RAAGs in Ham

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1 Introduction

For a graph Γ let $V(\Gamma), E(\Gamma)$ denote the vertex and edge sets of Γ. Let Γ be a graph with no loops and bigons. Define the Right Angled Artin group (RAAG) G_Γ with the Artin graph Γ by the presentation

$$\langle g_v, v \in V(\Gamma) | \quad [g_v, g_w] = 1, [vw] \notin E(\Gamma) \rangle.$$

We note that our definition is opposite to the usual one in the theory of Right Angled Artin groups (where one imposes the relators $[g_v, g_w] = 1$ for every $[vw] \in E(\Gamma)$). However, our convention is in line with the notation in the theory of finite Coxeter groups and Dynkin diagrams. We adopted this notation because it is most suitable for the purposes of this paper, while the usual definition leads to heavy notation.

Given a symplectic manifold (M, ω) we let $Ham(M, \omega)$ denote the group of Hamiltonian symplectomorphisms of (M, ω). Since, by Moser's theorem, for a closed surface M its symplectic structure is unique up to scaling, we will abbreviate $Ham(M, \omega)$ to $Ham(M)$ if M is a closed surface.

Our main result is:

Theorem 1.1. *For every finite Γ the group G_Γ embeds in $Ham(S^2)$.*

As a corollary of this result we establish the following result proven in the end of the paper:

Corollary 1.2. *For every finite Γ and every symplectic manifold (M, ω), the group G_Γ embeds in $Ham(M, \omega)$.*

Corollary 1.3. *Let $\Lambda \subset O(n, 1)$ be an arithmetic lattice of the simplest type, $n \geq 2$. Then Λ embeds in $Ham(M, \omega)$ for every symplectic manifold (M, ω).*

Proof. According to the result of Bergeron, Haglund and Wise [HW], Λ embeds in some RAAG G_Γ. Now, the result follows from Theorem 1.1. □

In contrast, suppose that M is a closed oriented surface with area form ω. Then L. Polterovich [P] proved that any irreducible nonuniform irreducible arithmetic groups of rank ≥ 2 does not embed in $Ham(M,\omega)$ (assuming M is not S^2). Moroever, Franks and Handel [FH] show that for any surface M, any homomorphism $\Lambda \to Diff(M,\omega)$ factors through a finite group, where Λ is commensurable to $SL(n,\mathbb{Z}), n \geq 3$ and $Diff(M,\omega)$ is the group of area-preserving diffeomorphisms of M.

Outline of the proof. Theorem 1.1 is proven in three steps.

Step 1. Let M be a closed connected oriented surface to which Γ embeds. For technical reasons, it will be convenient to assume that M is not the torus. We first prove

Theorem 1.4. *G_Γ embeds in $Ham(M)$. Moreover, each Artin generator g_v of G_Γ acts on M as an "N-iterated Double Dehn twist" $\Psi(g_v)$ supported in a homotopically trivial annulus in M.*

The key to verifying injectivity of $\Psi : G_\Gamma \to Ham(M)$ is that the action $G_\Gamma \curvearrowright M$ preserves a certain finite subset $P \subset M$, so that the restriction $G_\Gamma \curvearrowright M' = M - P$ projects to a faithful representation to the mapping class group of M', $G_\Gamma \to Map(M')$. Faithfulness of this representation follows from a special case of a theorem of L. Funar [F] (see also the more recent paper by T. Koberda [K]). This part of our paper is similar to the arguments by Crisp and Wiest [CW].

Step 2 (Lifting). If Γ were planar, Theorem 1.4 would imply Theorem 1.1. In general, of course, Γ need not be planar (or even admit a finite planar orbi-cover, see the Section 2), however, it has planar universal cover (i.e., disjoint union of simplicial trees). Suppose, therefore, that M has genus ≥ 2. We then lift the action $\Psi : G_\Gamma \curvearrowright M$ to the universal cover $\tilde{M}$ of M, which we identify with the hyperbolic plane, i.e., the unit disk D in $S^2 = \mathbb{C} \cup \infty$. We let ω_0 be the Euclidean area form on an open disk containing D; extend ω_0 smoothly to an area form ω_0 on S^2.

Let $D' = D - P'$ denote the punctured disk where P' is the preimage of P in D. Let $Ham(M,P)$ denote the subgroup of $Ham(M)$ fixing P pointwise. We have an (injective) homomorphism

$$\iota : Ham(M,P) \to Ham(D,P')$$

obtained by choosing an appropriate lifting of Hamiltonian diffeomorphisms. We thus obtain the lift $\tilde{\Psi} = \iota \circ \Psi$ of the homomorphism Ψ. We then show that $\tilde{\Psi}$ projects injectively to the mapping class group $Map(D')$.

Each generator g_v of G_Γ acts (via $\tilde{\Psi}$) on D as a product of infinitely many commuting N-iterated Double Dehn twists preserving the hyperbolic area form. However, $\tilde{\Psi}(G_\Gamma)$, of course, does not preserve ω_0. We then modify each of Double Dehn twist in the product decomposition of $\tilde{\Psi}(g_v)$ to obtain a new diffeomorphism $\rho_0(g_v)$ which is isotopic to $\tilde{\Psi}(g_v)$ on the punctured disk D' and is the time-N-map for the appropriately chosen function $H_v : D \to \mathbb{R}$ with respect to ω_0. It then follows that the resulting representation

$$\rho_0 : G_\Gamma \to Ham(D,\omega_0)$$

is again faithful. We will see that for each v, H_v extends by zero to a $C^{1,1}$-function on S^2 and $\rho_0(g_v)$ extends Lipschitz-continuously (by the identity) to the entire sphere, so we can think of it as a Lipschitz Hamiltonian symplectomorphism. However, the function H_v need not be C^2-smooth and $\rho_0(g_v)$ need not even be differentiable.

Step 3 (Approximation). The last step of the proof is an approximation argument: We approximate $H_v : S^2 \to \mathbb{R}$ by a mollifier, a smooth function $\eta_\epsilon H_v$ which depends analytically on $\epsilon > 0$ and converges to H_v uniformly on compacts in the open disk D as $\epsilon \to 0$. Each function $\eta_\epsilon H_v$ determines its own time-N map $\rho_\epsilon(g_v)$ and we obtain an analytic family of representations $\rho_\epsilon : G_\Gamma \to Ham(S^2)$, $\epsilon > 0$, which converge to ρ_0 as $\epsilon \to 0$. We then establish that the representations ρ_ϵ are injective for all bout countably many $\epsilon > 0$, thereby proving Theorem 1.1.

Questions.

Question 1.5. *Is it true that every G_Γ admits a quasi-isometric embedding $G_\Gamma \to Ham(S^2)$ with the L_2-metric on $Ham(S^2)$? See [CW] for the case of RAAGs with planar Γ.*

Question 1.6. *Let M be a closed oriented surface and let Λ be a Kähler group which is not virtually a surface group. Does Λ embed in $Ham(M)$? Does any infinite group with property T embed in $Ham(M)$?*

Question 1.7. *(A. Berenstein) Is it true that every Artin group (not necessarily right-angled) embeds in $Ham(S^2)$?*

Question 1.8. *(V. Kharlamov) Which RAAGs embed in $Diff(S^1)$?*

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2 Obricovers of graphs and embeddings of RAAGs

The *double* $D\Gamma$ of a graph Γ is defined to be the disjoint union of two copies of Γ.

If G_Γ is the RAAG with the Artin graph Γ, we call $G_{D\Gamma}$ the *double* of G_Γ. We then have the diagonal homomorphism

$$\delta : G_\Gamma \to G_{D\Gamma}, \quad \delta(g_v) = g_{v^+} g_{v^-}.$$

Note that since $[v^+, v^-] \notin E(\Gamma)$, the order in the product $g_{v^+} g_{v^-}$ is irrelevant.

Lemma 2.1. *The homomorphism $\delta : G_\Gamma \to G_{D\Gamma}$ is injective.*

Proof. We have the natural projection $\pi : G_{D\Gamma} \to G_\Gamma$, $\pi(g_{v^+}) = g_v, \pi(g_{v^-}) = 1$. Then, clearly, $\pi \circ \delta = id$. Hence, δ is injective. $\square$

This lemma is a special case of a more general result on embeddings of RAAGs proven below.

An *orbi-cover* (or a *branched-cover*) of a graph is a map of graphs $p : \Delta \to \Gamma$ which is a locally-surjective graph morphism. A *planar emulator* of a graph Γ is a finite orbi-cover $p : \Delta \to \Gamma$ with planar Δ. For instance, the graph K_6 is not planar but admits a planar 2-fold finite cover (since it embeds in $\mathbb{R}P^2$). Moreover, there are finite graphs Γ which admit finite planar emulators but admit no finite planar covers [RY]. On the other hand, if every vertex Γ has valence ≥ 6, then Γ does not admit a planar embedding; consequently, such graphs do not admit planar emulators. (I am grateful to Yo'av Rieck for this example.)

Given a RAAG G_Γ and a finite orbi-cover $p : \Delta \to \Gamma$, one defines a "diagonal" homomorphism $\delta = p^* : G = G_\Gamma \to \tilde{G} = G_\Delta$ by

$$p^*(g_v) := \prod_{x \in p^{-1}(v)} g_x.$$

Note that all the generators $g_x, x \in p^{-1}(v)$ of the group G_Δ commute (since $x, y \in p^{-1}(v)$ are never connected by an edge in Δ). It is immediate that $p^* : G_\Gamma \to G_\Delta$ is indeed a homomorphism: If $v, w \in V(\Gamma)$ and $[vw] \notin \Gamma$ then for any $x \in p^{-1}(v), y \in p^{-1}(w)$, $[xy] \notin \Gamma$.

Lemma 2.2. *$\delta : G \to \tilde{G}$ is injective.*

Proof. We will use the normal forms for the elements of RAAGs. We first order the vertices of Γ; we lift this order to a lexicographic order on $V(\Delta)$. Then a normal form of $g \in G$ is the product of generators

$$w = g_{v_1}^{\pm 1} ... g_{v_k}^{\pm 1}$$

with the condition that the word w contains no subwords of the form $g_v g_v^{-1}$ and if $[uv] \in E(\Gamma)$, $u < v$ and $g_v^{\pm 1}$ precedes $g_u^{\pm 1}$ in w, then between these letters in w there is a letter $g_z^{\pm 1}$ such that $[zv] \in E(\Gamma)$. Then every $g \in G$ admits a normal form and this normal form is unique. The reduction process of a word w to the normal form is as follows:

A pair of consecutive letters $g_y^{\pm 1} g_x^{\pm 1}$ in w is an *inversion* if $y > x$ and $[xy] \notin E(\Gamma)$. Then, in order to reduce w to its normal form use the commutation relation to reduce the number of inversions ("shuffling") and cancel appearances of the products $g_v g_v^{-1}$ ("cancellation"). We refer the reader to [HM] for the details.

Suppose now that $w = g_{v_1}^{\pm 1} ... g_{v_k}^{\pm 1}$ is a normal form for $g \in G$. The image $\delta(w) \in \tilde{G}$ defined as

$$\delta(g_{v_1}^{\pm 1}) ... \delta(g_{v_k}^{\pm 1})$$

need not be in normal form. We claim, however, that the length of the normal form of $\delta(w)$ is the same as the length of $\delta(w)$, i.e., no cancellations in the reduction process occur. Indeed, the only way we can get a cancellation is that w (or w^{-1}) contains

$$w = ...g_v...g_v^{-1}...$$

Then, since w is a normal form, between these appearances of g_v and g_v^{-1} there is some g_u (or g_u^{-1}) so that $[uv] \in E(\Gamma^c)$:

$$w = ...g_v...g_u...g_v^{-1}...$$

(We assume that the sub-word between g_v and g_v^{-1} is the shortest where a cancellation in $\delta(w)$ is possible.) Lifting w to $\tilde{G}$, we see that for each $x \in p^{-1}(v)$ and $y \in p^{-1}(u)$ there exists an edge $[xy] \in E(\Delta)$. Therefore, shuffling the generators of $\tilde{G}$ would not allow us to move any g_y^{-1} appearing in the lift of g_v^{-1} past the lift of g_u. Therefore, we would be unable to cancel any of these g_y^{-1} (in the lift of g_v) with any g_y (in the lift of g_v). Thus δ is injective and, moreover, is a quasi-isometric embedding $G \to \tilde{G}$. □

Remark 2.3. *Let* $D\Gamma$ *be the double of* Γ. *Since the natural map* $D\Gamma \to \Gamma$ *is a trivial 2-fold cover, Lemma 2.1 is a corollary of Lemma 2.2.*

We now observe that, if Γ is such that Γ admits a finite "planar emulator" $p : \Delta \to \Gamma$, then Theorem 1.4 implies that G_Γ embeds in $Ham(S^2)$, even though, Γ need not be planar. Thus, for G_Γ so that Γ admits a finite planar emulator, Steps 2 and 3 of the proof of Theorem 1.1 are not needed.

3 Proof of Theorem 1.4

Hamiltonian symplectomorphisms. Let (M, ω) be a symplectic manifold, $H : M \times \mathbb{R} \to \mathbb{R}$ a smooth function. Using the form ω one then converts the differential form $dH(x,t)$ to a time-dependent vector field $X_H(x,t)$ on M:

$$\omega(X_H, \xi) = dH(\xi), \quad \xi \in TM.$$

Consider the ODE

$$\frac{\partial F(x,t)}{\partial t} = X_H(x,t) \tag{1}$$

on the manifold M. Solutions $f_t(x) := F(x,t)$ of this equation are *Hamiltonian symplectomorphisms* of the manifold (M, ω). Note that for a non-compact manifold M the ODE (1) may not have solutions defined on the entire M for any $t > 0$. In general, Hamiltonian symplectomorphisms of (M, ω) form a pseudo-group $Ham(M, \omega)$. However, if M is closed, $Ham(M, \omega)$ is a group.

Double Dehn Twists. Recall that smooth manifolds with boundary satisfy the *Moser's Lemma*:

Theorem 3.1. *[GS, Lemmata 1 and 2] Suppose that M is a smooth manifold with boundary and ω, ω' are volume forms so that*

$$\int_M \omega = \int_M \omega'.$$

Then there exists a diffeomorphism f is isotopic to the identity, which carries ω to ω'. Moreover, if ω' varies continuously in C^∞ topology, f can be also chosen to vary continuously in C^∞ topology.

In particular, if (M, ω) is a symplectic surface and $A \subset M$ is an annulus with smooth boundary, then (A, ω) is symplectomorphic to a product annulus $A_a := S^1 \times [-a, a]$ with the product area form, where S^1 is the unit circle.

We define a *twist* Hamiltonian symplectomorphism $f : A_a \to A_a$ as follows. Pick a smooth function $H(s, t) = h(t)$, so that h vanishes (with all its derivatives) at $-a$ and a, and $h'(0) = 2\pi$. Let X_H be the associated Hamiltonian vector field on A_a. The field X_H is constant with respect to the s-coordinate and tangent to the circles $A^1 \times t, t \in [-a, a]$. Let $f : A_a \to A_a$ be the corresponding time-1 Hamiltonian symplectomorphism:

$$f(x) = F(x, 1), \quad F(x, 0) = id, \quad F = F(x, \tau), \quad \frac{\partial F}{\partial \tau} = X_H.$$

We let $A_a^\pm := S^1 \times [\pm a, 0]$ denote the subannuli in A_a with the common boundary circle $C = S^1 \times 0$, which we will call the *central circle* of A_a. We let $f_\pm$ denote the restrictions $f|A_a^\pm$ extended by the identity to the rest of A_a. Lifting H, X_H and f to the universal cover $\tilde{A}_a$ of A_a we see that the lift of f fixes the boundary lines of $\tilde{A}_a$ and acts on the line $\mathbb{R} \times 0$ (the lift of the central circle C) as the translation by 2π. Therefore, both $f_\pm$ are Dehn twists on A_a and f_- is isotopic to the inverse of f_+ relative to the boundary of A_a. By abusing the terminology, we will say that f is the *rotation by* 2π *along* C.

Let (M, ω) be a symplectic surface, $A \subset M$ be a smooth annulus which is symplectomorphic to some A_a. We will use the notation C_A (the *central circle* of A) for the circle in A corresponding to $C = S^1 \times 0 \subset A_a$. Using the symplectomorphism $A_a \to A$ we carry the maps $f, f_\pm : A_a \to A_a$, function H and the Hamiltonian vector field X_H to maps $f, f_\pm : A \to A$, function H and vector field X_H on A. The maps $f, f_\pm : A \to A$ extend by the identity to the rest of the surface M. We will use the notation $f_A, f_{A,\pm}$ for the extensions. Then $f_A : M \to M$ is a smooth symplectomorphism and $f_A = f_{A,+} \circ f_{A,-}$. The maps $f_{A,\pm}$ are Dehn twists on M which are, up to isotopy, inverses to each other. Moreover, f_A is a Hamiltonian symplectomorphism since the above function $H : A \to \mathbb{R}$ and its Hamiltonian vector field X_H extend by zero to the rest of M.

Pick a point $p \in C_A$, then the map $f_A : M \setminus p \to M \setminus p$ has infinite order in the mapping class group of this punctured surface, provided that the annulus A is *essential*, i.e., each component of $M \setminus A$ has negative Euler characteristic. We will refer to the map $f_A : M \to M$ as a *Double Dehn twist.*

Construction of homomorphisms of RAAGs to $Ham(M,\omega)$.

Let G_Γ be a RAAG with the Artin graph Γ. Since Γ is finite, there exists a closed oriented surface M which admits an embedding $j:\Gamma\to M$. Without loss of generality, we may assume that M is not the torus. We equip M with an area form ω. There exists a collection of closed disks $\mathcal{B}:=\{B(v):v\in V(\Gamma)\}$ so that:

1. Each $B(v)$ has smooth boundary.
2. $B(v)\cap B(w)\neq\emptyset$ (for $v\neq w$) iff $[vw]\in\Gamma$.
3. Whenever $B(v)\cap B(w)\neq\emptyset$, their boundary circles C_v, C_w intersect transversally and in exactly two points.
4. Triple intersections of discs are empty.

Remark 3.2. *One can construct such $\mathcal{B}$ as follows: Let Γ' be the barycentric subdivision of Γ. For each $v\in V(\Gamma)$ take a sufficiently small smooth disk neighborhood $B(v)\subset M$ of $j(Star(v))$, where $Star(v)$ is the star of v in Γ'.*

We then thicken each circle C_v to an annulus $A(v)$ in such a way that the nerve of the resulting collection of annuli $\{A(v):v\in V(\Gamma)\}$ is still isomorphic to Γ. We identify each annulus $A(v)$ with the corresponding symplectomorphic product annulus A_a, where a depends on v. Accordingly, we carry all the notation introduced for A_a to the annulus $A(v)$. We will identify the circles C_v in the above construction with the central circles $C_{A(v)}$ of the annuli $A(v)$. We note that all annuli $A(v)$ are inessential in M, since $M\setminus A(v)$ contains the disk $D(v)\setminus A(v)$.

We now define a certain finite subset $P\subset M$ fixed by all the double Dehn twists in the annuli $A(v)$. The points of P will serve as punctures on M. For each vertex $v\in V(\Gamma)$ we pick a 2-element set $P_v\subset C_v$, contained in the connected component of

$$C_v\setminus\bigcup_{w\in V(\Gamma),w\neq v}A(w).$$

Set

$$P_1:=\bigcup_{v\in V(\Gamma)}P_v$$

We let P_2 denote the subset of

$$M\setminus\bigcup_{v\in V(\Gamma)}A(v).$$

containing two points in each component of this surface. Lastly, pick some $q\in M$ which does not belong to any of the annuli $A(v)$ and any of the disks $B(w)$. We set

$$P:=P_1\cup P_2\cup\{q\}$$

and let $M':=M\setminus P$. See Figure 1. Now, all the circles in

$$\bigcup_{v\in V(\Gamma)}\partial A(v)$$

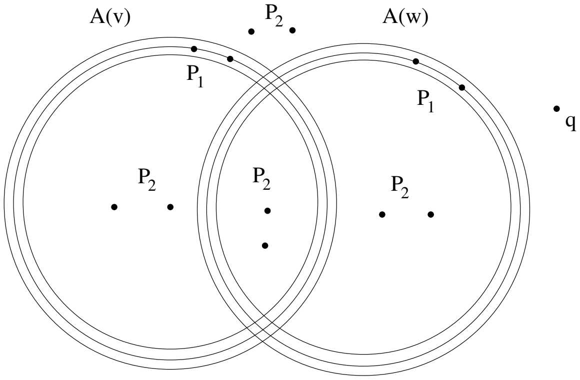

Figure 1:

are essential and pairwise non-isotopic in M'.

We let $f_v := f_{A(v)}, f_{v,\pm} := f_{A(v),\pm}$ denote the Hamiltonian Double Dehn twists and Dehn twists of (M, ω) determined by the symplectic annuli $(A(v), \omega)$ and the functions $H_{A(v)} : A(v) \to \mathbb{R}$ corresponding to the function $H : A \to \mathbb{R}$. It is clear from the construction that each Dehn twist $f_{v,\pm}, v \in V(\Gamma)$, fixes the set P pointwise. Moreover, each $f_v|M'$ is isotopic to the product of two commuting (isotopically) nontrivial Dehn twists $f_{v,\pm} : M' \to M'$ (which we regard as Dehn twists in the boundary circles of $A(v)$). Since each the boundary circles of $A(v)$'s are essential and non-isotopic in M', it follows that the maps $f_{A(v),\pm}$ are pairwise non-isotopic on M' and, moreover, generate distinct cyclic subgroups of the mapping class group $Map(M')$.

Since $A(v) \cap A(w) = \emptyset$ for $[vw] \notin E(\Gamma)$, f_v commutes with f_w whenever $[vw] \notin E(\Gamma)$. It follows that the map $\psi : g_v \to f_v$ determines a homomorphism

$$\psi : G_\Gamma \to Ham(M, \omega).$$

The image of ψ is contained in $Diff(M, P)$, the subgroup of fixing P pointwise. Moreover, for each natural number N we have a homomorphism

$$\psi_N : G_\Gamma \to Ham(M, \omega), \quad \psi_N(g_v) = f_v^N.$$

The diffeomorphism f_v^N is the time-N map of the Hamiltonian $H_{A(v)}$.

In what follows we will need a theorem of L. Funar [F] formulated below.

Let $\mathcal{A} := \{a_1, ..., a_m\}$ be a system of simple closed oriented loops on a compact surface S with at least one boundary component. We require that these loops have the least intersection number in their isotopy classes. One says that the system of loops $\mathcal{A}$ is *sparse* if for some choice of paths γ_i connecting q to a_i, the loops

$$b_i := \gamma_i^{-1} a_i \gamma_i$$

based at q generate a free subgroup of rank m in $\pi_1(S, q)$. We next note that a simple sufficient condition for a system of loops $\mathcal{A}$ to be sparse is that they define a linearly independent system of elements of $H_1(S)$ (or, equivalently, of F). Here and below, we use homology with real coefficients. Indeed, since $\pi_1(S)$ is free, the group generated by the loops b_i is necessarily free. Its rank equals the rank of the subspaces in $H_1(S)$ spanned by the elements $[a_i]$. We now assume that $\mathcal{A}$ is *sparse*.

Let D_{a_i} denote the Dehn twist (right or left) in a_i. Define the RAAG G_Γ, where Γ is the incidence graph of the collection of loops $\mathcal{A}$, i.e., $V(\Gamma) = A$, $[a_i, a_j] \in E(\Gamma)$ iff $a_i \cap a_j \neq \emptyset$. Let q be a point in the interior of S, disjoint from the curves in $\mathcal{A}$. We recall also that for the surface $S' := S \setminus \{q\}$, the *Mapping class group* $Map(S')$ is the group $Homeo(S, \partial S)/Homeo_0(S, \partial S)$, where $Homeo(S, \partial S)$ is the group of homeomorphisms fixing the boundary of S pointwise, while the normal subgroup $Homeo_0(S, \partial S) < Homeo(S, \partial S)$ is its identity component.

Theorem 3.3. *[L. Funar] Under the above conditions, for every $N \geq 2$, the natural homomorphism $\phi_N : G_\Gamma \to Map(S \setminus \{q\})$, $\phi_N : g_{a_i} \mapsto D^N_{a_i}$, is injective.*

We will apply this theorem in the case of punctured surfaces as follows. Given a surface S with boundary, we attach to each boundary circle of S a disk with at least two punctures. Let F denote the resulting noncompact surface. We observe that $Map(S)$ injects in $Map(F)$.

Proposition 3.4. *For $N \geq 2$ the homomorphism $\Psi := \psi_N : G_\Gamma \to Ham(M, \omega)$ is injective.*

Proof. Clearly, it suffices to show that the composition

$$G_\Gamma \overset{\psi}{\to} Diff(M') \overset{\pi}{\to} Map(M')$$

is injective, where $Map(M')$ is the Mapping Class Group of M'. We let $D\Gamma$ denote the double of Γ and $G_{D\Gamma}$ be the corresponding double Artin group. We then have natural homomorphisms

$$\phi_N : G_{D\Gamma} \to Diff(M'), \quad \phi_N(g_{v^\pm}) = f^N_{v^\pm}.$$

Then $\psi_N = \phi_N | G_\Gamma$, where G_Γ is embedded in $G_{D\Gamma}$ via the diagonal embedding δ as in Lemma 2.1. We let $\bar{\phi}_N : G_{D\Gamma} \to Map(M')$ denote the compositions $\pi \circ \phi_N$. We observe that the homomorphism $\bar{\phi} : G_{D\Gamma} \to Map(M')$ has the property that each Artin generator $g_{v^\pm}$ of $G_{D\Gamma}$ acts as a Dehn twist along the boundary curve $\alpha^\pm_v$ of $A(v)$. We claim that for every $N \geq 2$ the homomorphism $\bar{\phi}_N : G_{D\Gamma} \to Map(M')$ is injective. In view of Lemma 2.1, this would imply injectivity of $\bar{\psi}_N$, and, hence, of ψ_N as well.

We will derive injectivity of $\bar{\phi}_N$ from Funar's theorem above. We define a compact surface S, as in Funar's theorem, as follows.

Set

$$A := \bigcup_{v \in V(\Gamma)} A(v)$$

and define the compact surface T to be the union of components of $M \setminus int(A)$ which are not contained in any of the disks $B(w)$. Recall that for every $v \in V(\Gamma)$ the set

$P_1 \cap C_v$ is a 2-element subset P_v contained in a connected component of $(M \setminus A) \cup A(v)$. We let $\beta_v \subset C_v$ the arc connecting the points of P_v which is disjoint from $A \setminus A(v)$. Clearly, all the Dehn twists $D_{\alpha_u^\pm}$ fix each arc $\{\beta_v\}$.

Recall that the set of punctures P_2 in M contains some points in T. For each component T_j of T we pick a disk $U_j \subset int(T_j)$ containing $T_j \cap P_2$ and not containing the point q. Thus, each U_j contains at least two points of P_2. Set $R := A \cup T \setminus int(U)$. Lastly, cut R open along the arcs β_{vw} defined above and let S denote the resulting surface. See Figure 2.

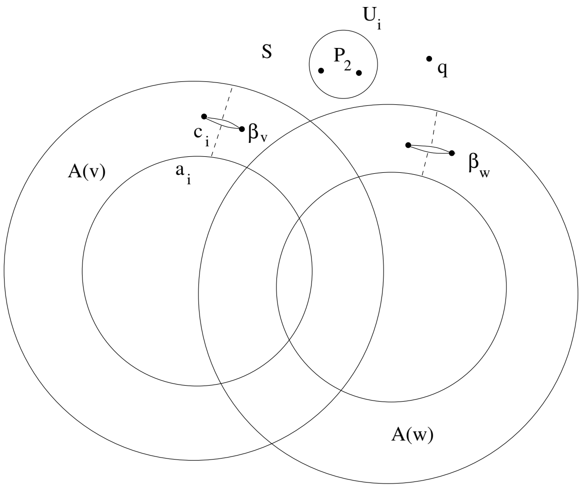

Figure 2: In this example, S is the sphere with 6 holes.

We orient the loops $\alpha_{v^\pm}$ in an arbitrary fashion. We claim that the system $\mathcal{A}$ of curves $\alpha_{v^\pm}$ in S is linearly independent in $H_1(S)$. Indeed, for each loop $a_i := \alpha_{v^\pm}$ there exists a properly embedded arc $c_i \subset S$ which intersects a_i in exactly one point, the end-point of c_i, and intersects the boundary circle corresponding to β_v at another end-point. Moreover, the arc c_i is disjoint from all arcs in $\mathcal{A}$ different from a_i, see Figure 2. The relative cycles $[c_i] \in H_1(S, \partial S) \cong H^1(S, \partial S)$ are then Poincaré dual to $[a_i] \in H_1(S, \partial S)$, which implies linear independence of $\mathcal{A}$ in $H_1(S)$.

Lastly, we observe that the surface M' is obtained from S by attaching punctured disks with at least two punctures each and that the point $q \in P$ belongs to S. Therefore, by Theorem 3.3, the homomorphism

$$\bar{\phi}_N : G_{D\Gamma} \to Map(S \setminus \{q\}) \hookrightarrow Map(M')$$

is injective. We conclude that ψ_N is injective as well. $\square$

Remark 3.5. *As an alternative to the above argument, one could use the results of [K], which, however, do not provide an explicit estimate on N.*

This finishes the proof of Theorem 1.4. $\square$

4 Lifting to the universal cover

We continue with the notation introduced in the previous section, where N is any integer ≥ 2. Our next goal is to lift the double Dehn twists f_v to the universal cover of the surface M. Without loss of generality, we may assume that M has genus ≥ 2.

Recall that the group of outer automorphisms, $Out(\Pi)$, of a group Π is the quotient $Aut(\Pi)/Inn(\Pi)$, where $Inn(\Pi)$ consists of inner automorphisms of Π. By Nielsen's theorem, if $\Pi = \pi_1(M)$, where M is a closed surface, then $Map(S) \cong Inn(\Pi)$. If M is not a closed surface, then the natural homomorphism $Map(M) \to Out(\Pi)$ (given by the action of homeomorphisms on the fundamental group) is injective but is not (in general) surjective.

Lemma 4.1. *Let $\Lambda \triangleleft \Pi$ be a nontrivial normal subgroup of a hyperbolic surface group Π. Let $[g] \in Out(\Pi)$ be an element stabilizing Λ. Then g induces a nontrivial element of $Out(\Lambda)$.*

Proof. Suppose that g induces the trivial automorphism of Λ, i.e., it commutes with all elements of Λ. We realize Π and Λ as Fuchsian subgroups of $PSL(2,\mathbb{R})$ acting on the unit circle. Since Λ is normal in Π and $\Lambda \neq 1$, it follows that the limit set of Λ is the entire circle S^1. The automorphism $g \in Aut(\Pi)$ is induced by a homeomorphism $h : S^1 \to S^1$. Since g commutes with all elements of Λ, and the limit set of Λ is S^1, it follows that h fixes S^1 pointwise. However, then g is a trivial automorphism of Π as well. $\square$

We now let $p : \tilde{M} \to M$ be the universal cover. We equip M with hyperbolic structure, whose area form is ω. Then $\tilde{M}$ is naturally identified with the hyperbolic plane, which we realize as the open unit disk D in the 2-sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and $\Pi = \pi_1(S)$ acts on D as a subgroup of $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$. We let $D' \subset D$ denote $p^{-1}(M') \subset D$, then D' is a disk with infinitely many punctures at the points of $P' := p^{-1}(P)$. The group $\Lambda = \pi_1(D')$ is a (nontrivial) normal subgroup in Π.

Recall that we have a homomorphism $\psi : G_\Gamma \to Ham(M,\omega)$, so that ψ_N is injective and $\psi_N(g_v) = f_v^N$, the N-times iterated double Dehn twists supported on the annulus $A(v) \subset M$. Moreover, ψ_N projects to an injective homomorphism $G_\Gamma \to Map(M')$. Since G_Γ maps trivially to $Map(M)$, it follows that it preserves the normal subgroup $\Lambda \triangleleft \Pi$. Therefore, by Lemma 4.1, for every $g \in G_\Gamma$, any lift of $\psi_N(g)$ to D is *not* isotopic to the identity as a map $D' \to D'$.

A diffeomorphism $f : M \to M$ does not have a unique lift to a homeomorphism $\tilde{M} \to \tilde{M}$ (lifts differ by a post-composition with elements of Π). However, the subgroup $Diff_0(M)$ consisting of diffeomorphisms isotopic to the identity has an isomorphic lift

$$\iota : Diff_0(M) \to Diff(D),$$

namely, lift the isotopy $F : M \times [0,1]$ from $f \in Diff_0(M)$ to id_M to an isotopy $\tilde{F} : [0,1] \times D \to D$ which ends at id_D. Then $\iota(f) = F(\cdot, 0)$.

Thus, each $\psi(g_v)$ has a canonical lift $\iota(\psi(g_v))$ which preserves each annulus in $p^{-1}(A(v))$. Clearly, this lift is an infinite product of commuting double Dehn twists supported in the (annular) connected components of $p^{-1}(A(v))$. The map $\iota(\psi(g_v))$ is the Hamiltonian symplectomorphism of $(D, \tilde{\omega})$ with respect to the function $D \to \mathbb{R}$ which is the lift of $H_{A(v)}$ to D. It is then clear that the map

$$g_v \mapsto \tilde{\Psi}(g_v) = \iota \circ \Psi(g_v)$$

determines a homomorphism $\tilde{\Psi} : G_\Gamma \to Ham(D, \tilde{\omega})$, where $\tilde{\omega}$ is the lift of ω to D. Since ω can be taken to be the hyperbolic area form on M,

$$\tilde{\omega} = \frac{dx \wedge dy}{(1 - (x^2 + y^2))^2}.$$

In view of the above observations, the homomorphism $\tilde{\Psi}$ is injective and, moreover, by Lemma 4.1, it projects to an injective homomorphism

$$G_\Gamma \to Map(D').$$

The problem, however, is that the symplectic structure $\tilde{\omega}$ does not extend to a symplectic structure on the entire sphere S^2. We, therefore, have to replace it with a symplectic structure ω_0 on S^2, which restricts to the Euclidean area form on an open neighborhood of the closure of D. Our next goal is to replace $\tilde{\Psi}(g_v)$ with another infinite product of commuting double Dehn twists which are Hamiltonian with respect to ω_0. Of course, this will also require correcting the functions on D which are lifts of the functions $H_{A(v)}$.

For each annulus $A(v) \subset M$ we fix its (homeomorphic) lift in D which we will again denote $A(v) \subset D$. For each $\sigma \in \Pi$ we let $\omega_{v,\sigma}$ be the symplectic form on $A(v)$ defined by taking pull-back of ω_0 via

$$\sigma : A(v) \to \sigma(A(v))$$

and then rescaling by some $\lambda_{v,\sigma}^{-2} \in \mathbb{R}_+$, so that

$$\int_{A(v)} \omega_{v,\sigma} = 1.$$

Clearly,

$$\lim_{\ell(\sigma)\to\infty} \lambda_{v,\sigma}^2 = 0,$$

where ℓ is a word metric on Π. The constants $\lambda_{v,\sigma}$ are $\asymp |\sigma'(z)|, z \in A(v)$, where

$$a \asymp b \iff a = O(b) \text{ and } b = O(a).$$

Lemma 4.2. *The forms $\omega_{v,\sigma}$ form a precompact set in C^∞ topology.*

Proof. Since $\sigma \in PSL(2, \mathbb{C})$, we have

$$\omega_{v,\sigma} = |\nu_{v,\sigma}(z)|\omega_0$$

where $\nu_{v,\sigma}(z) = \lambda_{v,\sigma}^{-2}\sigma'(z)^2$ is a function holomorphic on $A(v)$ and having unit L_1-norm. Such functions C^∞ subconverge to a holomorphic function on a neighborhood of the closed annulus $A(v)$.

Remark 4.3. *One can make the above argument more explicit as follows. Each linear-fractional transformation $\sigma = \sigma(z)$ is of the form*

$$\sigma(z) = e^{i\theta}\frac{z-a}{-\bar{a}z+1}, \quad |a| < 1, a = a_\sigma = -e^{-\theta}\sigma(0),$$

hence,

$$\sigma'(z) = \frac{(1-|a|^2)e^{i\theta}}{(-\bar{a}z+1)^2}$$

and

$$\lambda_{v,\sigma} = Const_{v,\sigma}|\sigma'(0)|^2,$$

where $Const_{v,\sigma} > 0$ is bounded away from 0 and ∞. From this, it is easy to see that the functions $\nu_{v,\sigma}(z)$ subconverge (uniformly on compacts in D) to a function of the form

$$Const \cdot \frac{1}{(1-e^{it}z)^4}, \quad t \in \mathbb{R},$$

where e^{-it} is the limit of a convergent subsequence of (a_σ).

□

We retain the notation C_v for the circle in $A(v) \subset D$ which covers the central circle $C_v \subset A(v) \subset M$. The circle C_v divides the hyperbolic area of the annulus $A(v)$ in half but this need not be the case with respect to the form $\omega_{v,\sigma}$. Nevertheless, by the above compactness lemma in conjunction with Moser's lemma, we can choose a C^∞ precompact family of area-preserving diffeomorphisms

$$(A(v), \omega_{v,\sigma}) \to A = A_{1/2} = S^1 \times [-1/2, 1/2]$$

which carry the circle C_v to the round circles $S^1 \times b_{v,\sigma} \subset A$, where $b_{v,\sigma}$ form a precompact subset of the open annulus $S^1 \times (-1/2, 1/2)$. We now repeat the construction of Hamiltonian Double Dehn twists on the annulus, except we will insist on having a rotation by 2π along the circles $S^1 \times b_{v,\sigma}$ instead of $S^1 \times 0$. To this end, we will be using hamiltonians $\hat{H}_{v,\sigma} : A \to \mathbb{R}$ so that $\hat{H}_{v,\sigma}(s,t) = h(t)$, $h'(b_{v,\sigma}) = 2\pi$. Pull-back these functions to the annuli $(A(v), \omega_{v,\sigma})$. We obtain a C^∞-precompact family of functions on $A(v)$. The corresponding Double Dehn twists on $A(v)$ will rotate C_v by 2π.

Note however that $\sigma^*(\omega_0) = \lambda^2_{v,\sigma}\omega_{v,\sigma}$ and, hence, we cannot use the above hamiltonians to define Double Dehn twists with respect to the forms $\sigma^*(\omega_0)$ since the resulting time-1 maps would rotate C_v by $2\pi\lambda^{-2}_{v,\sigma}$. Therefore, the correct family of functions $\tilde{H}_{v,\sigma} : A(v) \to \mathbb{R}$ is given by the pull-back of

$$\lambda^2_{v,\sigma}\hat{H}_{v,\sigma}$$

via the symplectomorphisms $(A(v), \omega_{v,\sigma}) \to A$. Clearly, the functions $\tilde{H}_{v,\sigma}$ converge to zero in C^∞ topology on the annulus $A(v)$.

We now define the function $H_{v,\sigma} : \sigma(A(v)) \to \mathbb{R}$ by $\tilde{H}_{v,\sigma} \circ \sigma^{-1}$. Every such function defines a Hamiltonian Double Dehn twist $f_{\sigma(A(v))}$ on the annulus $\sigma(A(v))$ which is clearly isotopic (rel. boundary and the punctures on C_v) to the lift of the Double Dehn twist $\iota\psi(g_v) = \tilde{\psi}(g_v))$.

We define $H_v : S^2 \to \mathbb{R}$ by

$$H_v|_{\sigma(A(v))} = H_{v,\sigma}, \quad H_v(z) = 0 \ \text{ for } \quad z \in S^2 \setminus \bigcup_{\sigma\in\Pi} \sigma(A(v).$$

Accordingly, we extend the maps $f_{\sigma(A(v))}$ by the identity on the complement of $\sigma(A(v))$ in S^2 and use the notation f_v for the product of the resulting commuting double Dehn twists:

$$f_v = \prod_{\sigma\in\Pi} f_{\sigma(A(v))}.$$

Since

$$\lim_{\ell(\sigma)\to\infty} diam(\sigma(A(v))) = 0,$$

it is clear that $f_v : S^2 \to S^2$ is a homeomorphism. Since

$$\lim_{\ell(\sigma)\to\infty} \|H_{v,\sigma}\|_{C^0} = 0,$$

it is clear that $H_v : S^2 \to \mathbb{R}$ is continuous. Clearly, $f_v|D$ is smooth and is the time-1 map of $H_v|D$ with respect to ω_0. Moreover, $[f_u, f_v] = 1$ provided that $[uv] \in E(\Gamma)$, since the support sets of the maps $f_u, f_v : D \to D$ are disjoint.

We thus obtain a homomorphism

$$\rho_0 : G_\Gamma \to Ham(D, \omega_0), \quad \rho_0(g_v) = f_v^N.$$

Since each f_v is isotopic to $\iota(\psi(g_v))$ on D', it follows that the homomorphisms

$$\tilde{\Psi}, \rho_0 : G_\Gamma \to Diff(D')$$

have the same projection to $Map(D')$. Since the projection of $\tilde{\Psi}$ to $Map(D')$ was 1-1, it follows that ρ_0 also projects injectively. In particular, ρ_0 is 1-1 as well.

Our next goal is to analyze smoothness of the functions H_v and maps $\rho_0(g_v)$. The following lemma (and its corollary) is not need for the proof of Theorem 1.1 and we include the proof only for the sake of completeness and as a warm-up for the proof of Lemma 4.7 which will play an important role in smoothing the functions H_v.

Lemma 4.4. *For every $v \in V(\Gamma)$, $H = H_v : S^2 \to \mathbb{R}$ is $C^{1,1}$-smooth, i.e., it has Lipschitz differential.*

Proof. We only have to verify smoothness on the boundary circle S^1 of the unit disk D. The function H on the annulus $A_{v,\sigma}$ equals $\lambda^2_{v,\sigma}\hat{H}_{v,\sigma}\circ\sigma^{-1}$. Since the derivative of σ on $A(v)$ is of the order of $\lambda_{v,\sigma}$, we conclude that $dH|_{\sigma(A(v)}$ converges uniformly to zero as $\ell(\sigma) \to \infty$. Moreover, the second derivatives of H are uniformly bounded (by the upper bound on the C^2-norm of $\hat{H}_{v,\sigma}$). It remains to check that dH vanishes at the boundary points $\xi \in S^1$ of the unit disk. Observe that the distance from the annulus $A_{v,\sigma}$ to S^1 is $\asymp \lambda_{v,\sigma}$. Therefore, if $d(\xi, z) = R$, where $z \in A_{v,\sigma}$, then $R \geq C_1 \lambda_{v,\sigma}$; this implies that

$$\frac{H(z)}{R} \leq C_1 C_2 \lambda_{v,\sigma}^{-1}$$

where

$$\|\hat{H}_{v,\sigma}\|_{C^0} \leq C_2.$$

It follows that H has vanishing derivative on S^1. The statement that dH is Lipschitz on the closed disk follows from the above bound on the 2-nd derivative. □

Corollary 4.5. *For each $g \in G_\Gamma$, $\rho(g)$ is Lipschitz on S^2.*

Corollary 4.6. *The homomorphism $\rho : G_\Gamma \to Homeo(S^2)$ is injective. Its image consists of bilipschitz symplectomorphisms of (S^2, ω_0) which are Hamiltonian with respect to $C^{1,1}$ functions on S^2.*

This proves a vesrion of Theorem 1.1 but with very low regularity of symplectomorphisms of S^2. Our goal is to replace these bilipschitz symplectomorphisms with infinitely differentiable ones while preserving injectivity of the homomorphism $G_\Gamma \to Ham$. In order to do so, we will need an estimate on the growth of partial derivatives of the functions H_v at the unit circle.

Lemma 4.7. *For every $v \in V(\Gamma)$ and each $n = k + m$, the function $H = H_v$ satisfies*

$$\left| \frac{\partial^n}{\partial z^m \partial \bar{z}^k} H(z) \right| = O(r^{-(n-2)})$$

where $r = 1 - |z|$. In particular, all n-th order derivatives of H blow up at S^1 at most polynomially fast.

Proof. The proof repeats the argument in the above lemma. Suppose that $z \in A_{v,\sigma}$. Set $w = \sigma^{-1}(z) \in A(v)$. Then $H(z) = \lambda_{v,\sigma}^2 \hat{H}_{v,\sigma} \circ \sigma^{-1}(z)$, where

$$\lambda_{v,\sigma}^{-1} \asymp \left| \frac{d}{dz} \sigma^{-1}(z) \right|.$$

We have

$$\frac{\partial^k}{\partial \bar{z}^k} H(z) = \lambda_{v,\sigma}^2 \frac{\partial^k}{\partial \bar{z}^k} \hat{H}_{v,\sigma}(w)$$

are uniformly bounded (with respect to σ) for each k. On the other hand, since all derivatives of orders $\leq k$ of the functions $\hat{H}_{v,\sigma}$ are uniformly bounded in σ, we have

$$\left| \frac{\partial^m}{\partial z^m} H(z) \right| \asymp \left| \lambda_{v,\sigma}^2 \right| \cdot \left| \frac{\partial^m}{\partial z^m} \sigma(w) \right|^{-1}.$$

We then observe that

$$\left| \frac{\partial^m}{\partial z^m} \sigma(w) \right| \asymp \lambda_{v,\sigma}^m \asymp r^m.$$

Lemma follow. $\square$

It is clear however that the above calculations cannot get better than $C^{1,1}$-smoothness for the function H_v. In order to embed G_Γ in $Ham(S^2)$ which consists of smooth Hamiltonian diffeomorphisms, we will use an approximation argument.

5 Approximation

The mollifiers. We define a family of C^∞ functions $\eta_\epsilon(z), \epsilon > 0$ (the mollifiers) on S^2 so that:

- For every $\epsilon > 0, z \notin D$, $\eta_\epsilon(z) = 0$. Moreover, $\eta_\epsilon(z)$ (and its derivatives of all orders) vanishes exponentially fast on $S^1 = \partial D$.
- For every fixed $z \in D$ the functions
$$\epsilon \mapsto \eta_\epsilon(z), \quad \epsilon \mapsto d\eta_\epsilon(z)$$
are real-analytic.
- $$\lim_{\epsilon \to \infty} \eta_\epsilon(z) = 1$$
in C^∞ topology uniformly on compacts in D.

Explicitly, one can take

$$\eta_\epsilon(z) = \varphi_\epsilon(|z|), \quad |z| < 1,$$
$$\eta_\epsilon(z) = 0, \quad |z| \geq 1.$$

where $\varphi_\epsilon(x)$ is the composition of

$$\exp(-\epsilon y^2)$$

and

$$y = \tan\left(\frac{\pi x}{2}\right).$$

Now, set $H_v^{(\epsilon)} := \eta_\epsilon H_v$. Since derivatives of all orders of H_v blow up on S^1 at most polynomially fast (Lemma 4.7), it follows that the functions $H_v^{(\epsilon)}$ are C^∞ on S^2. Clearly, for $[uv] \in E(\Gamma)$, the supports of $H_u^{(\epsilon)}, H_v^{(\epsilon)}$ in D are disjoint. Moreover, the functions $H_v^{(\epsilon)}$ (and their derivatives) depend analytically on ϵ and converge to H_v in C^∞-topology uniformly on compacts in the open disk D. Therefore, the corresponding time-N Hamiltonian maps $\rho_\epsilon(g_v) := f^N_{v,\epsilon}$ (with respect to the form ω_0) converge to $\rho_0(g_v)$ as well (uniformly on compacts in D). Moreover, for each v and z, the function

$$\epsilon \mapsto f^N_{v,\epsilon}$$

is real-analytic, for $\epsilon > 0$.

We therefore obtain a family of representations $\rho_\epsilon : G_\Gamma \to Ham(S^2)$ which send the generators g_v to $\rho_\epsilon(g_v)$ as above.

Lemma 5.1. *For all but countably many ϵ, the representations ρ_ϵ are faithful.*

Proof. For a fixed $g \in G_\Gamma \setminus 1$ the set E_g of $\epsilon > 0$ for which $g \in Ker(\rho_\epsilon)$ is either countable or the entire $\mathbb{R}_+$ (since $\rho_\epsilon(g)$ depends real-analytically on ϵ). If all the sets E_g are countable, we are done. Otherwise, there exists $g \in G_\Gamma - 1$ which maps trivially by all ρ_ϵ. Then its limit as $\epsilon \to 0$, $\rho_0(g)$, is also the identity on D. However, it contradicts faithfulness of ρ_0. $\square$

This concludes the proof of Theorem 1.1. $\square$

Higher-dimensional symplectic manifolds.

Corollary 5.2. *Let (M, ω) be a symplectic manifold. Then $Ham(M, \omega)$ contains every RAAG G_Γ.*

Proof. Let $2n = dim(M)$, consider a polydisk $(2D)^n \subset M$, where $2D \subset \mathbb{C}$ is the disk of radius 2, embedded in M so that restriction of the symplectic structure ω on $(2D)^n$ splits as the sum

$$c \cdot \omega_0 \oplus ... \oplus \omega_0$$

where ω_0 is the area form on the j-copy on $2D$ and c is a sufficiently small positive constant. Take a faithful representation $\rho_\epsilon : G_\Gamma \to Ham(2D, \omega_0) \subset Ham(S^2, \omega_0)$ constructed in the proof of Theorem 1.1. Then the group $\rho_\epsilon(G_\Gamma)$ fixes $2D - D$ pointwise. The images of the generators $\rho_\epsilon(g_v)$ are time-N-maps of functions $H_v^{(\epsilon)}$ supported in $cl(D)$. We identify the disk $2D$ with the first factor of the polydisk $(2D)^n$ and extend the functions $H_v^{(\epsilon)}$ to the rest of the factors of the by zero. We extend the resulting function h_v by zero to the rest of the manifold M. We retain the notation h_v for the extension. Then the time-N map $\rho(g_v)$ of h_v is supported in the polydisk D^n and has the form

$$\rho(g_v) := \rho_\epsilon(g_v) \times id ... \times id$$

on this polydisk. It is then clear that $g_v \to \rho(g_v)$ determines a monomorphism $G_\Gamma \to Ham(M, \omega)$, since its composition with the restriction to the first factor of the polydisk D^n is $1 - 1$. $\square$

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