb{R}^n$ is called the coefficient of quasiconformality of f. A mapping f is called K-*quasiconformal* if $K(f) \le K$.

Note: The map f is differentiable almost everywhere, so the derivative and Jacobian in (2) make sense pointwise a.e.. The assumption (1) can be replaced by the assumption that f is ACL, i.e., that f is absolutely continuous on a.e. line parallel to the n coordinate directions. The above analytical definition of quasiconformality for maps of $\mathbb{R}^n$ turns out to be equivalent to four other definitions given below.

(1) *Quasiconformal* (metric definition): Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. For $r \in \mathbb{R}^+$, define the following:

$$L_f(x,r) := \sup\{|f(y) - f(x)| : |y-x| = r\}$$

$$\ell_f(x,r) := \inf\{|f(y) - f(x)| : |y-x| = r\}$$

$$H_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{\ell_f(x,r)} \text{ (the \textit{metric dilatation} of } f)$$

Then, f is *quasiconformal* iff $H_f(x)$ is uniformly bounded by some $H = H(f) \ge 1$.

(2) *Quasiconformal* (geometric definition): Let Γ be a family of paths in $\mathbb{R}^n$. We say a Borel function $\rho : \mathbb{R}^n \to [0,\infty)$ is *admissible* for Γ iff for every $\gamma \in \Gamma$, we have $\int_\gamma \rho\, ds \ge 1$.

Define $\text{mod}_n(\Gamma) := \inf\left\{\int_{\mathbb{R}^n} \rho^n(x)\, dx : \rho \text{ is admissible for } \Gamma\right\}$. A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is *quasiconformal* iff there is some constant $K = K(f) \ge 1$ such that for every path family Γ, we have

$$\frac{1}{K}\text{mod}_n(\Gamma) \le \text{mod}_n(f(\Gamma)) \le K\text{mod}_n(\Gamma).$$

(3) A homeomorphism $f : X \to Y$ between metric spaces X and Y is *quasisymmetric* iff there exists a homeomorphism $\eta : (0,\infty) \to (0,\infty)$ such that for all triples of distinct points $x, y, z \in X$, the following inequality holds:

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right).$$

(4) A homeomorphism $f : X \to Y$ between metric spaces X and Y is *quasi-Moebius* iff there exists a homeomorphism $\eta : (0,\infty) \to (0,\infty)$ such that for all quadruples of distinct points $x, y, z, w \in X$, the following inequality holds:

$$[f(x), f(y), f(z), f(w)] \le \eta\left([x,y,z,w]\right).$$

Note that Definitions 1–4 make sense in the context of general metric spaces, see below for details. If X is noncompact then quasi-symmetric maps are the same as quasi-Moebius maps. However, for compact metric spaces quasi-Moebius is a more appropriate (although more cumbersome) definition. One can rectify this problem by redefining quasi-symmetric maps for compact metric spaces as follows. A map

$f : X \to Y$ is quasi-symmetric if X admits a finite covering by open spaces $U_i \subset X$ so that the restriction $f|U_i$ is quasi-symmetric in the above sense for each i.

One defines a *quasi-symmetric equivalence* for metric spaces by

$$X \sim_{qs} Y$$

if there exists a quasisymmetric homeomorphism $X \to Y$.

Let (X, d, μ) be a metric measure space, where μ is a Borel measure. Then X is called *Ahlfors Q-regular*, if there exists $C \geq 1$ so that

$$C^{-1}R^Q \leq \mu(B_R(x)) \leq CR^Q$$

for each $R \leq diam(X)$.

Remark 8. Let X be a metric space with the Hausdorff dimension $Hdim(X) = Q$. Then the most natural measure to use is the Q-Hausdorff measure on X. This is the measure to be used for the boundaries of hyperbolic groups. Then (X, d) is called *Ahlfors regular* if it is Ahlfors Q-regular with $Q = Hdim(X)$.

Given two compact continua E, F in a metric space X define their *relative distance*

$$\Delta(E, F) := \frac{d(E, F)}{\min(\mathrm{diam}(E), \mathrm{diam}(F))}.$$

Here $d(E, F) := \min\{d(x, y) : x \in E, y \in F\}$.

Given an Ahlfors Q-regular metric measure space X, define $mod_Q(E, F)$ to be

$$mod_Q(\Gamma)$$

where Γ is the set of all curves in X connecting E to F.

Definition 2. A metric space X is called *Q-Loewner* if it satisfies the inequality

$$mod_Q(E, F) \leq \phi(\Delta(E, F)),$$

for a certain function ψ.

Problem 35 (Mario Bonk)**.** Are diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ dense in the space of all quasiconformal maps?

Remark 9 (Juha Heinonen). The answer is known to be "yes" for $n = 2, 3$ due to Moise's theorem.

Remark 10 (Misha Kapovich). In fact, the answer is also known to be "yes" for quasiconformal diffeomorphisms of $\mathbb{R}^n$, $n > 4$. This was proven by Connell [20] for stable homeomorphisms of $\mathbb{R}^n$, $n \geq 7$, improved by Bing [7] to cover dimensions ≥ 5. Lastly, it was shown by Kirby [39] that all orientation-preserving homeomorphisms of $\mathbb{R}^n$ are stable for $n \geq 5$. Note that the proof in the case of quasiconformal homeomorphisms is easier since quasi-conformal homeomorphisms are differentiable a.e. and the stable homeomorphism conjecture was known for $n \geq 5$ prior to Kirby's work. However the problem appears to be open in the case $n = 4$. On the other hand, Kirby observed that for sufficiently large n there are open connected subsets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ and a homeomorphism $f : \Omega_1 \to \Omega_2$, which cannot be approximated by diffeomorphisms $f_j : \Omega_1 \to \Omega_2$.

The problem becomes more subtle if we require approximating diffeomorphisms to be globally quasiconformal:

Problem 36 (Misha Kapovich)**.** Let $f : B^n \to B^n$ be a quasiconformal homeomorphism. Can f be approximated by globally quasiconformal diffeomorphisms $f_j : B^n \to B^n$? Can this be done so that f_j's are K-quasiconformal for all j?

Note that all the maps in problem will extend quasiconformally to the closed n-ball.

Problem 37 (Mario Bonk)**.** Find good classes of spaces such that the infinitesimal metric condition (for quasiconformality) implies the local condition. (This is generally true in Loewner spaces.)

Problem 38 (Kim Ruane)**.** Outside of the boundaries of Fuchsian buildings, what boundaries have the Loewner property?

Remark 11. Loewner spaces are good for analytic tools: have a cotangent bundle, so can "do calculus"; also, can do PDEs, etc.

Problem 39 (Juha Heinonen)**.** Let X be a non-smoothable closed simply connected 4-manifold. Does it admit an Ahlfors 4-regular linearly locally contractible metric? This is wide open; unknown even for examples, like E_8.

Remark 12. The non-smoothable closed simply-connected 4-manifolds like E_8 are known not to admit a *quasiconformal* atlas, [25]. In dimensions ≥ 5 Sullivan [55] proved that every topological manifold admits a quasiconformal atlas and, moreover, quasiconformal structure is unique. An alternative proof of Sullivan's theorem and its generalization was given by J. Luukkainen in [42], his proof avoids the construction of almost parallelizable hyperbolic manifolds; see also [59]. It was observed by Tom Farrell that a detailed proof of the fact that all closed hyperbolic n-manifolds are virtually almost parallelizable (and much more) is contained in the paper by B. Okun [48].

Hence in dimension $n \geq 5$ one would ask for Ahlfors n-regular linearly locally contractible metrics on the *unresolvable* homology manifolds, see [14]. The broad goal here is to find an analytic framework for studying *exotic* topological and homology manifolds.

Problem 40 (Uri Bader)**.** Develop a theory for analysis on the ideal boundaries of relatively hyperbolic groups, as it is done for hyperbolic groups.

Problem 41 (Bruce Kleiner)**.** In what generality does quasiconformal imply quasisymmetric? (Specifically, of interest are self-similar spaces which are connected, without local cut points; or visual boundaries of hyperbolic groups.)

Definition. Call a metric space X *quasi-isometrically cohopfian* if each quasi-isometric embedding $f : X \to X$ is a quasi-isometry. (Examples include Poincaré duality groups, solvable groups, Baumslag-Solitar groups.) X is *quasisymmetrically cohopfian* if every quasisymmetric embedding $X \to X$ is onto.

The above definition is a coarse analogue of the notion of *cohopfian groups* from group theory.

Fact: a hyperbolic group G is quasi-isometrically cohopfian iff $\partial_\infty G$ is quasisymmetrically cohopfian, cf. [51].

Problem 42 (Ilia Kapovich)**.** Take your favorite metric fractal. Is it quasisymmetrically cohopfian? What about the boundaries of hyperbolic groups?

Subproblem: What about the case of (round) Sierpinski carpets and Menger spaces which appear as boundaries of hyperbolic groups.

Background. *Round Sierpinski carpets* are the ones which are bounded by round circles. Such sets arise as the ideal boundaries of fundamental groups of compact hyperbolic manifolds with nonempty totally geodesic boundary. It is known that if G, G' are such groups which are not commensurable then their ideal boundaries are not quasisymmetric to each other. There is a similar rigidity theorem (due to Marc Bourdon and Herve Pajot [10]) for a certain class of Menger curves, i.e. the ones which appear as visual boundaries of 2-dimensional Fuchsian buildings. Quasisymmetric cohopfian property is open in both cases.

Remark 13 (Danny Calegari). As an example for the previous problem: the limit set L of a leaf of a taut foliation of a hyperbolic 3-manifold with 1-sided branching is a dendrite in S^2 which is nowhere dense, has Assouad dimension 2, and for any point p in L and any neighborhood U of p in S^2, L can be embedded by a *conformal* automorphism of S^2 into $L \cap U$.

Remark 14 (Juha Heinonen). If X is the *standard* "square" Menger space $M_{k,n}$ then it is clearly not quasisymmetrically cohopfian.

Problem 43 (Conjecture: Juha Heinonen)**.** If $\partial_\infty G$ is Loewner, then it is quasisymmetrically cohopfian. (Boundaries of Fuchsian buildings provide a good test case for this conjecture.)

Problem 44 (Bruce Kleiner)**.** If G is a hyperbolic group and $\partial_\infty G$ is connected with no local cut points, is there a natural measure class which is quasisymmetrically invariant? (That is, invariant under quasisymmetric homeomorphisms $\partial_\infty G \to \partial_\infty G$.)

Motivation: rigidity theorems rely on absolute continuity of quasisymmetric maps as a foundational ingredient.

Remark 15. If $\partial_\infty G$ is Loewner, then the answer is "yes." But there are examples of Bourdon and Pajot [11] whose boundary is not Loewner for each metric which is quasisymmetric to a Gromov-type metric.

In Bourdon—Pajot examples, Patterson-Sullivan measure works because there are relatively few quasisymmetric maps.

Problem 45 (Kim Ruane)**.** Can you do analysis on CAT(0) boundaries? With no natural metric, is there any structure beyond topology?

Remark 16 (Bruce Kleiner). "Pushing in" the visual sphere gives pseudo-metrics on $\partial_\infty X$, where X is the CAT(0) space acted on by G. Consider the radial projection $\partial_\infty X \to S_R(0)$ to spheres of radius R; then $d_R = Pr^{-1}(d_X | S_R(0))$ are the pseudometrics. But then for a function ϕ going quickly enough to zero,

$$\sum_{R \in \mathbb{N}} \phi(R) d_R$$

is a metric on $\partial_\infty X$.

Remark 17 (Damian Osajda). One can define a family d_A of metrics on $\partial_\infty X$ as follows. Pick $A > 0$ and choose a base-point $o \in X$. Let α, β be geodesic rays emanating from o and asymptotic to points $\xi, \eta \in \partial_\infty X$. Let a be such that

$$d(\alpha(a), \beta(a)) = A.$$

If such a does not exists (i.e. $\alpha = \beta$), then set $a := \infty$. Finally, set

$$d_A(\xi, \eta) := \frac{1}{a}.$$

Problem 46 (Bruce Kleiner)**.** $G = \mathrm{Isom}(X)$ acts on $\partial_\infty(X)$. Is this action "nice" with respect to the metrics in the previous remark?

Problem 47 (Marc Bourdon)**.** If D is the boundary of a hyperbolic group and D is connected, has no local cut points, and is not Loewner, is there a quasisymmetrically invariant nontrivial closed equivalence relation $\sim$ on D so that $D/\sim$ is Hausdorff and is a boundary of G relative to a collection of parabolic subgroups?

Remark 18. In Bourdon–Pajot examples [11], the answer to the above problem is positive.

Problem 48 (Jeremy Tyson)**.** Study relationships between different notions of conformal structure on $\partial_\infty(G)$ for hyperbolic G. Here is an (incomplete) list of such notions:

(1) 1-quasiconformal in the metric sense, i.e. $H(f) = 1$.
(2) preserving modulus of curves joining two compacta .
(3) η-quasisymmetric with η as close to linear as we like. (η are functions of the point $x \in \partial_\infty G$ where we test f for conformality)
(4) if Poincaré inequality holds for $\partial_\infty G$, then, using Cheeger cotangent bundle $T^*\partial_\infty G$, can give a notion of measurable bounded conformal structure μ such that

$$\overline{\mathrm{Conf}(\partial_\infty G, \mu)}$$

is a convergence group.

Remark 19. Good notions of quasiconformality should have the convergence property, and metric notion does not, so its usefulness would be if (1) $\Longrightarrow$ (2), since (1) is checkable and (2) is not.

Recall that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism for $n \geq 2$, then

$$k - \text{quasiconformal} \implies \text{quasisymmetric} \implies (\text{balls} \mapsto \text{quasiballs})$$

Problem 49 (Juha Heinonen)**.** Is the same true for Hilbert spaces? (All known proofs of above type use geometry, not analysis.)

It follows from work of Mario Bonk and Oded Schramm that there are quasi-isometric embeddings of $\mathbb{HH}^n$ (quaternionic hyperbolic space) into $\mathbb{HH}^m$ which are very far from isometric embeddings. (One can construct such examples with $m \approx c \cdot n$ for a constant c which is no less than 16.)

Problem 50 (David Fisher)**.** Can one do this with smaller m? Say $m = n + 1$? (Same problem valid for complex hyperbolic space.)

Subproblem (Misha Kapovich): Consider $X = \partial_\infty \mathbb{HH}^n$ sitting inside of $Y = \partial_\infty \mathbb{HH}^{n+1}$. Is X *locally quasi-symmetrically rigid* in Y? More precisely, is it true that each quasisymmetric embedding $f : X \to Y$ which is sufficiently close to the identity is induced by an isometry of $\mathbb{HH}^{n+1}$?

Remark 20. This subproblem might be easier to settle than Problem 50, since one can try to use infinitesimal tools like quasiconformal vector-fields.

David Fisher and Kevin Whyte have constructed some "exotic" quasi-isometric embeddings for higher-rank symmetric spaces that are "algebraic," in the sense that $\pi : A_1 N_1 \to A_2 N_2$.

Problem 51 (David Fisher)**.** Are all quasi-isometric embeddings between higher-rank symmetric spaces either isometries or algebraic in this way?

Problem 52 (Misha Kapovich)**.** Let G be a hyperbolic group. Is it true that G admit a uniformly quasiconformal discrete action on S^n (for some n)?

The answer is probably negative. It is reasonable to expect that every group satisfying Property (T) which admits such an action must be finite. However the usual proofs that infinite discrete subgroups of $\mathrm{Isom}(\mathbb{H}^{n+1})$ never satisfy Property (T) do not work in the quasiconformal category.

9. Problems related to Cannon's Conjecture

Problem 53 (Cannon's Conjecture, Version I)**.** If G is a (Gromov) hyperbolic group with $\partial_\infty G$ homeomorphic to S^2, then G acts geometrically on $\mathbb{H}^3$.

Problem 54 (Cannon's Conjecture, Version II)**.** Under the same assumptions on G,

$$(\partial_\infty G, \text{visual metric}) \sim_{qs} (S^2, \text{standard}).$$

Remark 21. Perelman's proof of Thurston's geometrization conjecture implies that the Cannon's conjecture is equivalent to the finding that such G is commensurable to a 3-manifold group: There exists an exact sequence

$$1 \to F \to G_0 \to \pi_1(M^3) \to 1$$

with F finite and $[G : G_0] < \infty$.

Remark 22. If G is hyperbolic and torsion-free then $\partial_\infty G \cong S^2$ iff G is a PD^3 group (a 3-dimensional Poincaré duality group), see [6].

Problem 55 (Conjecture of C.T.C. Wall)**.** Every PD^3 group is a 3-manifold group.

Problem 56 (Cannon's Conjecture, Relative Version I)**.** If G is hyperbolic and $\partial_\infty G$ is homeomorphic to the Sierpinski carpet, then G acts geometrically on a convex subset of $\mathbb{H}^3$.

Remark 23. This follows from the Cannon's Conjecture by doubling.

Problem 57 (Cannon's Conjecture, Relative Version II)**.** For G hyperbolic relative to $\{H_1, \ldots, H_k\}$ for a collection of virtually-$\mathbb{Z}^2$ subgroups H_i, if the Bowditch boundary $\partial_{\text{Bow}} G$ is S^2, then G is commensurable to the fundamental group of a hyperbolic 3-manifold of finite volume.

Remark 24. The same problem could be posed allowing the boundary to be S^2 or Sierpinski carpet.

Problem 58 (Cannon's Conjecture, Analytic Version)**.** If G is a hyperbolic group with $\partial_\infty G$ homeomorphic to the Sierpinski carpet, then the visual metric on $\partial_\infty G$ is quasisymmetric to some round Sierpinski metric:

$$(\partial_\infty G, \text{visual metric}) \sim_{qs} (\text{Sierpinski}, \text{round}).$$

Recent work of Mario Bonk gives simplifications and partial answers here.

Problem 59. Prove Cannon's conjecture under additional assumptions, such as

- $G = \pi_1(M^3)$, in Haken case (without appealing to Thurston's proof of the hyperbolization theorem)
- G a PD^3 group that splits over a surface group
- G acts on a CAT(0) cube complex
- ...

Remark 25. Cannon's Conjecture is known for Coxeter groups G (a work by Mario Bonk and Bruce Kleiner). This follows of course from Andreev's theorem, but the point here is to give a proof which only uses the geometry of the ideal boundary of G.

Problem 60 (Misha Kapovich)**.** Give positive solution to Problem 57 assuming the absolute case by doing "hyperbolic Dehn surgery" (see Groves–Manning, Osin). Namely, add relators R_i, $i = 1, ..., k$, where $R_i \in H_i$. For sufficiently long elements R_i the quotient $G := \Gamma / << R_1, ..., R_k >>$ are known to be hyperbolic.

a. Prove that if R_i's are sufficiently long then G is an (absolute) PD^3 group, by, say, computing $H^*(G, ZG)$.

b. Assuming that each G is a 3-manifold group, show that Γ is a 3-manifold group as well.

To motivate a possible approach to Cannon's conjecture recall the following:

Theorem 2 (M. Bonk, B. Kleiner, [8])**.** Suppose that G is a hyperbolic group, $G \curvearrowright Z$ is a uniformly quasi-Moebius action on a metric space which is topologically conjugate to the action of G on its ideal boundary. Assume that Z is Ahlfors n-regular and has topological dimension n. Then Z is quasi-symmetric to the round n-sphere. In particular, G acts geometrically on $\mathbb{H}^{n+1}$.

Therefore, given a hyperbolic group G with $Z = \partial_\infty G$ homeomorphic to S^2 one would like to replace the visual metric d on $\partial_\infty G$ with a quasisymmetrically equivalent one, which has Hausdorff dimension 2. Since Hausdorff dimension ($Hdim$) of a metric compact homeomorphic to S^2 is ≥ 2, one could try to *minimize Hausdorff dimension* in the *quasi-conformal gauge* of (Z, d), i.e. the collection $\mathcal{G}(Z, d)$ of metric spaces (Z, d') which are quasisymmetric to (Z, d). This motivates the following:

Definition 3. For a metric space Z, define its *Pansu conformal dimension*

$$PCD(Z) := \inf\{Hdim(Y) : Y \in \mathcal{G}(Z)\}.$$

Likewise, *Ahlfors regular Pansu conformal dimension* of X is

$$ACD(Z) := \inf\{Hdim(Y) : Y \in \mathcal{G}(Z),\ Y \text{ is Ahlfors regular}\}.$$

The importance of the latter comes from

Theorem 3 (M. Bonk, B. Kleiner, [9])**.** Suppose that G is a hyperbolic group, $Z = \partial_\infty G$ is homeomorphic to S^2. If the $ACD(Z)$ is attained then G acts geometrically on $\mathbb{H}^3$.

Remark 26. Bourdon–Pajot examples [11] show that the ACD for the boundaries of hyperbolic groups is not always attained.

Generally, $ACD(Z)$ attained iff there is a Loewner metric in $\mathcal{G}(Z)$ (which is then minimizing).

Problem 61 (Conjecture of Bruce Kleiner)**.** For a hyperbolic group G,

$$ACD(\partial_\infty G) = \inf_{G \curvearrowright X}\{Hdim(\partial_\infty X, \text{visual})\},$$

where the infimum is taken over all geometric actions of G on metric spaces X. A bolder conjecture would be that, when the infimum is attained, it is attained by a visual metric.

Problem 62. What is ACD of the standard Sierpinski carpet? In particular, does the above conjecture hold?

Problem 63 (Juha Heinonen)**.** Under what assumptions on hyperbolic groups G with Q-Loewner boundary $\partial_\infty G$ does it admit a 1-Poincaré inequality for the boundary?

Cannon's conjecture has a generalization to *nonuniform* convergence group actions on compacts. Here is one of such

Suppose L is the support of a measured lamination on a surface S and $S \setminus L$ consists of topological disks. Lift this lamination to a lamination Λ in the unit disk $D \subset S^2$ and define the following equivalence relation $\sim$:

1. The closure of each component of $D \setminus \Lambda$ in the closed disk $\bar{D}$ is an equivalence class.

2. If $\gamma \subset \Lambda$ is a geodesic which is not on the boundary of a component of $D \setminus \Lambda$, then the closure of γ in $\bar{D}$ is an equivalence class. The rest of the points of S^2 are equivalent only to themselves.

Note that $\sim$ is G-invariant and that the equivalence classes are cells. Therefore the quotient $S^2/\sim$ is homeomorphic to S^2 and the group G acts on $S^2/\sim$ by homeomorphisms. One can check that this is a convergence group action.

More generally, one can form an equivalence relation using a pair of transversal laminations and make the corresponding G–invariant quotient.

Problem 64. [Cannon–Thurston] Is this action conjugate to a conformal action?

The situation here is, in many ways, more complicated than in Cannon's conjecture since there is no a priori a useful metric structure on $Z = S^2/\sim$. It is not even clear that there exists a Gromov-hyperbolic space X with the ideal boundary Z so that the action $G \curvearrowright Z$ extends to a uniformly quasi-isometric quasi-action $G \curvearrowright X$.

One can reformulate this problem using theory of Kleinian groups as follows. According to the Ending Lamination Conjecture, there exists a discrete embedding $\iota(G) \subset Isom(\mathbb{H}^3)$ so that the ending lamination of G is L.

Problem 65. The limit set of the Kleinian group $\iota(G)$ is locally connected.

In the presence of two geodesic laminations, the limit set of $\iota(G)$ is the entire 2-sphere, so local connectedness is meaningless. Then the correct reformulation is as follows:

Problem 66. Is there an equivariant continuous map (called *Cannon–Thurston map*) from the unit circle S^1 (the ideal boundary of G as an abstract group) to S^2?

Then Problem 64 is equivalent to 66.

Remark 27. Positive solution of Problem 66 is known in certain cases. For instance, Jim Cannon and Bill Thurston showed this for laminations which are stable for a pseudo-Anosov homeomorphism. Yair Minsky [45] proved this under the assumption that the injectivity radius of $\mathbb{H}^3/\iota(G)$ is bounded away from zero. Curt McMullen proved this in the case when G is the fundamental group of once punctured torus of quadruply punctured sphere, [44]. A complete solution of this problem is claimed in the recent preprint of Mahan Mj (Mahan Mitra) [47].

Problem 67 (Mahan Mitra)**.** Let $H \subset G$ be a hyperbolic subgroup of a hyperbolic group (we do not assume that H is quasiconvex). Is it true that there exists an equivariant continuous map

$$\partial_\infty H \to \partial_\infty G.$$

See [46] for partial results in this direction.

10. Poisson boundary

Problem 68 (Vadim Kaimanovich)**.** What is the Poisson boundary of the free group with an arbitrary measure?

Let (Y, d) be a metric space and let $C(Y)$ denote the space of continuous functions on Y equipped with the topology of uniform convergence on bounded subsets. Fixing a basepoint $y \in Y$, the space Y is continuously injected into $C(Y)$ by

$$\Phi : z \mapsto d(z, \cdot) - d(z, y).$$

If Y is proper, then $\overline{\Phi(Y)}$ is compact. The points on the boundary $\overline{\Phi(Y)} \setminus \Phi(Y)$ are called *horofunctions* (or Busemann functions).

Problem 69 (Conjecture of Anders Karlsson)**.** There almost surely exists a horofunction h such that

$$\lim_{n\to\infty} -\frac{1}{n} h(x_n) = A,$$

where $A = \lim\limits_{n\to\infty} \frac{1}{n} d(x_0, x_n)$.

A theorem of Karlsson states that $\forall\epsilon > 0$ there exists a horofunction h_ϵ such that

$$A - \epsilon \leq -\frac{1}{n} h_\epsilon(x_n) \leq A + \epsilon$$

for all $n \geq N(\epsilon)$.

Remark 28. This works for any finitely generated group.

Problem 70 (Anders Karlsson)**.** For any proper metric space it is possible to associate a kind of incidence geometry at infinity via horofunctions, halfspaces and their limits called stars. For the CAT(0) case, this structure is intimately connected with the Tits geometry, and for Teichmüller space it should relate well with the curve complex. In which situations do homomorphisms induce "incidence preserving" maps between these geometries at infinity? Same problem for quasi-isometries.

Problem 71 (Anders Karlsson)**.** Consider the compactification of a finitely generated group constructed in the usual Stone-Čech way using the first l^2 (or some other function space) cohomology. Is the associated incidence geometry at infinity always trivial (i.e., hyperbolic)? (This is related to problems of Gromov in *Asymptotic invariants* in the chapter on l^p cohomology.)

Kaimanovich and Masur proved the following: for a measure μ on the mapping class group Γ (μ can be any finite first moment, finite entropy probability measure such that the group generated by its support is non-elementary), there exists a measure ν on $\mathcal{PMF}$ so that

$$\text{Poiss}(\Gamma, \mu) = (\mathcal{PMF}, \nu).$$

The measure ν is called *a μ-stationary measure on $\mathcal{PMF}$*, this measure is unique. Here $\text{Poiss}(\Gamma, \mu)$ is the Poisson boundary.

Problem 72 (Moon Duchin)**.** Characterize the hitting measure ν on $\mathcal{PMF}$ obtained from the random walk by mapping classes on Teichmüller space. Is it absolutely continuous with respect to visual measure (that is, Lebesgue measure on the visual sphere of directions)?

Problem 73 (Moon Duchin)**.** What is the Poisson boundary of Outer space?

11. Asymptotic cones

A geodesic metric space X (e.g. Cayley graph of a finitely-generated group) is Gromov-hyperbolic if and only if all asymptotic cones of X are trees. There are examples of finitely generated[1] groups G so that *some* asymptotic cones of G are trees but G is not Gromov-hyperbolic, see [58]. Call such groups *lacunary hyperbolic* following Olshansky, Osin and Sapir see [49]. All such groups are *limits* of hyperbolic groups in the sense that G admits an infinite presentation

$$G =< x_1, ..., x_n | R_1, R_2, R_3, ... >$$

so that each $G_k =< x_1, ..., x_n | R_1, ...R_k >$ is hyperbolic.

Problem 74 (Misha Kapovich)**.** Is there are meaningful structure theory for lacunary hyperbolic groups? Can one define a useful boundary for such groups? Is it true that either $Out(G)$ is finite or G splits over a virtually cyclic subgroup?

[1]Such groups are never finitely-presented

Remark 29. A counter-example to the last problem is known to M. Sapir.

It is known that for each relatively hyperbolic group G, all asymptotic cones of G have cut points.

Problem 75 (Cornelia Drutu)**.** To what extent is the reverse implication true?

Remark 30. Some counterexamples are known; for instance, the mapping class group and fundamental groups of graph manifolds are weakly relatively hyperbolic but not strongly.

Problem 76 (Mario Bonk)**.** The study of asymptotic cones has been non-analytic (they have been studied up to homeomorphism). What analytic tools could be developed?

12. Kleinian groups

Problem 77 (Misha Kapovich)**.** For the fundamental group G of a closed hyperbolic n-manifold consider a short exact sequence

$$1 \to \mathbb{Z}_p \to \Gamma \to G \to 1.$$

Is the group Γ residually finite? In other words, is there a finite-index subgroup G' in G so that the restriction map

$$H^2(G, \mathbb{Z}_p) \to H^2(G', \mathbb{Z}_p)$$

is zero? Remarkably, positive answer is presently known only for $n = 2$. Same problem makes sense also for the fundamental groups of complex-hyperbolic and quaternionic-hyperbolic manifolds.

Problem 78 (Misha Kapovich)**.** Let G be as above. Is there a finite-index subgroup $G' \subset G$ so that the restriction map

$$H^3(G, \mathbb{Z}_2) \to H^3(G', \mathbb{Z}_2)$$

is zero?

This problem is interesting because $H^3(G, \mathbb{Z}_2)$ classifies PL structures on the hyperbolic manifold $\mathbb{H}^n/G$.

Problem 79 (Misha Kapovich)**.** Let G be a Gromov-hyperbolic Coxeter group. Does G admit a discrete embedding in $Isom(\mathbb{H}^n)$ for large n?

Note that the Coxeter generators are not assumed to act as reflections on $\mathbb{H}^n$. Otherwise, there are counter-examples, see [31].

Problem 80. (Misha Kapovich) Let $G \subset PU(2, 1)$ be a convex-cocompact subgroup of isometries of complex-hyperbolic 2-space. Can the limit set of G be homeomorphic to the Sierpinski carpet?

Problem 81 (Misha Kapovich)**.** Let $G \subset Isom(\mathbb{H}^n)$ be a discrete torsion-free finitely-generated subgroup without abelian subgroups of rank ≥ 2. Is it true that

(a)

$$cd_{\mathbb{Z}}(G) \leq Hdim(\Lambda_c(G)) + 1 \quad ?$$

Here Λ_c is the conical limit set. The answer is known [36] to be positive if one considers homological rather than cohomological dimension.

(b) In the case of equality, is it true that the limit set of G is the round sphere and G? This is known to be true in the case when G is geometrically finite [36].

(c) If $Hdim(\Lambda_c(G)) < 2$, is it true that G is geometrically finite?

(d) If $Hdim(\Lambda_c(G)) < 1$, does it follow that G is a classical Schottky-type group? (I.e. the one whose fundamental domain is bounded by round spheres.) See [34] for partial results.

Problem 82 (Lewis Bowen)**.** Let $G \subset Isom(\mathbb{H}^4)$ be a Schottky group (or, more generally, a free convex-cocompact group). Can Hausdorff dimension of the limit set of G be arbitrarily close to 3?

Problem 83 (Misha Kapovich)**.** Let G be a finitely-generated discrete group of isometries of a Gromov-hyperbolic space X so that the limit set of G is connected. Is it true that the limit set of G is locally connected?

Consider a representation $\rho : G \to \mathrm{Isom}(\mathbb{H}^n)$. This action of G on the hyperbolic space determines a class function

$$\ell_\rho : G \to \mathbb{R}_+,$$

so that $\ell_\rho(g)$ is the displacement for the isometry $\rho(g)$ of $\mathbb{H}^n$, i.e.,

$$\ell_\rho(g) = \inf_{x\in\mathbb{H}^n} d(\rho(g)(x), x).$$

Problem 84. Suppose that ρ_1, ρ_2 are discrete and faithful representations as above so that there exists $C > 0$ for which we have

$$C^{-1} \le \frac{\ell_{\rho_1}(g)}{\ell_{\rho_2}(g)}) \le C, \quad \forall g \in G.$$

Does it follow that there exists a quasiconformal map $f : \Lambda(\rho_1(G)) \to \Lambda(\rho_2(G))$ which is equivariant with respect to the isomorphism $\rho_2 \circ \rho_1^{-1}$? Can one choose f which is K-quasiconformal for $K = K(C)$?

If $n = 3$ and G is finitely generated, then the answer to the first part of the problem is positive and follows from the solution of the ending lamination conjecture.

A constructive proof of Rips compactness theorem. Let G be a finitely-presented group which does not split as a graph of groups with virtually abelian edge groups. For every n define the space

$$\mathcal{D}_n(G)$$

of conjugacy classes of discrete and faithful representations of G into $\mathrm{Isom}(\mathbb{H}^n)$. We assume that G is not virtually abelian itself. Then Rips' theory of group actions on trees implies that $\mathcal{D}_n(G)$ is compact.

Problem 85. Find a "constructive" proof of the above theorem. More precisely, consider a finite presentation $\langle g_1, .., g_k | R_1, .., R_m\rangle$ of G. Given $[\rho] \in \mathcal{D}_n(G)$ define

$$B_n([\rho]) := \inf_{x\in\mathbb{H}^n} \max_{i=1,...,k} d(x, \rho(g_i)(x)).$$

Find an explicit constant C, which depends on n, k, m and the lengths of the words R_i, so that the function $B_n : \mathcal{D}_n(G) \to \mathbb{R}$ is bounded from above by C.

Remark 1. Y. Lai [41] found such an explicit constant C can be for Coxeter groups; moreover, in this case C depends only on n and the number of Coxeter generators.

One possible application of the solution of Problem 85 is in producing nontrivial algebraic restrictions on Kleinian groups.

An abstract *Kleinian* group is a group which admits a discrete embedding in $\mathrm{Isom}(\mathbb{H}^n)$ for some n.

All currently known algebraic restrictions on finitely-generated Kleinian groups can be traced to the following:

1. Every Kleinian group has the Haagerup property: They admit isometric properly discontinuous actions on some Hilbert space. See for instance [19].

2. If π is the fundamental group of a compact Kähler manifold, then every homomorphism $\rho : \pi \to \mathrm{Isom}(\mathbb{H}^n)$ wither factors through a group commensurable to a surface group or $\rho(\pi)$ preserves a point or a pair of points in $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$. See [17].

Problem 86. Find new restrictions on Kleinian groups.

Recall that a group G is called *coherent* if every finitely-generated subgroup of G is finitely-presented.

Problem 87 (M. Kapovich, L. Potyagailo, E.B. Vinberg)**.** Prove that every arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^n)$ $(n \geq 4)$ is non-coherent.

See [38] for some partial results in this direction.

It is well-known that every lattice in $\mathrm{Isom}(\mathbb{HH}^n)$ $(n \geq 2)$ has Property T.

Problem 88. Suppose that $G \subset \mathrm{Isom}(\mathbb{HH}^n)$ is a discrete subgroup satisfying Property T. Does it follow that G preserves a totally-geodesic subspace H in $\mathbb{HH}^n$ and acts on H as a lattice?

The main motivation for this problem comes from the fact that the obvious constrictions of discrete groups of isometries are by various graphs of groups and hence these groups do not have Property T. One can try to use triangles of groups:

Problem 89. Suppose that Δ is a developable triangle of groups, where all the cell-groups have Property T and so that all the links in the universal cover of T have $\lambda_1 > 1/2$. Does it follow that $\pi_1(\Delta)$ has Property T?

Problem 90. Generalize Bestvina-Feighn combination theorem from graphs of groups to complexes of groups.

Background: Let $\mathcal{G}$ be a graph of groups, so that vertex and edge groups are hyperbolic and the edge subgroup are quasiconvex in the vertex groups. Bestvina and Feighn [5] found some sufficient conditions for $\pi_1(\mathcal{G})$ to be hyperbolic. Hammenstädt has some partial results towards solving this problem.

Discrete subgroups in other Lie groups.

A *reflection* in a complex-hyperbolic space $\mathbb{CH}^n$ is an isometry of finite order which fixes a (complex) codimension 1 hyperplane. A *reflection group* in $\mathbb{CH}^n$ is a subgroup of $PU(n, 1)$ generated by reflections. These concepts generalize the notion of reflections and reflection groups acting on $\mathbb{H}^n$. Vinberg [60] proved that there for

$n \geq 30$ there are no uniform lattices in $O(n,1)$ which are reflection groups. This result was extended by Prokhorov [52] who proved nonexistence of reflection lattices in $O(n,1)$ for $n \geq 996$.

Problem 91. Generalize Vinberg's finiteness theorem for reflection groups to complex-hyperbolic reflection groups, i.e., prove that there exists a number N such that for $n \geq N$, there are no lattices in $PU(n,1)$ which are generated by reflections.

Problem 92 (Misha Kapovich)**.** There is a theory of quasi-convex groups acting on Gromov hyperbolic spaces, generalizing the theory of convex-compact groups of isometries of the real hyperbolic space. Develop a theory of geometric finiteness in CAT(0) spaces.

Remark 31. It is *a priori* unclear what to take as the definition of geometric finiteness in the context of CAT(0) spaces (even in the case of symmetric spaces). Taking quotients of the convex hull is a bad idea, as shown by a theorem of Bruce Kleiner and Bernhard Leeb: There are only few convex subsets in symmetric spaces of rank ≥ 2.

A better definition replacing convex-cocompactness could be:

A finitely-generated group $G \subset \mathrm{Isom}(X)$ is *undistorted* if the induced map from the Cayley graph of G to X is a quasi-isometric embedding.

In the case of Gromov hyperbolic spaces, *undistorted* is equivalent to *quasi-convex*.

There are examples of undistorted free Zariski dense subgroups of $SL(n,\mathbb{R})$, generalizing the Schottky construction.

Is there an interpretation of the notion of *undistorted* groups in terms of the group actions on limit sets?

F. Labourie [40] introduced another notion of convex-cocompactness that he calls an *Anosov structure*, for group representations $\rho : \Gamma \to G$, where G is a semisimple Lie group. In the case when Γ is a surface group and $G = SL(n+1,\mathbb{R})$, this notion can be reformulated in terms of action of $\rho(\Gamma)$ on its limit set in $\mathbb{R}P^n$, i.e. existence of a $\rho(\Gamma)$-invariant hyperconvex curve in $\mathbb{R}P^n$.

Problem 93 (Anna Wienhard)**.** Extend this relation of Anosov structure and dynamics on the limit set to representations of other hyperbolic groups.

Problem 94 (Anna Wienhard)**.** Generalize holomorphic chain patterns in $\partial_\infty \mathbb{CH}^n$ in order to prove rigidity results for embeddings of lattices in $PU(n,1)$ into other higher rank Lie groups.

Background. Ideal boundaries of totally-geodesic subspaces $\mathbb{CH}^1 \subset \mathbb{C}H^n$ define *holomorphic chains* in $\partial_\infty \mathbb{CH}^n$. These circles are characterized by the property that three points belong to such a chain if and only if they span an ideal triangle in $\mathbb{CH}^n$ of maximal (symplectic) area. The incidence relation between holomorphic chains in $\partial_\infty \mathbb{CH}^n$ determines a "building-like" structure where chains serve as apartments: Every two points belong to a chain. Given a measurable map

$$\partial_\infty \mathbb{CH}^n \to \partial_\infty \mathbb{CH}^m, m \geq n,$$

which induces a measurable morphism of these "building-like" structures, is induced by a holomorphic embedding $\mathbb{CH}^n \to \mathbb{CH}^m$. This, in turn, can be used to reprove Corlette's rigidity theorem [21] for representations of lattices in $PU(n,1)$ into $PU(m,1)$. The motivation for the Problem 94 is to extend Corlette's rigidity result to representations of $PU(n,1)$ to other Lie groups.

Problem 95 (Anna Wienhard)**.** Obtain new rigidity results for embeddings of real-hyperbolic lattices into higher-rank semisimple Lie groups in terms of the boundary maps.

13. Miscellaneous problems in geometric group theory

Problem 96 (Kevin Whyte: Homotopy Nielsen realization)**.** If X is a compact polyhedron and G is a discrete group of simple homotopy equivalences $X \to X$, is there a compact space X', homotopy equivalent to X, such that G can be realized as a group of homeomorphisms of X'.

Remark 32. It is a long-standing open problem to determine if the exact sequence

$$1 \to Homeo_0(S) \to Homeo(S) \to Mod(S) \to 1$$

is split. Here S is a compact surface of genus ≥ 2 and $Homeo_0(S)$ denotes the connected component of the identity in the group of homeomorphisms.

Moreover, there are examples due to George Cooke of spaces X and finite groups G of simple homotopy-equivalences of X for which the answer to Problem 96 is "No." However all known examples occur in dimensions ≥ 5 and require the group to have torsion.

Problem 97 (Ilia Kapovich)**.** Consider finite cell complexes X. Is there an algorithm to determine if X is contractible?

Remark 33. The triviality problem is known to be unsolvable for finitely presented groups. However, their presentation complexes are never contractible, since the presentation is unbalanced. On the other hand, for complexes of dimension $n \geq 4$, there is no algorithm to determine contractibility (S. Weinberger [62]). Indeed, take a triangulated closed n-manifold M for which it is impossible to decide if M is homeomorphic to S^n. Let X be the complement to an open n-simplex in M. Then contractibility of X is undecidable, since it is equivalent to M being a homotopy sphere.

Remark 34 (Daniel Groves). It is an open problem if the triviality of a group is algorithmically solvable for groups with balanced presentation. (A presentation is called *balanced* if the number of generators equals the number of relators.)

Problem 98 (Kevin Whyte)**.** For a word-hyperbolic G not splitting over any virtually cyclic group, can an infinite-index subgroup and a finite-index subgroup be isomorphic?

Remark 35. This asks for something slightly stronger than the cohopfian property.

Problem 99 (Misha Kapovich)**.** Consider Teichmüller space $T(S)$ with Teichmüller metric. Does it have quadratic isoperimetric inequality?

Background: If $dim_{\mathbb{C}}(T(S)) \geq 2$, $T(S)$ is known to be non-hyperbolic. However the Mapping Class Group is bi-automatic, therefore the "thick part" of $T(S)$ is semihyperbolic. One can ask a similar question for the outer space.

Curt McMullen defined "inflexibility" for Kleinian groups, [43].

Problem 100 (Danny Calegari)**.** Is there a similar statement to this inflexibility result this with no group specified—that is, for subsets $\Lambda \subset S^2$ of the boundary sphere of $\mathbb{H}^3$?

Here is a possible setup for such problem: Define a random Beltrami differential as follows. Let τ be the tessellation of $\mathbb{H}^2$ by regular right-angled hyperbolic pentagons. (All such pentagons are isometric to a model pentagon P.) Let M be a compact (perhaps finite or even a singleton) set of Beltrami differentials on P having norm $1/2$ (or any fixed number < 1). For concreteness, suppose that $M = \{\mu_0\}$ is a singleton. For each pentagon $P' \in \tau$ choose a random isometry $g : P \to P'$. (There are 10 such isometries.) Then push-forward μ_0 from P to P' via g. This defines a *random Beltrami differential* μ on $\mathbb{H}^2$. Given a closed connected set $\Lambda \subset S^2$ observe that each complementary component $\Omega \subset S^2 \setminus \Lambda$ is simply-connected. Choose a Riemann mapping $R : \mathbb{H}^2 \to \Omega$; push-forward the random Beltrami differential μ from $\mathbb{H}^2$ to Ω. Repeat this for each component of $S^2 \setminus \Lambda$ and extend the resulting differential to Λ by zero. This defines a *Beltrami differential* μ_Λ on S^2. Let $q = q_\Lambda : S^2 \to S^2$ be the quasiconformal map which is a solution of the Beltrami equation

$$q_{\bar{z}} = \mu_\Lambda q_z.$$

Such a quasiconformal map has a natural *Thurston-Reimann* biLipschitz extension $Q_\Lambda : \mathbb{H}^3 \to \mathbb{H}^3$, [53].

Problem 101. Given $p \in \mathbb{H}^3$, estimate the biLipschitz constant of Q_Λ near p in terms of the distance d from p to the exterior of the convex hull of Λ.

More concretely: if Λ is a quasicircle, is the decay exponential in d? That is, are there positive constants C_1, C_2 such that

$$L(Q_\Lambda, p) \leq 1 + C_1 exp(-C_2 d(p, \partial_\infty K))$$

where L is the bilipschitz constant of Q_Λ restricted to the ball of some fixed radius (say radius 1) about p, K is the convex hull of Λ, and p is a point in the interior of K.

Problem 102 (Mladen Bestvina)**.** Are braid groups CAT(0)?

Remark 36. It is conjectured that all Artin groups are CAT(0).

Problem 103. Extend Rips' theory to higher-dimensional buildings, e.g. products of $\mathbb{R}$-trees.

Rank rigidity. Let X be a $CAT(0)$ metric space. The space X is said to be or rank $\geq n$ if every geodesic segment in X is contained in a subset E which is isometric to a flat n-dimensional parallelepiped. If Y is a locally $CAT(0)$ metric space, then Y is said to have rank $\geq n$ if its universal cover is of rank $\geq n$. The *rank rigidity theorem* proven by Ballmann [1] and by Burns and Spatzier [15, 16] states that:

If M is a compact nonpositively curved Riemannian manifold of rank ≥ 2, then either M admits a finite cover the universal cover of M splits (nontrivially) as a Riemannian direct product or M is a locally symmetric space.

Problem 104 (Werner Ballmann, Misha Brin)**.** Suppose that Y is a compact finite-dimensional locally $CAT(0)$ metric space of rank $n \geq 2$. Then either the universal cover of Y splits (nontrivially) as a Riemannian direct product or it is isometric to a Euclidean building.

This problem is most natural in the context of piecewise-Euclidean metric cell complexes. The conjecture was proven in the case of 2-dimensional and 3-dimensional complexes by Ballmann and Brin [2, 3].

Cogrowth. Let $H \subset G$ be a subgroup of a finitely-generated group G. The *cogrowth* of H in G is the growth of the Shreier graph Γ_G/H, where Γ_G is a Cayley graph of G.

Problem 105. Compute cogrowth for "interesting" subgroups. For instance:

1. Show that the cogrowth of $SL(n, \mathbb{Z})$ in $SL(n+1, \mathbb{Z})$ is exponential.
2. Compute cogrowth of special subgroups in Coxeter groups. (See [61] for partial results.)
3. Suppose that Γ_G/H is Gromov-hyperbolic. Is it true that the cogrowth is either constant, linear or exponential?

Coarse Whitehead Conjecture.

Problem 106 (Whitehead Conjecture)**.** Let X be an aspherical (i.e. with contractible universal cover) 2-dimensional complex. Is it true that every subcomplex of X is also aspherical? (See [**?**].)

A metric space Z is said to be *coarsely trivial* π_m if the following holds: There exists a function $\phi(R)$ so that for each $R \geq 0$ the map

$$Rips_R(Z) \to Rips_{\phi(R)}(Z)$$

induces zero map of the m-th homotopy groups. For instance, suppose that Z is the 0-skeleton of an m-connected simplicial complex X, which admits a cocompact free group action. Metrize Z by declaring each edge of X to have unit length. Then Z has coarsely trivial π_j for all $j \leq m$. Given a 2-dimensional contractible complex X as above and a connected subgraph $Y \subset X^{(1)}$, metrize $Y^{(0)}$ using the above path-metric on Y.

Problem 107 ("Coarse Whitehead Conjecture", Misha Kapovich)**.** Under the above assumptions, is it true that Y has coarsely trivial π_m for $m \geq 2$?

More restrictively one can consider the case when X is the Cayley complex of a finitely-presented group G and H is finitely-generated subgroup of G, identified with $Y^{(0)}$. (Then the metric on $Y^{(0)}$ is quasi-isometric to the word metric on H.) Note that if H is finitely-presented then (since its cohomological dimension ≤ 2) it has finite type and, thus H necessarily has coarsely trivial π_m for all m.

Problem 108. Does the Coarse Whitehead Conjecture hold if G is hyperbolic?

Note that there is an abundant supply of finitely generated non-finitely presented subgroups of 2-dimensional hyperbolic groups, given by the Rips construction [54].

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