Geometrization Conjecture and the Ricci Flow

Michael Kapovich

September 10, 2003

Contents

Abstract

The goal of this talk is to state Thurston's Geometrization Conjecture for 3-manifolds and outline the Ricci flow approach to this conjecture following Hamilton and Perelman.

1 Geometrization Conjecture for 3-manifolds.

In dimension 3 TOP=PL=DIFF (Moise), i.e. each topological 3-manifold admits a unique PL/smooth structure. Hence throughout I will be working in the category of differentiable manifolds, assuming for simplicity that all 2- and 3-manifolds are orientable.

Loosely speaking, the goal of the Geometrization Conjecture (GC) is to generalize the classification of surfaces by their genus.

Definition 1. *A* **geometry** *is a simply-connected homogeneous unimodular Riemannian manifold X. Unimodularity means that X admits a discrete group of isometries with compact quotient.*

A lá Felix Klein we will be identifying geometry with its group of isometries.

Definition 2. *A compact manifold M is called* **geometric** *if $int(M) = X/\Gamma$ has finite volume, where X is a geometry and Γ is a discrete group of isometries of X acting freely.*

3-dimensional geometries (the first 5 are symmetric spaces):

- $S^3, \mathbb{E}^3, \mathbb{H}^3$, are the constant (sectional) curvature geometries.
- $S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ are the product geometries.
- $Nil, Sol, \widetilde{SL_2}(\mathbb{R})$ are the twisted product geometries.

Note that only the spherical geometry is compact. The hyperbolic geometry is the most interesting one. See [20] for a detailed discussion of these geometries.

Decomposition of 3-manifolds:

Assume that M is closed (compact, no boundary).

Step 1: Connected sum decomposition of M into *prime* pieces (closed manifolds which cannot be decomposed further).

Step 2. If M is prime, consider a *toral* decomposition of M along *incompressible* [1] tori into *simple* pieces (the ones which cannot be decomposed further). Note that simple pieces typically have nonempty toral boundary.

Both decomposition processes terminate (Kneser, Haken: theory of normal surfaces).

Uniqueness of the decompositions: (1) Components of the connected sum decomposition are uniquely determined by M (Milnor). (2) The toral decomposition is unique up to isotopy if we consolidate simple pieces into maximal geometric pieces (Jaco, Shalen; Johannson).

Similar decompositions exist for compact manifolds with boundary.

Thurston's Geometrization Conjecture (GC): *Each prime closed 3-manifold M is either geometric or its simple pieces are geometric.*

A similar conjecture can be stated (and is proven by Thurston!) if M has nonempty boundary.

A restatement of the GC: Each closed prime 3-manifold is either geometric or it splits along disjoint incompressible tori as $M_{thick} \cup M_{thin}$, where M_{thick} is a disjoint union of hyperbolic manifolds, and M_{thin} is a *graph-manifold*, i.e. a manifold obtained by gluing along boundary tori of geometric 3-manifolds which are **not** modeled on $\mathbb{H}^3$.

Graph-manifolds are interesting and well-understood objects, they appear for instance in theory of complex surface singularities. Example of a graph-manifold: let Σ be a surface of genus ≥ 1 with one boundary circle, M_1, M_2 are copies of $\Sigma \times S^1$. Now glue M_1, M_2 along their boundary tori.

Omnibus Theorem (Thurston et al.):

(1) GC is equivalent to the conjunction of PC (Poincare conjecture), SSFC (spherical space form conjecture) and HC (Hyperbolization conjecture).

(2) (Thurston) GC holds if M is prime but not simple.

(3) (Thurston) GC holds for *Haken manifolds*[2]. (For proofs of this theorem, which includes (2) as a special case, see [16], [11].)

(4) If M is (prime) aspherical then GC holds for $M \iff$ GC holds for all manifolds finitely covered by $M \iff$ GC holds for all (prime) manifolds which are homotopy-equivalent to M. (See [7] for the proof in the most difficult case.)

(5) GC holds if $\pi_1(M)$ contains $\mathbb{Z} \times \mathbb{Z}$ or has infinite center. (See [26], [6, 2] for the key parts of the proof of this.)

Explanation:

PC: If M is homotopy-equivalent to the sphere then it is diffeomorphic to the sphere. Equivalently, if M is (closed) simply-connected, then $M = S^3$.

SSFC: If the universal cover of M is the 3-sphere then M admits a metric of (positive) constant curvature, i.e. it is geometric, modeled on S^3.

[1] I.e. π_1-injective.

[2] I.e. M is prime and contains an incompressible surface: a π_1-injective surface which is not S^2.

HC: If M is prime, aspherical (i.e. its universal cover is contractible) and $\pi_1(M)$ does not contain $\mathbb{Z} \times \mathbb{Z}$ then M is hyperbolic.

HC is the most interesting (although, not the most famous) of the 3 parts of the geometrization conjecture.

A confidence-building exercise: GC implies PC. Indeed, suppose that M is closed and simply-connected. Consider connected sum decomposition of M into prime components $M_1, ..., M_k$. Then each M_i is also closed and simply-connected. Since $\pi_1(M_i)$ is trivial, M_i contain no incompressible tori, hence, by GC, M_i is geometric. Since the only compact 3-dimensional geometry is spherical, we conclude that $M_i = S^3$ for each i. Hence $M = S^3$ as well.

A historic remark. The proof of Thurston's hyperbolization theorem for Haken manifolds splits in two cases: (a) Case of manifolds which fiber over the circle, (b) Generic case.

Part of the proof in the case (a) (the *Double Limit Theorem*) was covered in Thurston's unpublished preprint [24], the remaining part of the proof was given by McMullen in [14]. A different (and self-contained) proof of the hyperbolization theorem for manifolds fibering over S^1 was given by Otal in [16].

Parts of the proof in the case (b) were covered by Thurston in his paper [23], his unpublished preprint [25] and his lecture notes [22]. A key part of the argument in the case (b) (the *Bounded Image Theorem*) was proven by McMullen 1989, [13]; a year earlier Morgan and Shalen [15] proved a part of the Bounded Image Theorem. A complete proof in the case (b) is presented in [11].

2 Ricci Flow

The previous section described the status of the GC until November of 2002. In November of 2002 Perelman had posted a preprint [17] which is the first part of the proof of the entire GC. The second Perelman's paper [19] was posted in March of 2003, the third paper [18] was posted in July of 2003. The fourth paper (concerning collapse) should appear some time in the future, although it was essentially covered in the preprint of Shioya and Yamaguchi [21]. The goal of this section is to outline the approach (Ricci flow) to GC used in Perelman's papers.

Ricci flow was introduced by Hamilton in 1982 as a possible approach to GC. I refer the reader to [9], [3], [1], [4] for surveys of the Ricci flow.

We consider a closed 3-manifold M and a smooth family of Riemannian metrics $g(t), g \in [0, T), T \le \infty$, on M. This family is said to be *Ricci flow* if it satisfies the Ricci Flow Equation (RF):

$$g'(t) = -2Ric(g(t)),$$

where $Ric(g(t))$ is the *Ricci tensor* of $g(t)$.

Ricci flow typically does not preserve volume (e.g. it decreases the volume if the $g(0)$ has positive curvature); by rescaling both space and time one gets *Normalized* Ricci Flow Equation (NRF):

$$\widehat{g}'(t) = -2Ric(\widehat{g}(t)) + \frac{2}{3} r \widehat{g}(t).$$

Here $r = r(t)$ a scalar function, which is the *average scalar curvature* of the metric $\widehat{g}$. The metric $\widehat{g}$ has constant volume, this metric is called the *normalized solution* of the Ricci flow.

What is it good for? Suppose for a moment that $\widehat{g}$ is a fixed point of the NRF, then $\widehat{g}'(t) = 0$ and hence the Ricci tensor is a scalar multiple of the metric tensor, i.e. the metric $\widehat{g}$ is *Einsten.* In dimension 3, Einsten metrics are metrics of constant (sectional) curvature, hence M is geometric!

Ricci flow as an analogue of the heat flow. Consider the metric tensor $g_{ij}(x)$ in a normal coordinate ($x = (x^i)$) near zero, then

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ipjq}x^p x^q + O(|x|^3)$$

where R_{ipjq} is the Riemann curvature tensor (and I use the Einstein summation notation). Hence for the usual (Euclidean) Laplacian Δ we have:

$$\Delta g_{ij}(0) = -\frac{1}{3}Ric_{ij}.$$

Thus, up to the higher order terms, the Ricci flow equation is the heat equation

$$g'_{ij}(t) \approx -6\Delta g_{ij}$$

on the space of symmetric 3×3 matrices.

The good news:

1. Short-time existence theorem (Hamilton, 1982, [8]; de Turck, 1983, [5]): There exists $T > 0$ such that given the initial condition $g(0)$ the RF equation has a (unique) solution for $t \in [0, T)$.

2. Positive curvature solutions (Hamilton, 1982, [8]): If $g(0)$ has positive curvature then the RF equation has a solution for $t \in [0, \infty)$, the solution has positive curvature and the normalized solution converges to a constant curvature metric. In particular, M is geometric.

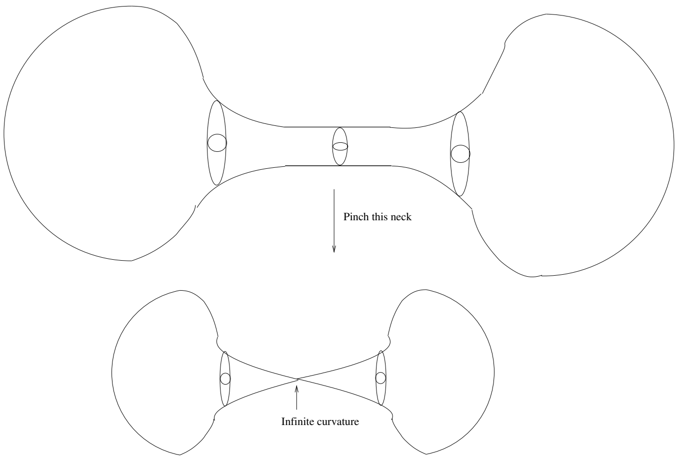

Figure 1: *Neck pinching.*

3. Geometric decomposition at infinity (Hamilton, 1999, [10]): Suppose that RF equation has a solution for all $t > 0$ and the curvature tensor Rm of the normalized solution $\widehat{g}$ has operator norm bounded by some time-independent constant: $|Rm(t)| \leq Const$ for all t. Then GC holds for M. Moreover, as $t \to \infty$, $(M, \widehat{g}(t))$ splits as $M_{thick} \cup M_{thin}$ along incompressible tori, where $(M_{thick}, \widehat{g}(t))$ converges to a disjoint union of finite-volume complete hyperbolic manifolds and M_{thin} *collapses* and is homeomorphic to a graph-manifold.

The bad news: If M contains essential spheres then NRF blows up in a finite amount of time.

Example: Take two copies of the round sphere S^3 and connect them by a thin neck. The neck will get pinched (under RF) in a finite time. Figure 1.

Hamilton-Perelman Approach to GC via (unnormalize) Ricci flow:

Figure 2: *Blow up times.*

Part 0. Without loss of generality (via Kneser's theorem) we can assume that the manifold M is irreducible.

Part 1. Show that the forming singularities of the normalized solution at the blow-up times T_i (say, near the first finite blow-up time T_1) are *standard*: "neck" or "cap". See Figure 3.

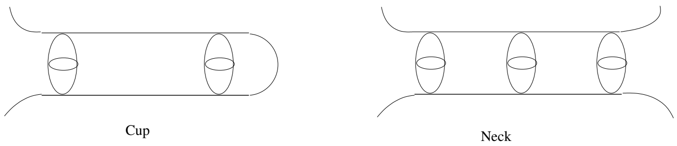

Figure 3: *Standard singularities.*

To identify the location of neck and caps look "around" points of $(M, g(t))$, $t = T_i - \epsilon$, where the norm $|Rm|$ is maximal.
Remark: Hamilton was unable to justify this step of the program, in particular, he was unable to rule out a singularity of the form $S^1\times$ 2-dimensional "cigar soliton" (steady-state) solution of the Ricci flow. Step 1 was done by Perelman in his 1-st paper, [17].

Part 2. Do the *surgery*: Cut out necks and cups from M near T_i and replace them with (carefully chosen) spherical cups of bounded curvature. See Figure 4.

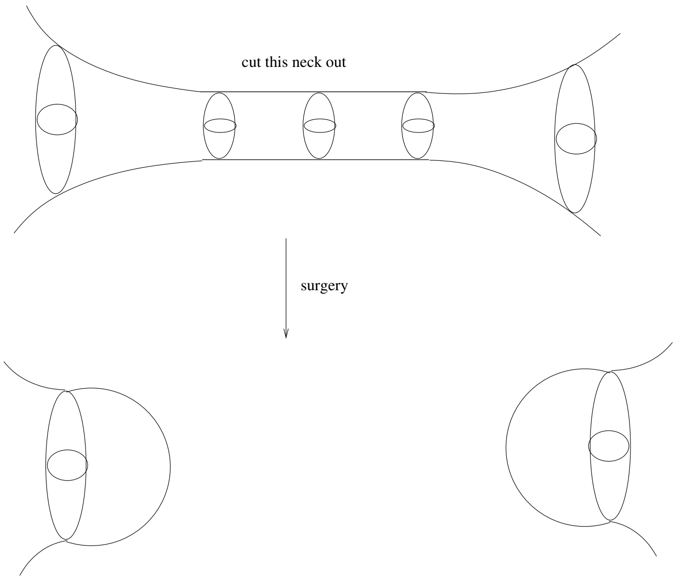

Figure 4: *Surgery.*

In the resulting manifold throw away components of positive sectional curvature; call the result M^{sur} (this manifold is connected by assumption).

Continue the flow on M^{sur}.
Remark. Throwing away components of positive sectional curvature does not change the topology of M (since M was assumed to be irreducible), except in the case $M^{sur} = \emptyset$. In the latter case we necessarily have: M admits a metric of positive sectional curvature and hence is

a spherical space-form (by Hamilton's theorem [8]). In the case $M^{sur} = \emptyset$ the Ricci flow with surgeries is said to become *extinct* in a finite amount of time.

Part 3. Show that there are only finitely many (Hamilton)/ locally finitely many (Perelman)[3] blow-up times T_i.

Part 4. Show that as $t \gg 0$, the manifold M^{sur} splits (along incompressible tori) into $M_{thick} \cup M_{thin}$ where the metrics on the components of the thick part converge to finite volume complete hyperbolic metrics; the thin part collapses as $t \to \infty$.

Part 5. Show that M_{thin} is a graph-manifold.

Part 1 is covered by the 1-st Perelman's paper [17]; parts 2, 3 and 4 are covered by the second paper [17]; part 5 should appear in the fourth paper by Perelman. Part 5 is essentially covered by the preprint of Shioya and Yamaguchi [21]. In their paper Shioya and Yamaguchi cannot handle the case when $\pi_1(M)$ is finite. However in this case the Ricci flow with surgeries becomes extinct in a finite amount of time:

In his third preprint [18] Perelman gives a simplified version of his argument for manifolds with finite fundamental groups. Namely, he shows that after a finite number of surgeries at times $T_1, ..., T_n$, as $t \to T_{n+1}$ the (scalar) curvature of the manifold M^{sur} blows up to $+\infty$ everywhere on M^{sur}. In this case the manifold M^{sur} has positive sectional curvature for $t \approx T_{n+1}$, hence the solution of the Ricci flow becomes extinct at $t = T_{n+1}$ and M is a spherical space-form. This solves the Poincaré conjecture and the spherical space-form conjecture for 3-manifolds. The discussion of collapse is not needed in this case.

I refer the reader to the notes by Kleiner and Lott [12] where they fill in some of the details in 1-st Perelman's paper [17].

3 Beyond the GC.

Assume that GC holds.

Corollary 1. *Suppose that M is aspherical ($\pi_i(M) = 0, i \geq 2$). Then:*

1. The universal cover of M is diffeomorphic to $\mathbb{R}^3$.

2. Homotopy-equivalence $M \approx N$ implies diffeomorphism $M = N$. Thus M is determined by its fundamental group.

Remark. *(1) fails (even in the topological setting) for manifolds of dimension ≥ 4, as was shown by Mike Davis.*

(2) fails for manifolds of dimension ≥ 4. Topological version of this is known in higher dimensions as Borel Conjecture. It was verified by Farrell and Jones in a number of cases.

Corollary 2. *Suppose that M is compact. Then:*

1. All three algorithmic problems for $\pi_1(M)$ (i.e. the word, conjugacy and isomorphism problem in the class of 3-manifold groups) are solvable.

2. The homeomorphism problem (PL setting) is solvable for 3-manifolds.

3. If $\pi_1(M)$ is amenable then it is "elementary amenable" (and, moreover, is virtually solvable).

Note that 1, 2 and 3 fail for 4-manifolds.

Corollary 3. *One can "order" hyperbolic 3-manifolds by their volume: This is analogous to ordering surfaces by their genus.*

[3] In the original version of [17] Perelman was saying that there are only finitely many blow-up times. Later, he modified the claim to local finiteness.

The problem with this "order" is that given v there could be several (but only finitely many) hyperbolic manifolds with the volume v. Also, the set of hyperbolic volumes in $\mathbb{R}_+$ is not discrete, although the appearance of accumulation points is relatively well-understood (hyperbolic Dehn surgery).

Several remaining open problems:

1. Relate topology of a hyperbolizable 3-manifold M with the geometric properties of the hyperbolic metric on M. For instance: Kashaev's conjecture; how to predict Margulis tubes; how to predict arithmeticity of $\pi_1(M)$, etc.

2. Virtual problems, e.g. the virtual $b_1(M) > 0$ problem, i.e. if M is hyperbolic then it admits a finite cover with positive first Betti number. This problem is still open even in the case of arithmetic π_1, although much was proven in the works of Millson, Li, Clozel, Lubotzky and others.

3. $PD(3)$ groups: Show that 3-dimensional Poincare duality groups G are 3-manifold groups. Much of the *Coarse Approach* (see [11, Chapter 20]) to the GC makes sense in this setting, but since we have no smooth structure to speak of, analytic methods are not available. Note however that even "Haken" case (i.e. when G splits as an amalgam) is open, even assuming that G is, say, amenable!

4. Algorithmic aspects: Once GC is known to hold for M, there is a rigorous but extremely inefficient algorithm for constructing geometric structures on the simple pieces of M. The most difficult case here is when M is hyperbolic. On the other hand, there are very efficient algorithms (Weeks; Casson) for constructing hyperbolic structure on M. However there is no theoretical justification for their work (can a combinatorial version of Ricci flow help here?). *Find a (justified and efficient) algorithm for geometrizing 3-manifolds.* The same applies to the recognition problem for 3-manifolds: The fastest (probabilistic) algorithm to tell two 3-manifolds apart is to check that they are hyperbolic, compute their volumes and show that the volumes are different. Can this be converted to a rigorous procedure? (One needs of course more invariants in addition to the volume.)

References

[1] H.-D. Cao and B. Chow, *Recent developments on the Ricci flow*, Bull. Amer. Math. Soc. (N.S.), 36 (1999), pp. 59–74.

[2] A. Casson and D. Jungreis, *Convergence groups and Seifert-fibered 3-manifolds*, Inventiones Mathematicae, 118 (1994), pp. 441–456.

[3] B. Chow, *Ricci flow and Einstein metrics in low dimensions*, in Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, pp. 187–220.

[4] ——, *A survey of hamilton's program for the ricci flow on 3-manifolds*. math.DG/0211266, 2002.

[5] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom., 18 (1983), pp. 157–162.

[6] D. Gabai, *Convergence groups are Fuchsian groups*, Annals of Math., 136 (1992), pp. 447–510.

[7] D. Gabai, R. Meyerhoff, and N. Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Ann. of Math., 157 (2003), pp. 335–431.

[8] R. Hamilton, *Three-manifold with positive Ricci curvature*, Journal of Differential Geometry, 17 (1982), pp. 255–306.

[9] ——, *The formation of singularities in the Ricci flow*, in "Surveys in differential geometry", Vol. II (Cambridge, MA, 1993), Internat. Press, 1995, pp. 7–136.

[10] ——, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom., 7 (1999), pp. 695–729.

[11] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Birkhäuser Boston Inc., Boston, MA, 2001.

[12] B. Kleiner and J. Lott, *Notes of Perelman's papers.* http://www.math.lsa.umich.edu/research/ricciflow/perelman.html, 2003.

[13] C. McMullen, *Iterations on Teichmüller space*, Inventiones Mathematicae, 99 (1989), pp. 425– 454.

[14] ——, *Renormalization and 3-manifolds which fiber over the circle*, vol. 142 of Annals of Mathematics Studies, Princeton University Press, 1996.

[15] J. Morgan and P. Shalen, *Degenerations of hyperbolic structures, III: Actions of 3-manifold groups on trees and Thurston's compactness theorem*, Annals of Math., 127 (1988), pp. 457– 519.

[16] J.-P. Otal, *Le théorème d'hyperbolisation pour les variétès fibrées de dimension 3*, Astérisque 235, Société mathématique de France, 1996.

[17] G. Perelman, *Entropy formula for the ricci flow and its geometric applications.* http://www.arXiv.org/abs/math.DG/0211159, 2002.

[18] ——, *Finite extinction time for the solutions to the ricci flow on certain three-manifolds.* http://www.arXiv.org/abs/math.DG/0307245, 2003.

[19] ——, *Ricci flow with surgery on 3-manifolds.* http://www.arXiv.org/abs/math.DG/0303109, 2003.

[20] P. Scott, *The geometry of* 3*-manifolds*, Bull. of the LMS, 15 (1983), pp. 401–487.

[21] T. Shioya and T. Yamaguchi, *Volume collapsed three-manifolds with a lower curvature bound.* http://www.arXiv.org/abs/math.DG/0304472, 2003.

[22] W. Thurston, *Geometry and topology of* 3*-manifolds.* Princeton University Lecture Notes, 1978–1981.

[23] ——, *Hyperbolic structures on 3-manifolds, I*, Ann. of Math., 124 (1986), pp. 203–246.

[24] ——, *Hyperbolic structures on 3-manifolds, II.* Preprint, 1987.

[25] ——, *Hyperbolic structures on 3-manifolds, III.* Preprint, 1987.

[26] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math., 391 (1988), pp. 1–54.