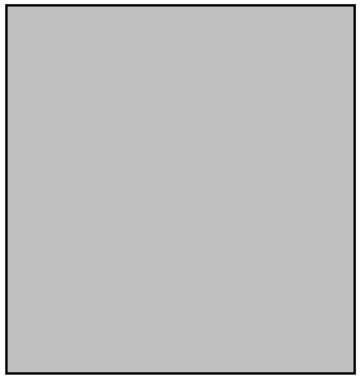
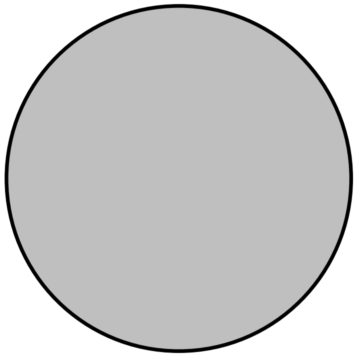
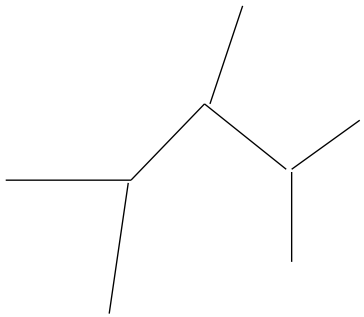
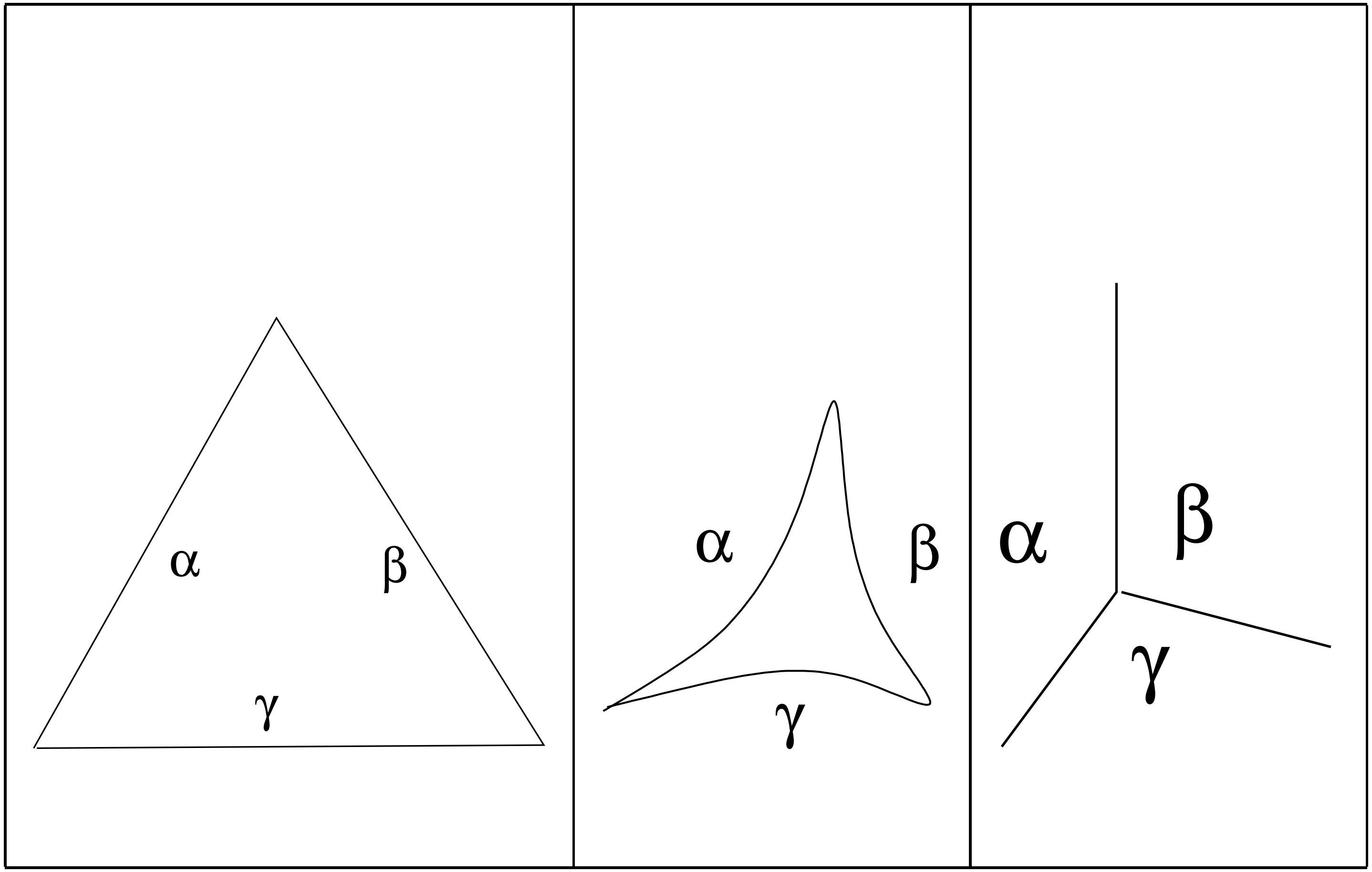
Generalized triangle inequalities and their applications

Misha Kapovich
with Bernhard Leeb and John Millson

GEOMETRY

Rank 1 Spaces:	$\mathbb{R}^2$ 	$\mathbb{H}^2$ 	Tree 
Distances:	$\mathbb{R}_+$	$\mathbb{R}_+$	$\mathbb{R}_+$

Triangles



Triangle inequalities: $\gamma \leq \alpha + \beta$.

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Example 1: $X = Sym_m$, the space of positive-definite symmetric $m \times m$ matrices. Metric at 1 is $\langle A, B \rangle = tr(AB)$. The isometry group is $GL(m, \mathbb{R})$, stabilizer of 1 is $K = O(m)$.

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Example 2: $X = \mathbb{H}^2$, positive-definite symmetric 2×2 matrices of unit determinant.

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Example 2: $X = \mathbb{H}^3$, $X' = \mathbb{R}^3$, $Aut(X') = O(3) \ltimes \mathbb{R}^3$.

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2. Every two points belong to an apartment, an isometrically embedded copy of the Euclidean m-space A. The number m is called the rank of X.
3. The way the apartments are glued together is governed by a finite reflection group W (Weyl group) operating on A isometrically.

Apartments

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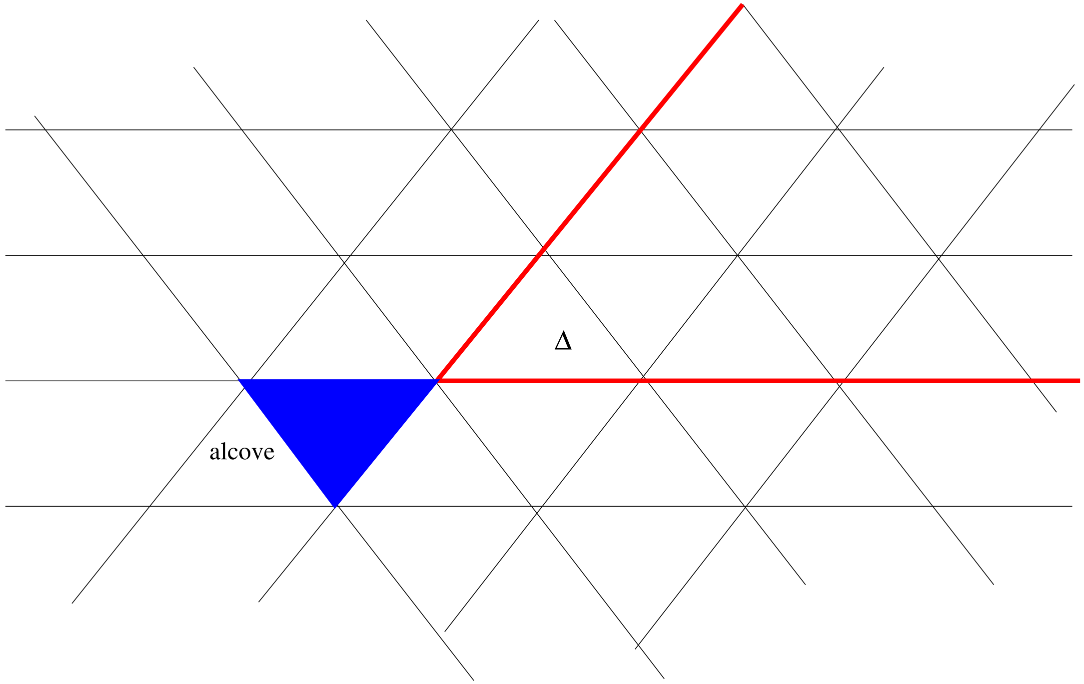

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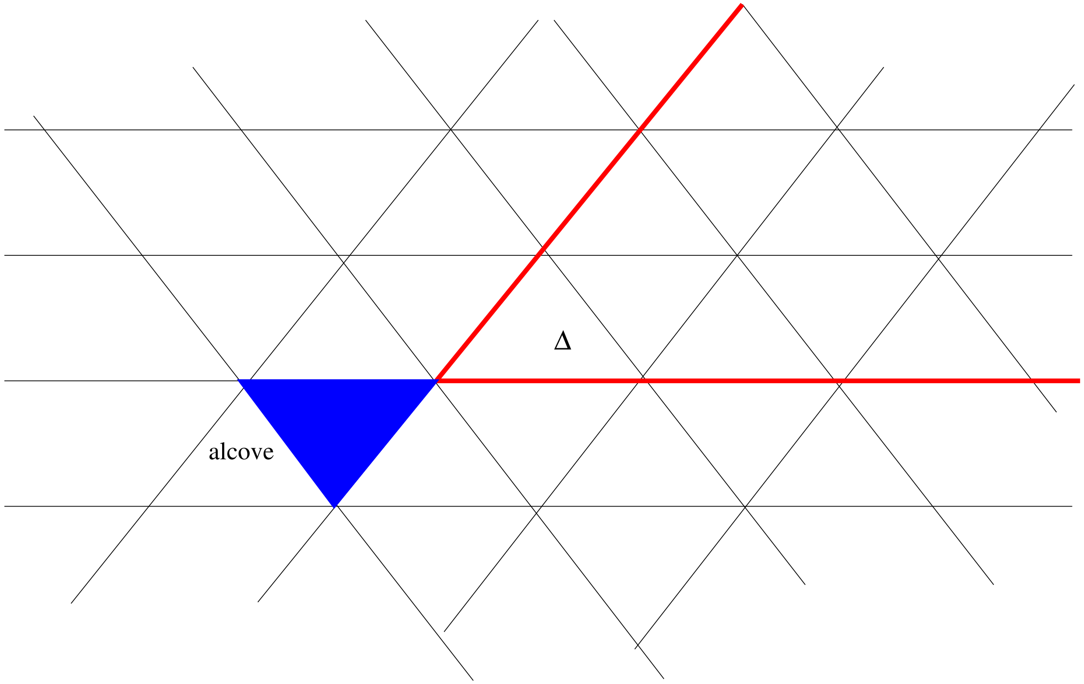
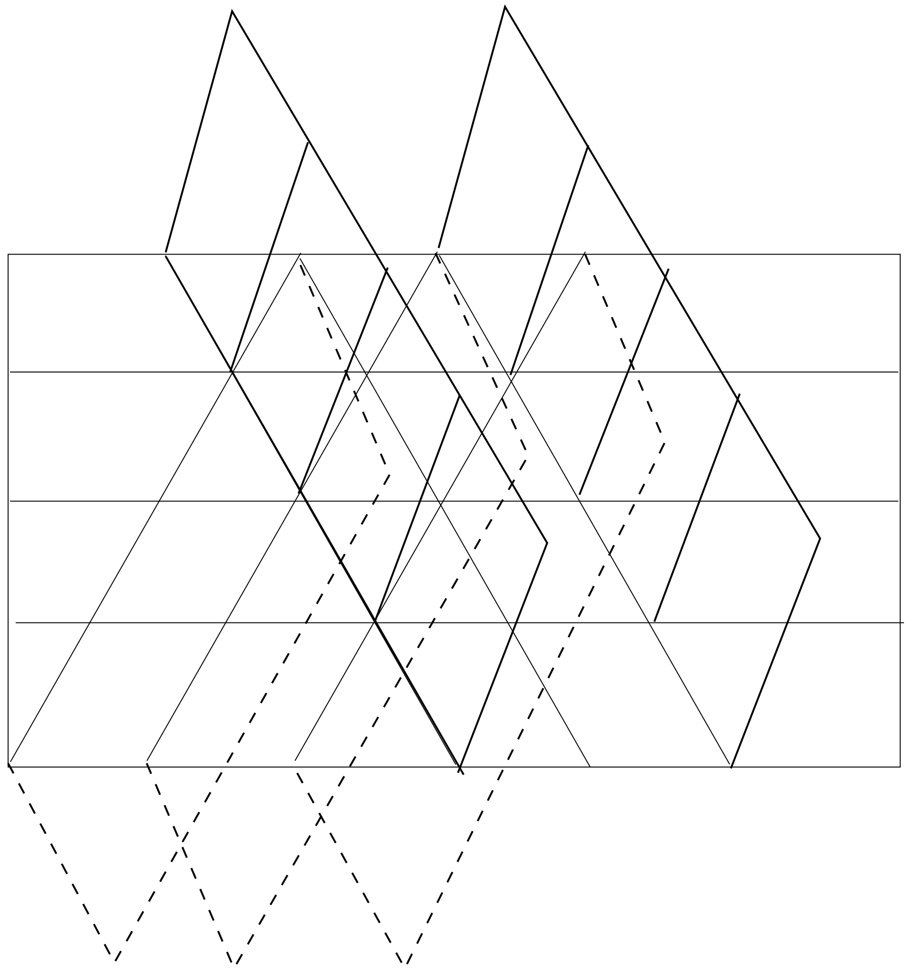


Example: $X = \mathbb{H}^2$ or a tree. Apartments are lines. Chambers are rays. $W = \mathbb{Z}/2$

Buildings

A lame attempt to draw a rank 2 Euclidean building.



Δ-valued distance function

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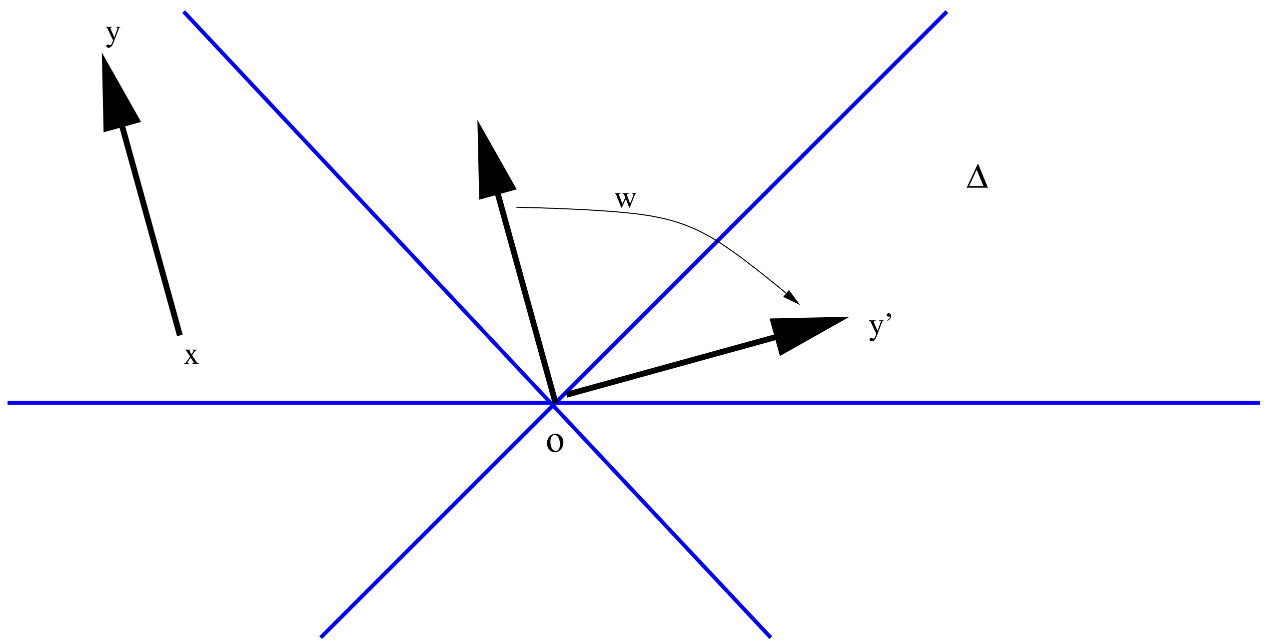

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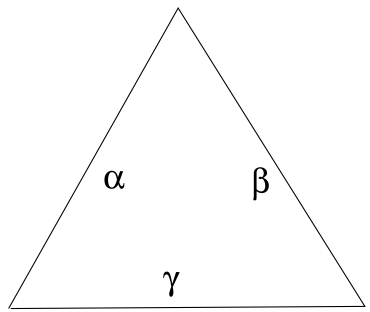
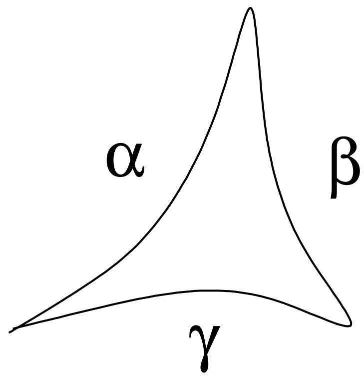
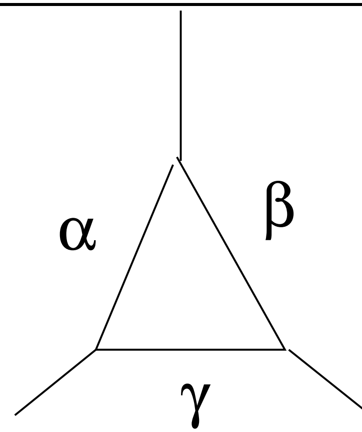
Example 3. $X = Sym_m$, symmetric space. Then $d(1, x)$ is the set of singular values of x arranged in the descending order.

Summary

$Spaces:$	$\mathfrak{p} = T_o X$	$X = G/K$	building
Automorphisms:	$\mathfrak{p} \rtimes K$	G	G_p
Δ-Distances:	Δ	Δ	Δ
Triangles:			

In the case of symmetric and infinitesimal symmetric spaces, $d_\Delta(x, y)$ is a complete congruence invariant of $[x, y]$.

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More generaly:

$$2. \quad w\alpha \leq_{\Delta^*} w\beta + \gamma, \forall w \in W$$

and permutations of α, β, γ. In rank 2 case this is enough.

Solution

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Schubert calculus: computation of the cohomology rings $H^*(G/P, \mathbb{Z})$, where P is a maximal parabolic subgroup.

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P2. Singular values of a product. Give necessary and sufficient conditions on α, β and γ in order that there exist matrices A, B and C in $GL(m, \mathbb{R})$ the logarithms of whose singular values are α, β and γ, respectively, so that

$$ABC = 1.$$

Linear algebra digression

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The matrix $Diag(p^{-\lambda_1}, ..., p^{-\lambda_m})$ is called Smith normal form of A.
The integers $\lambda_1 \geq ... \geq \lambda_m$ are called invariant factors of A. ($-log_p$ of the diagonal entries.)

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Below $\alpha, \beta, \gamma \in \Delta \cap \mathbb{Z}^m$ are dominant weights of $SL(m, \mathbb{C})$:

2 discrete problems

P3. Invariant factors of a product. Give necessary and sufficient conditions on the integer vectors α, β and γ in order that there exist matrices A, B and C in $GL(m, \mathbb{Q}_p)$ with invariant factors α, β and γ, respectively, so that

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$$\iff V_\gamma^* \subset V_\alpha \otimes V_\beta.$$

Generalization to other groups

The above algebra problems make sense for other (reductive) Lie groups G, like $Sp(m), O(m)$. So we have problem $P1, P2, P3, P4$ for groups $G(\mathbb{C}), G(\mathbb{R}), G(\mathbb{Q}_p)$ (e.g. $O(m, \mathbb{C}), Sp(m, \mathbb{Q}_p)$).

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Let $Sol(Pi, G)$ denote the solution set of the problem Pi for the group G.

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So far we have discussed problems, it is time to talk about the answers!

Relation between algebra and geometry

Associate with $g_1, g_2, g_3 \in G$ the polygonal chain in X with the (four) vertices

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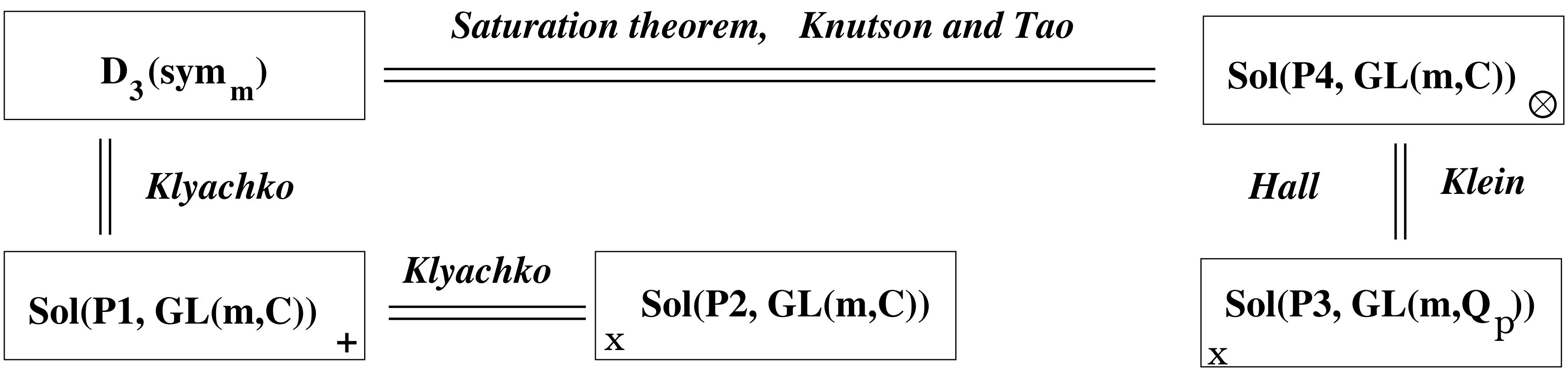
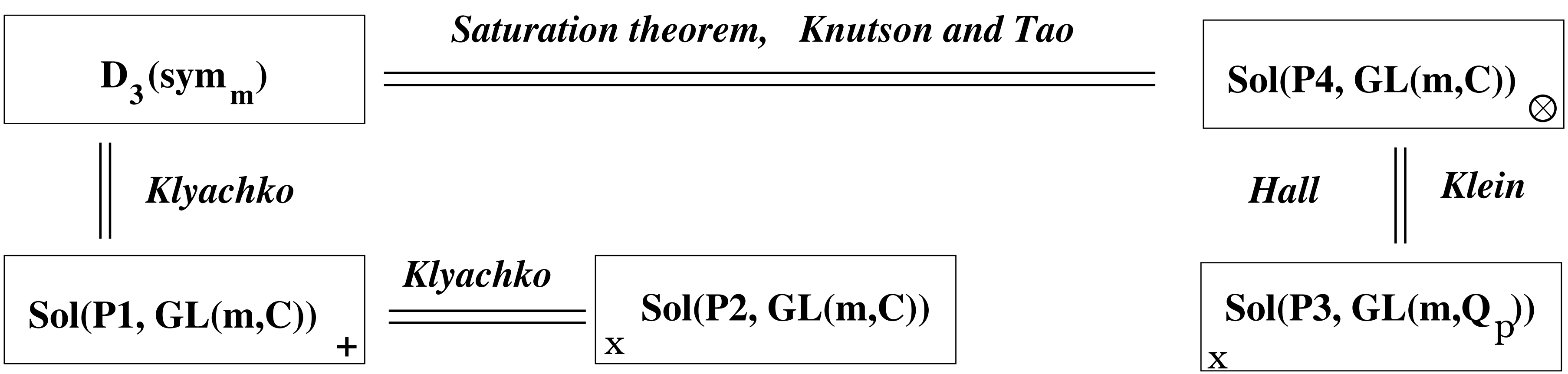
$$D_3(X') = Sol(P1, G),$$

for the infinitesimal symmetric space $X' = T_oX$.

GL(m) case


$D_3(sym_m)$
Saturation theorem, Knutson and Tao
$Sol(P4, GL(m,C))_{\otimes}$
Klyachko
Hall
Klein
$Sol(P1, GL(m,C))_{+}$
Klyachko
$_{x}Sol(P2, GL(m,C))$
$_{x}Sol(P3, GL(m,Q_p))$

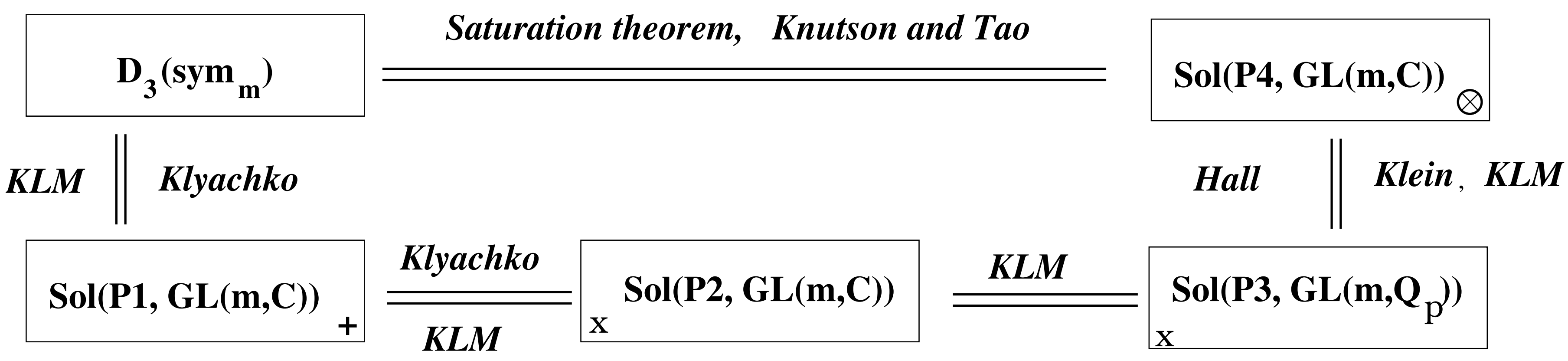
GL(m) case



Recall that $Sol(P3)$, $Sol(P4)$ are contained in certain lattices. Namely:

$$\alpha, \beta, \gamma \in L, \alpha + \beta + \gamma \in Q(R^{\vee}).$$

GL(m) case


$D_3(sym_m)$
Saturation theorem, Knutson and Tao
$Sol(P4, GL(m,C))_{\otimes}$
KLM
Klyachko
Hall
Klein, KLM
$Sol(P1, GL(m,C))_+$
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KLM
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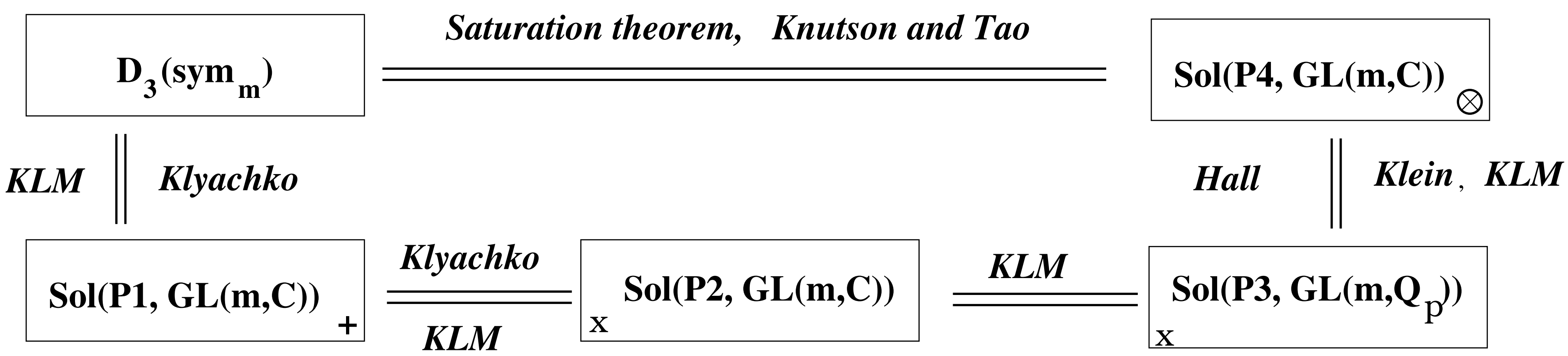
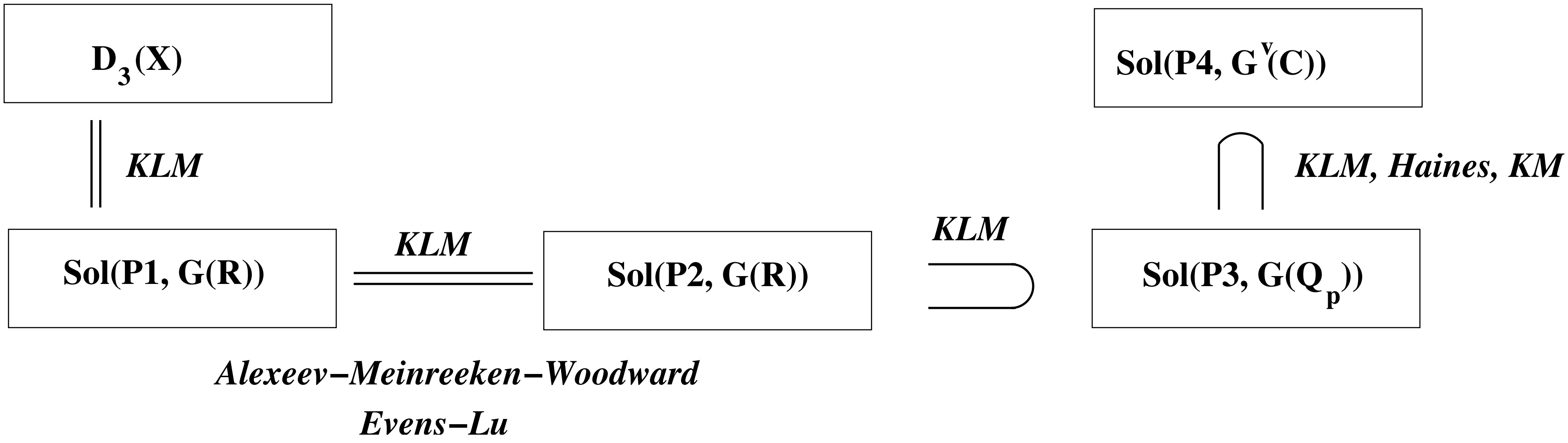
KLM
$Sol(P3, GL(m,Q_p))_x$

GL(m) case

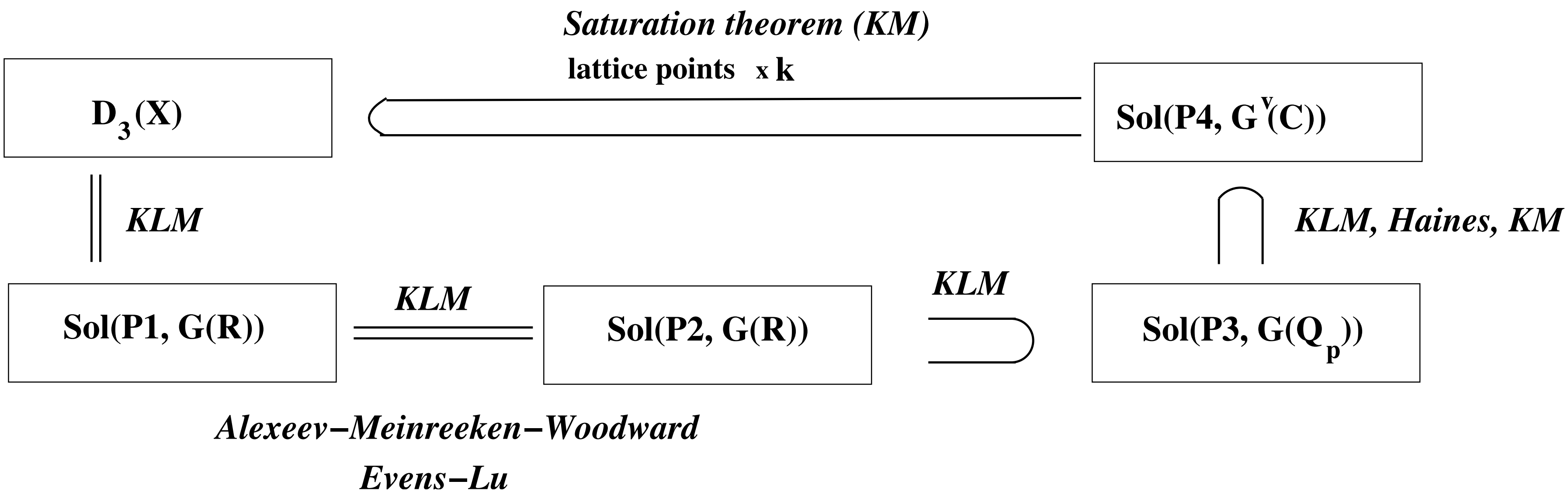


Knutson—Tao, Hall and Klein— combinatorics.
KLM $P4 \subset P3$— Satake correspondence (a kind of Fourier transform).
$P2 = P3$— geometry.

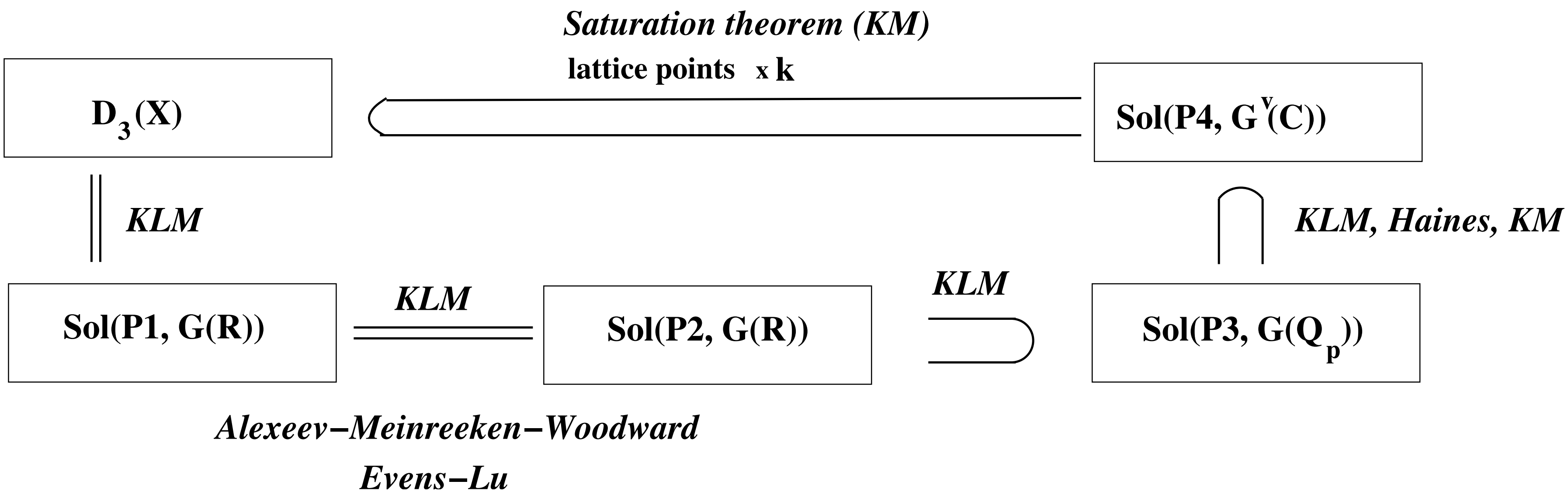
General case



General case

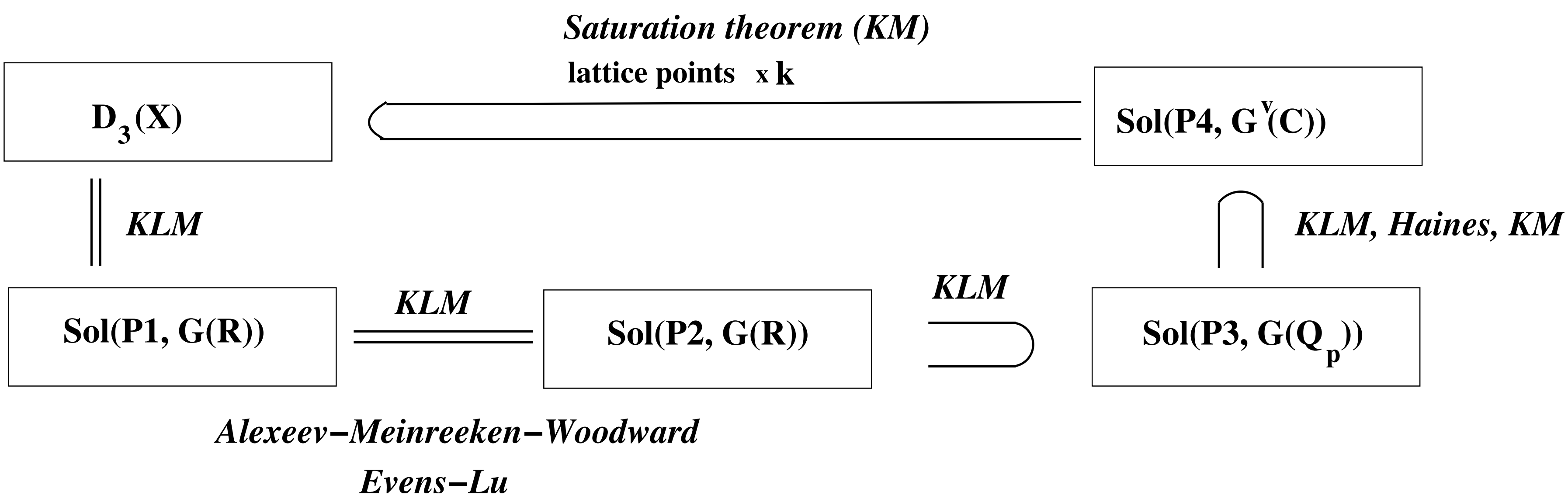


General case



The number k is the *saturation constant*, depends only on the group G. For $GL(m)$, $k = 1$!
The current values of k are described on the next slide.

General case



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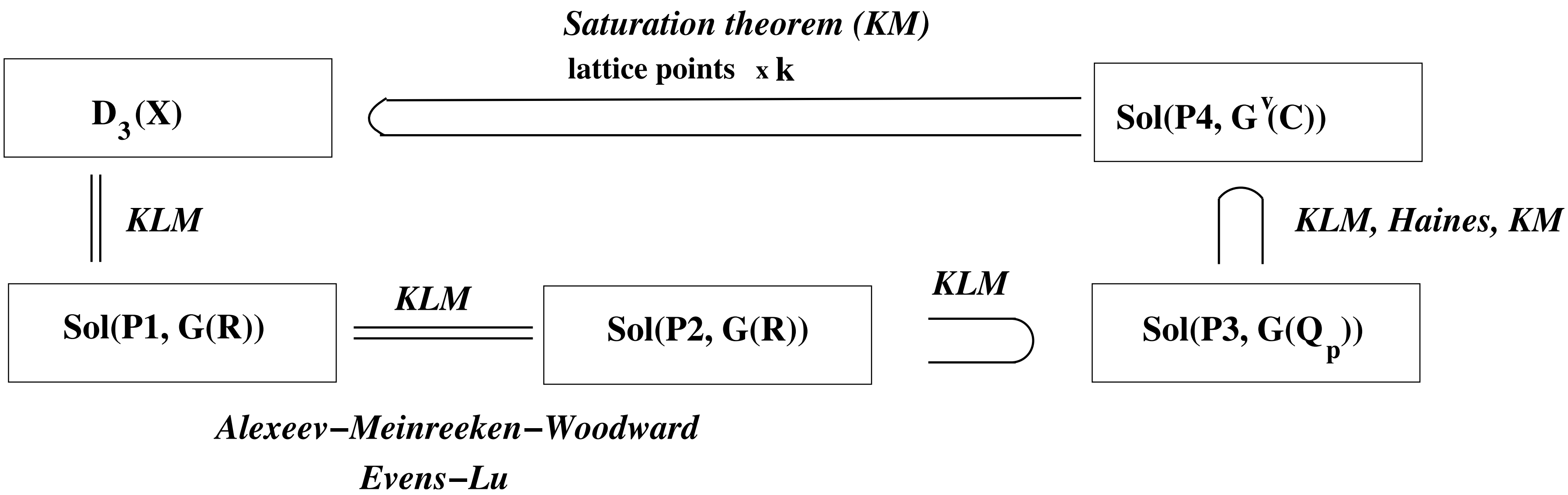
Multiplication by 2 is needed for all non-simply laced groups.

General case



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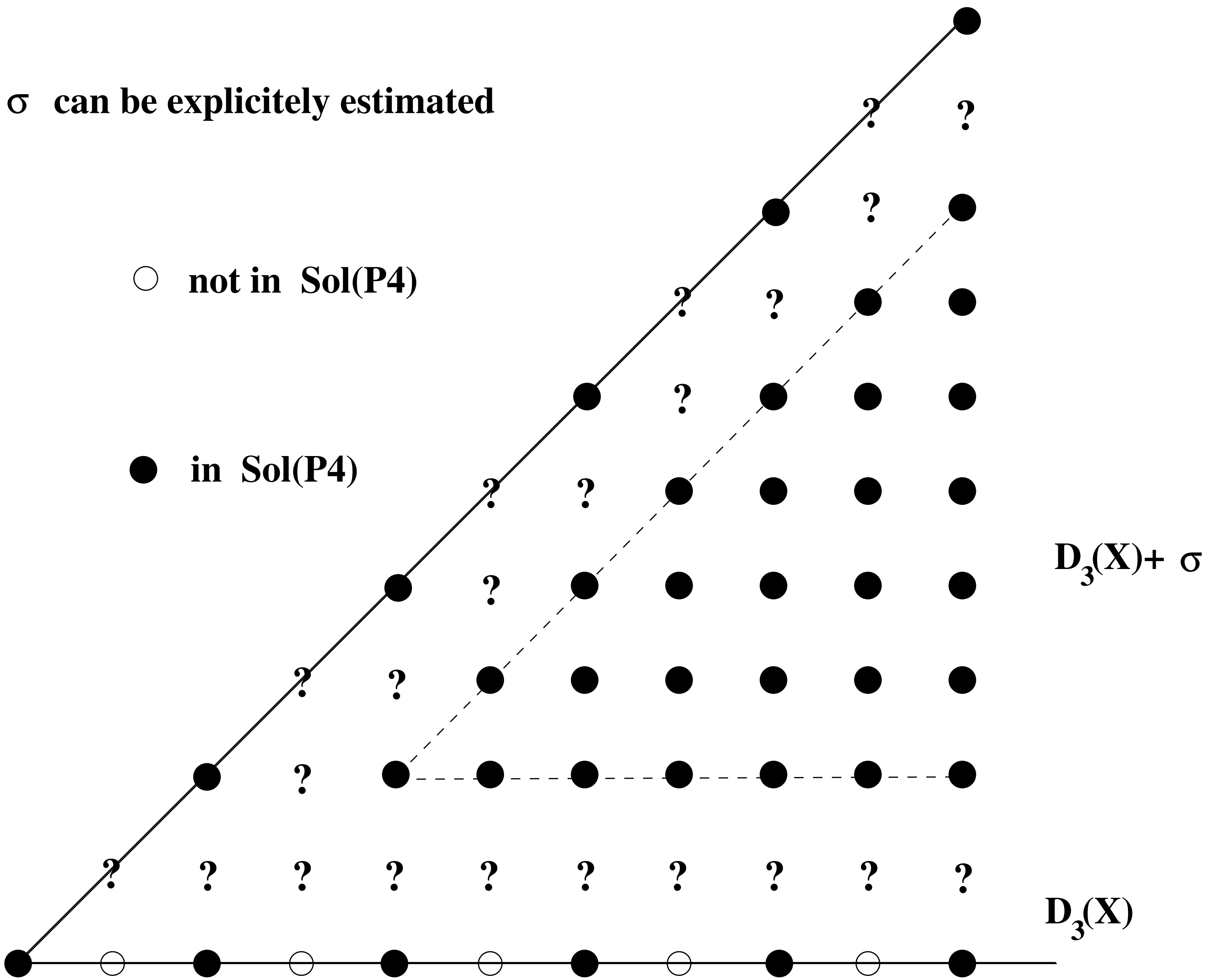
The current values of k are described on the next slide.
Multiplication by 2 is needed for all non-simply laced groups.

Conjecture: One can take $k = 1$ in the simply laced and $k = 2$ in the non-simply laced case.

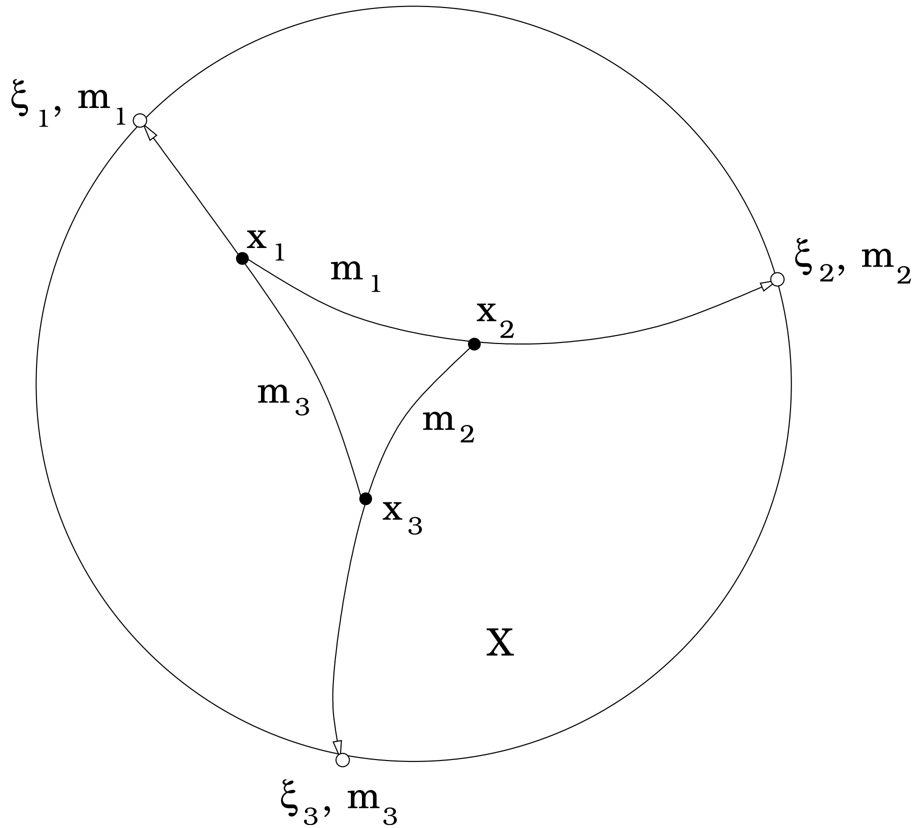
Picture of $Sol(P4)$



Current saturation factors k:

Root system R	Group G	k
A_m	$SL(m+1), GL(m+1)$	1
B_m	$SO(2m+1), Spin(2m+1)$	2
C_m	$Sp(2m), PSp(2m)$	2
D_m	$Spin(2m), SO(2m)$	2
G_2	G	2
F_4	G	12
E_6	G	6
E_7	G	12
E_8	G	60

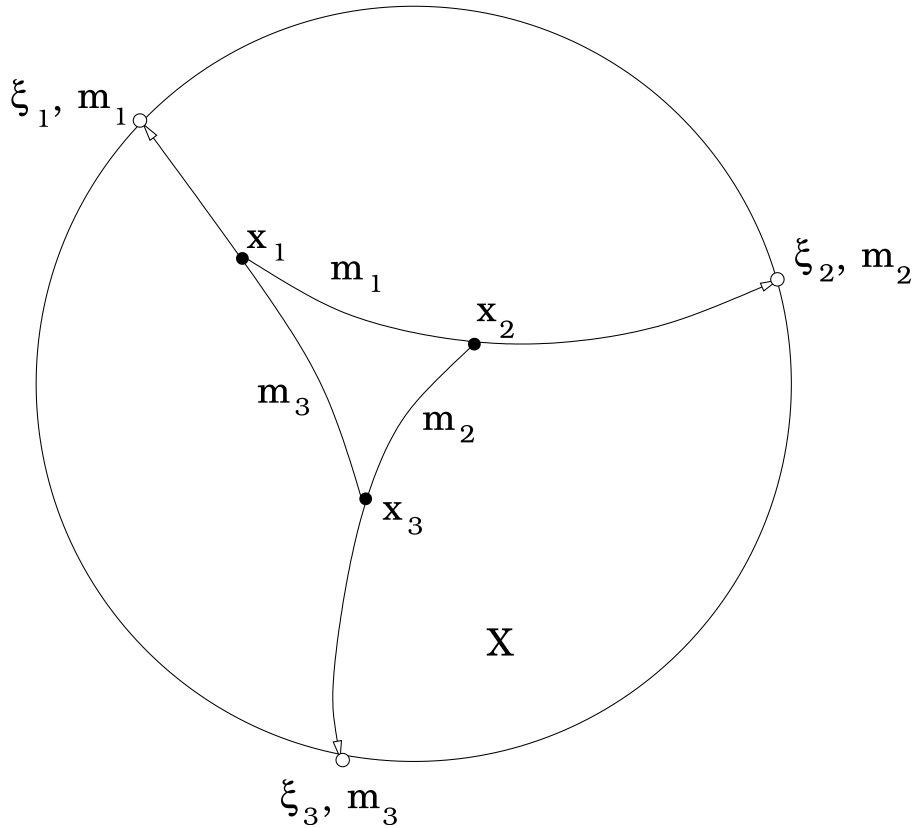
Gauss map



Gauss: Triangles $\longrightarrow$ weighted configurations at infinity.

$[x_1, x_2, x_3] \longrightarrow ((\xi_1, m_1), (\xi_2, m_2), (\xi_3, m_3))$

Gauss map



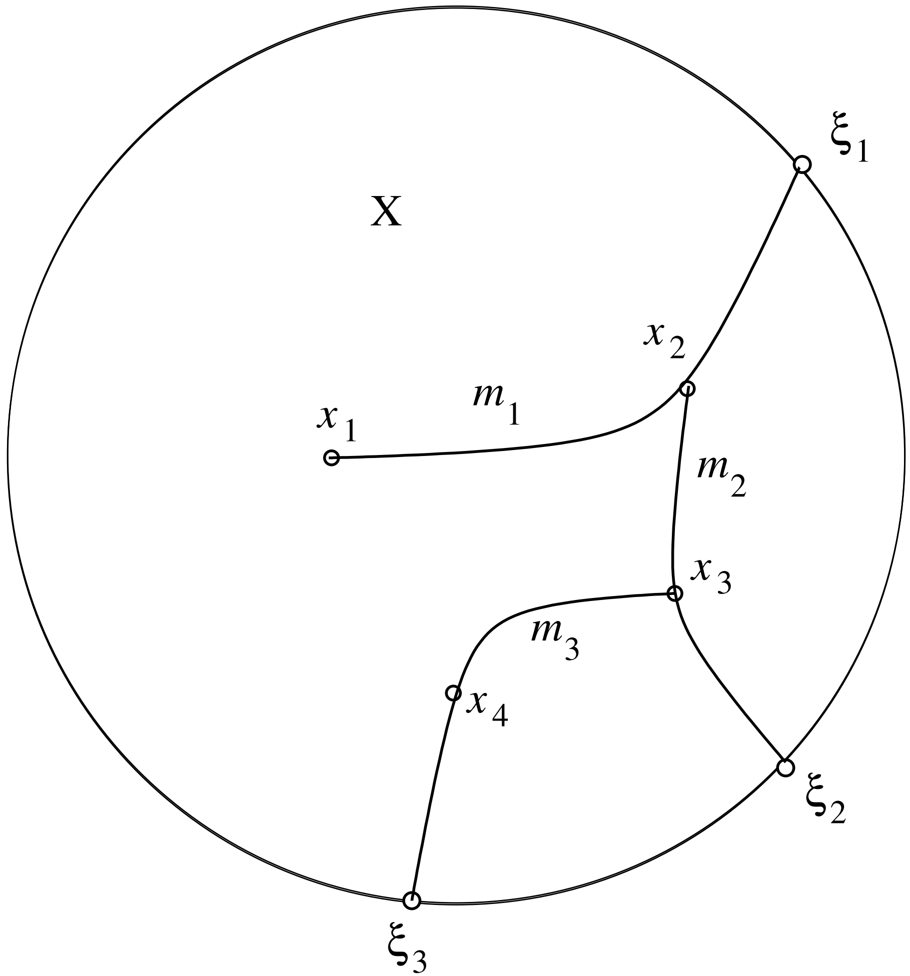
Gauss: Triangles $\longrightarrow$ weighted configurations at infinity.

$[x_1, x_2, x_3] \longrightarrow ((\xi_1, m_1), (\xi_2, m_2), (\xi_3, m_3))$

We would like to invert this map.

Fixed point problem

A fixed point problem for the map $\Phi : x_1 \to x_4$. Existence $\iff$ the weighted configuration at infinity is *nice semistable*.



Proof that D_3 depends on W only

How to transport triangles from one symmetric space (building) to another.

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Let X and X' have the same (finite) Weyl group.

Proof that D_3 depends on W only

Triangles in X	$\overset{?}{\longleftrightarrow}$	Triangles in X'
$\updownarrow Gauss$		$\updownarrow Gauss$
Semistable configurations in $\partial_\infty X$		Semistable configurations in $\partial_\infty X'$
$\updownarrow Gauss$		$\updownarrow Gauss$
Triangles in $Cone(X)$	$\overset{transfer}{\longleftrightarrow}$	Triangles in $Cone(X')$

References

1. M. Kapovich, B. Leeb, J. J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, Preprint, 2004.

2. M. Kapovich, J. J. Millson and B. Leeb, *Polygons in buildings and their side-lengths*, Preprint, 2004.

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4. M. Kapovich, J. J. Millson, *A path model for geodesics in Euclidean buildings and its applications to the representation theory*, Preprint, 2004.