Invent math
DOI 10.1007/s00222-013-0453-4

Dirichlet fundamental domains and topology of projective varieties

Michael Kapovich

Received: 24 January 2012 / Accepted: 10 January 2013

Abstract We prove that for every finitely-presented group G there exists a 2-dimensional irreducible complex-projective variety W with the fundamental group G, so that all singularities of W are normal crossings and Whitney umbrellas.

1 Introduction

It is well-known that fundamental groups of compact Kähler manifolds satisfy many restrictions, see e.g. [1]. On the other hand, C. Simpson proved in [23] that every finitely-presented group G appears as the fundamental group of a (singular) irreducible complex-projective variety. In the same paper Simpson asked the following question which is a variation on a problem about fundamental groups of irreducible projective varieties originally posed by D. Toledo:

Question 1.1 Is it true that every finitely-presented group G is isomorphic to the fundamental group of a irreducible complex-projective variety whose singularities are normal crossings only?

In our previous paper with János Kollár [17] we proved that the answer to this question is positive provided one does not require irreducibility. Although we do not know how to answer Simpson's original question, in this paper we prove

M. Kapovich (✉)
Department of Mathematics, University of California, Davis, CA 95616, USA
e-mail: kapovich@math.ucdavis.edu

Theorem 1.2 *Let G be a finitely-presented group. Then there exists a 2-dimensional irreducible complex-projective variety W with the fundamental group G, so that the only singularities of W are normal crossings and Whitney umbrellas. Furthermore, if G is isomorphic to the fundamental group of a compact 3-dimensional hyperbolic manifold with (possibly empty) convex boundary, then all singularities of W are normal crossings.*

In other words, we get W with "controlled" singularities (unlike the ones which appear in Simpson's proof in [23]). The proof of Theorem 1.2 is a blend of hyperbolic and algebraic geometry. The key tools in our proof are the recent "universality" theorem by Petrunin and Panov, and a certain genericity result for Dirichlet fundamental domains of discrete isometry groups of the hyperbolic 3-space.

Theorem 1.3 (D. Panov, A. Petrunin [20]) *Let G be a finitely-presented group. Then there exists a discrete cocompact subgroup $\Gamma < PO(3,1)$ so that*:

1. *The only nontrivial finite subgroups of Γ are isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
2. *For each order 2 element of Γ its fixed-point set in the hyperbolic 3-space has dimension 0 or 1.*
3. *The fundamental group of the quotient $M := \mathbb{H}^3/\Gamma$ is isomorphic to G.*

Note that M is a 3-dimensional complex which is a manifold away from a finite subset, where the singularities are cones over projective planes. We will need a minor variation on their construction:

Theorem 1.4 *Let G be a finitely-presented group. Then there exists a discrete nonelementary subgroup $\tilde{\Gamma} < PO(3,1)$ so that*:

1. *All nontrivial finite subgroups of $\tilde{\Gamma}$ are isomorphic to $\mathbb{Z}_2$, each has a single fixed point in $\mathbb{H}^3$. In other words, every nontrivial finite subgroup of $\tilde{\Gamma}$ is generated by a Cartan involution of $\mathbb{H}^3$.*
2. *The group $\tilde{\Gamma}$ is convex-cocompact (every convex fundamental domain in $\mathbb{H}^3$ of $\tilde{\Gamma}$ has only finitely many faces and $\tilde{\Gamma}$ contains no parabolic elements).*
3. *The fundamental group of the quotient $\mathbb{H}^3/\tilde{\Gamma}$ is isomorphic to G.*

We will refer to the class of subgroups of $PO(3,1)$ satisfying property 1 in this theorem as *class* $\mathcal{K}$ and to the class of groups satisfying properties 1 and 2 as the *class* $\mathcal{K}^2$.

For a discrete subgroup $\Gamma < PO(3,1)$ and a point $x \in \mathbb{H}^3$ (not fixed by any nontrivial element of Γ) we define the *Dirichlet tiling* $\mathcal{D}_x$ of $\mathbb{H}^3$ to be the Voronoi tiling of $\mathbb{H}^3$ corresponding to the orbit $\Gamma \cdot x$. The *tiles* of $\mathcal{D}_x$ are the

Dirichlet fundamental domains

$$D_{\gamma x} = \{p \in \mathbb{H}^3 : d(p, \gamma x) \leq d(p, \alpha(x)), \forall \alpha \in \Gamma \setminus \{\gamma\}\}.$$

Conjecture 1.5 *For $\Gamma < PO(3,1)$ of class $\mathcal{K}$, for generic choice of x the tiling $\mathcal{D}_x$* simple, *i.e., the dual cell-complex to $\mathcal{D}_x$ is a simplicial complex.*

Conjecture 1.5 was stated as a theorem (for torsion-free groups Γ) in the paper by Jorgensen and Marden [15]. However, their proof has a serious gap noted by Diaz and Ushijima in [9]: The trouble with [15] is confusion between algebraic and semi-algebraic sets. In [15] one of the key claims (Corollary 3.1) is that certain semi-algebraic sets in $\mathbb{H}^3$ have empty interiors, while all what is proven is that these are proper subsets of $\mathbb{H}^3$. (The sets in question are subsets $\mathcal{E}(\underline{A}) \subset \mathbb{H}^3$ consisting of points x such that quadruple intersections of bisectors

$$\bigcap_{i=1}^{4} Bis(x, A_i x),$$

are transversal in $\mathbb{H}^3$. Here $\underline{A} = \{A_1, \ldots, A_4\}$, where $A_i \in \Gamma$ are fixed pairwise distinct and nontrivial elements.) We will actually see in Sect. 7 that some of the sets $\mathcal{E}(\underline{A})$ could have non-empty interiors. The paper [9] proves an analogue of Conjecture 1.5 for torsion-free orientation-preserving discrete subgroups of $PO(2,1)$. We do not know how to prove Conjecture 1.5 either. Nevertheless, we will prove a weaker result that will suffice for our purposes:

Theorem 1.6 *Suppose that $\Gamma < PO(3,1)$ is a subgroup of class $\mathcal{K}$. Then*: (1) *for a generic choice of $x \in \mathbb{H}^3$ the Dirichlet tiling $\mathcal{D}_x$ is simple away from its vertex set $\mathcal{D}_x^{(0)}$. Moreover,* (2) *only points in the interiors of* 2*-dimensional faces of $\mathcal{D}_x$ can be fixed by Cartan involutions in Γ.*

Once Theorems 1.4 and 1.6 are established, the proof of Theorem 1.2 follows closely the arguments in [17], by complexifying a certain hyperbolic polyhedral complex $\mathcal{C}$ (obtained by taking a quotient of $\mathcal{D}_x \setminus \mathcal{D}_x^{(0)}$) and then blowing up "parasitic subspaces" of the complexification. Using $\mathcal{C}$ one constructs a (reducible) projective variety X and a finite group Θ acting on X, so that the only singularities of X are normal crossings, the projective variety $V = X/\Theta$ is irreducible and $\pi_1(V) \cong G$. Irreducibility of V comes from the fact that all facets of $\mathcal{D}_x$ are equivalent under the $\tilde{\Gamma}$-action (unlike the Euclidean polyhedral complexes used in [17] which have many facets). The projective surface W is obtained by applying Lefschetz hyperplane section theorem to V. Whitney umbrella singularities of W correspond to the fixed points of the action on $\mathbb{CP}^3$ of Cartan involutions in $\tilde{\Gamma}$.

2 Preliminaries

Notation 2.1 Throughout the paper we will use the topologist's convention: $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Let $\mathbb{R}^{n,1}$ denote the Lorentzian space, it is $\mathbb{R}^{n+1}$ equipped with nondegenerate inner product $x \cdot y$ of the signature $(n, 1)$. We will be mostly interested in the case $n = 3$, but our proofs are more general. We will refer to the inner product $x \cdot x$ as the *Lorentzian norm* of x. The *light cone* $\mathcal{L}$ of $\mathbb{R}^{n,1}$ consists of vectors of negative Lorentzian norm. This cone has two components, we fix one of these components $\mathcal{L}^{\uparrow}$; we will refer to $\mathcal{L}^{\uparrow}$ as the *future light cone*. We let C denote the boundary of $\mathcal{L}^{\uparrow}$ and C^+ the closure of $\mathcal{L}^{\uparrow}$. The *future* (or the "upper") sheet of the hyperboloid

$$\{x | x \cdot x = -1\}$$

is the intersection H of this hyperboloid with $\mathcal{L}^{\uparrow}$. Then H is the Lorentzian model of the hyperbolic n-space $\mathbb{H}^n$: Restriction of the Lorentzian inner product to the tangent bundle of H is a Riemannian metric of the sectional curvature -1 on H. For a subset $E \subset \mathbb{R}^{n+1}$ we let $\mathbb{P}E$ denote its projection to $\mathbb{RP}^n$. We will identify $\mathbb{H}^n$ with the projectivization $\mathbb{P}H$ of H (and of $\mathcal{L}^{\uparrow}$). The projectivization $\mathbb{P}C^+$ of the cone C^+ is the standard compactification of $\mathbb{H}^n$: $\mathbb{P}C^+ = \mathbb{H}^n \cup S^{n-1}$, where $S^{n-1} = \mathbb{P}C$. For a subset X of $\mathbb{H}^n$ we define its *ideal boundary* $\partial_\infty X$ by:

$$\mathbb{P}\big(cl(X) \cap C\big).$$

In other words $\partial_\infty X$ is the accumulation of X on the boundary sphere S^{n-1} of $\mathbb{H}^n$.

For $x, y \in H$ we let $d(x, y)$ denote their hyperbolic distance. Then (see e.g. [21])

$$x \cdot y = -\cosh\big(d(x, y)\big). \tag{1}$$

In particular, $x \cdot y = -1$ iff $x = y$.

Lemma 2.2 *Let $u, v, w \in H$. Then for any $t, s \in \mathbb{R}$ such that $st \neq 0$, $s + t \neq 0$,*

$$su + tv \neq (s + t)w$$

unless $u = v = w$.

Proof Note that it suffices to show that $u = v$ (since $s + t \neq 0$). Computing the Lorentzian norms of both sides of the equation

$$su + tv = (s + t)w$$

we get:

$$uv = -1.$$

Since $u, v \in H$, it follows that $u = v$. Since $t + s \neq 0$, it follows that $u = v = w$. $\square$

For $x, y \in \mathbb{H}^n$ the *bisector* $Bis(x, y)$ is the hyperplane

$$Bis(x, y) = \left\{p \in \mathbb{H}^n : d(x, p) = d(y, p)\right\}$$

In view of (1), bisectors are described by

$$Bis(x, y) = \{p \in H : x \cdot p = y \cdot p\}.$$

We extend this definition to the entire $\mathbb{R}^{n,1}$, then the *extended* bisector $\widetilde{Bis}(x, y)$ is the hyperplane

$$\widetilde{Bis}(x, y) = \left\{p \in \mathbb{R}^{n+1} : p \cdot (x - y) = 0\right\} = (x - y)^{\perp}.$$

Recall that extended bisectors $\widetilde{Bis}(x, y_i)$, $i = 1, \ldots, k$ are transversal if and only if their intersection has codimension k in $\mathbb{R}^{n+1}$. Thus, these bisectors are transversal if and only if the normal vectors $(x - y_i)$, $i = 1, \ldots, k$ are linearly independent.

Isometry group We let $O(n, 1) < GL(n, \mathbb{R})$ denote the automorphism group of $\mathbb{R}^{n,1}$. This group has index 2 subgroup $O(n, 1)^{\uparrow}$ preserving the future light cone $\mathcal{L}^{\uparrow}$. Thus,

$$O(n, 1) = O(n, 1)^{\uparrow} \times \mathbb{Z}_2$$

where $\mathbb{Z}_2 = \{\pm I\}$ and $I \in GL(n + 1, \mathbb{R})$ is the identity matrix. In particular, $O(n, 1)^{\uparrow}$ is isomorphic to $PO(n, 1)$, the isometry group of $\mathbb{H}^n$. This isomorphism will allow us to identify subgroups of $PO(n, 1)$ with subgroups of $O(n, 1)$.

Classification of nontrivial elements of $PO(n, 1)$ 1. An element $A \in O(n, 1)^{\uparrow}$ is *elliptic* if it has a fixed vector in H. If this fixed vector is unique and $n = 3$, then A necessarily has order 2 and reverses orientation. Regarding $H = \mathbb{H}^3$ as a symmetric space, such elliptic elements are *Cartan involutions* in $\mathbb{H}^3$. For arbitrary n, Cartan involutions are characterized by the property that each has a unique fixed point in $\mathbb{H}^n$ (and order 2). If $A \in PO(n, 1)$ has finite order, it is necessarily elliptic.

2. An element $A \in O(n, 1)^{\uparrow}$ is *parabolic* if it has a unique, up to a multiple, (nonzero) fixed vector $p \in C^+$ and, furthermore, p belongs to C.

3. The rest of the isometries of $\mathbb{H}^n$ are *loxodromic*. These elements $A \in O(n,1)^{\uparrow}$ are characterized by the property that each has exactly two (up to multiple) eigenvectors e_+, e_- in C and the corresponding eigenvalues λ, λ^{-1} are different from 1. The span of these eigenvectors is a plane $E_A \subset \mathbb{R}^{n+1}$ invariant under A. The intersection $E_A \cap H$ is a hyperbolic geodesic L invariant under A, it is called the *axis* of A. The restriction of A to its axis is a nontrivial translation. The eigenvectors $e_\pm$ project to the points in $\mathbb{P}C \cong S^{n-1}$ fixed by A. (These are the only fixed points that A can have.)

There is a finer classification of loxodromic isometries. Every loxodromic $A \in O(n,1)^{\uparrow}$ preserves the orthogonal complement $E_A^{\perp}$ of E_A. The restriction of the Lorentzian inner product to $E_A^{\perp}$ is necessarily positive-definite. If A fixes $E_A^{\perp}$ pointwise, it is called *hyperbolic*; otherwise, it is called *strictly loxodromic*. For $n = 3$, this can be described more precisely: Each strictly loxodromic element acts on $E_A^{\perp}$ as a nontrivial rotation R_θ (by the angle θ) or a reflection (in case A reverses orientation on H). The angle θ is the *angle of rotation* of A. Intrinsically, in terms of the geometry of $\mathbb{H}^n$, hyperbolic isometries are characterized as compositions $\tau_1 \circ \tau_2$ of distinct Cartan involutions. It we trivialize the normal bundle of the axis L by parallel vector fields, then each hyperbolic isometry acts on the normal bundle as the translation along the axis, while a strictly loxodromic element is a composition of the translation and a (nontrivial) orthogonal transformation of a normal plane.

Discrete subgroups Suppose that $\Gamma < PO(n,1)$ is a discrete subgroup. If $A_1, A_2 \in \Gamma$ are loxodromic which share a common fixed point in $S^{n-1} = \mathbb{P}C$, i.e., they have a common eigenvector $e_+ \in C$. Then A_1, A_2 share the other eigenvector e_- in C as well (see e.g. [21, Theorem 5.5.4]) and, hence, have the common axis L in $\mathbb{H}^n$, the unique geodesic connecting the common fixed points of A_1, A_2. If $A_1, A_2 \in SO(3,1)$ then they necessarily commute in this situation, otherwise, they (typically) do not commute. However, if they do not commute, then the group $\langle A_1, A_2\rangle$ generated by A_1, A_2 does not act faithfully on L, i.e., it contains a nontrivial elliptic element $[A_1, A_2]$ fixing L pointwise. Thus, such noncommuting pairs of loxodromic elements with common axis cannot belong to a group $\Gamma < PO(n,1)$ of the class $\mathcal{K}$.

Fundamental domains Let D be a closed convex domain in $\mathbb{H}^n$ and Γ a discrete group of isometries of $\mathbb{H}^n$. Then D is said to be a *fundamental domain* of Γ if the following hold:

(1) $\Gamma \cdot D = \mathbb{H}^n$.
(2) For every $\gamma \in \Gamma \setminus \{1\}$, $\gamma D \cap D \neq \emptyset$ unless $\gamma D \cap D$ is contained in the boundary of D.
(3) The covering $\{\gamma D, \gamma \in \Gamma\}$ of $\mathbb{H}^n$ is *locally finite*, i.e., every compact in $\mathbb{H}^n$ intersects only finitely many domains γD.

In particular, if D_1, D_2 are fundamental domains of Γ and $D_1 \subset D_2$ then $D_1 = D_2$.

The key example of a fundamental domain is the *Dirichlet fundamental domain* with the center at $x \in \mathbb{H}^n$, where x is not fixed by any $\gamma \in \Gamma \setminus \{1\}$:

$$D_x := \{p \in \mathbb{H}^n : d(p, x) \le d\big(p, \gamma(x)\big), \forall \gamma \in \Gamma\}.$$

The fundamental domain D_x and its images $D_{\gamma x}$ under $\gamma \in \Gamma$ form a Γ-invariant tiling $\mathcal{D}_x(\Gamma)$ of $\mathbb{H}^n$, called the *Dirichlet tiling*.

Convex-cocompact subgroups A discrete subgroup $\Gamma < PO(n, 1)$ is called *convex-cocompact* if the following holds:

(a) Γ contains no parabolic elements.
(b) One (equivalently, every) Dirichlet fundamental domain D_x of Γ is a polyhedral domain in $\mathbb{H}^n$ with finitely many faces.

The reader can find detailed discussion and alternative characterizations of convex-cocompact subgroups of $PO(n, 1)$ in [4, 6].

Classes $\mathcal{K}$ and $\mathcal{K}^2$ We say that a subgroup $\Gamma < PO(n, 1)$ belongs to the class $\mathcal{K}$ if it is discrete and every elliptic element of Γ is a Cartan involution. We define the class $\mathcal{K}^2$ to consists of all convex–cocompact discrete subgroups $\Gamma < PO(n, 1)$ which belong to the class $\mathcal{K}$.

Elementary and nonelementary groups A subgroup $\Gamma < PO(n, 1)$ is called *elementary* if it either has a fixed point in the compactification $\mathbb{H}^n \cup S^{n-1}$ of $\mathbb{H}^n$ (this compactification is the projectivization $\mathbb{P}C^+$ of C^+) or has an invariant geodesic in $\mathbb{H}^n$. Clearly, elementary groups can have nontrivial center, e.g., we can take Γ to be abelian. Furthermore, if Γ is discrete and elementary then it is *virtually abelian*, i.e., contains an abelian subgroup of finite index.

Lemma 2.3 *If Γ is nonelementary and belongs to the class $\mathcal{K}$, then Γ has trivial center.*

Proof Let $\gamma \in \Gamma$ be a nontrivial central element. Then Γ preserves the fixed-point set $Fix(\gamma)$ of γ in $\mathbb{P}C^+$. If this fixed set is finite, then Γ is elementary. Otherwise, $Fix(\gamma)$ has to contain a hyperbolic geodesic, so γ is an elliptic element that cannot be a Cartan involution. Contradiction. □

3 Hyperbolic polyhedral complexes

This section essentially repeats the definitions given in [17] in the context of Euclidean polyhedral complexes except that we weaken one of the axioms of the polyhedral complex.

A *convex polyhedral cone* in $\mathbb{R}^{n+1}$ is a set given by finitely many strict and non-strict linear inequalities. A convex (real) projective polytope in $\mathbb{RP}^n$ is a projectivization of a convex polyhedral cone in $\mathbb{R}^{n+1}$.

A (convex) *hyperbolic polyhedron* is a subset P of $\mathbb{H}^n$ given as the intersection of finitely many open and closed hyperbolic half-spaces. Equivalently, if we identify $\mathbb{H}^n$ with the upper sheet H of the hyperboloid in $\mathbb{R}^{n,1}$, then P is given as the intersection of H with some convex polyhedral cone $\widetilde{P} \subset \mathbb{R}^{n+1}$. Projectivizing $\widetilde{P}$ we obtain a convex projective polytope $\widehat{P} \subset \mathbb{RP}^n$. Note, however, that $\widehat{P}$ could have faces which are completely disjoint from the projection of H to $\mathbb{RP}^n$. Since we would like to preserve combinatorics of our polyhedra, the solution is to remove from $\widehat{P}$ all the faces which are disjoint from the projection of H. Let $\check{P}$ denote the resulting convex projective polytope. Thus we obtain a map

$$P \mapsto \widetilde{P} \mapsto \widehat{P} \mapsto \check{P}$$

from the set of convex hyperbolic polyhedra $P \subset \mathbb{H}^n$ to the set of convex projective polytopes $\check{P} \subset \mathbb{RP}^n$.

The *real-projective span* $Span_{\mathbb{R}}(P)$ of a convex projective polytope P of a hyperbolic polyhedron P is the smallest projective subspace in $\mathbb{RP}^n$ containing P or $\widehat{P}$ respectively. The *dimension* of P is its topological dimension, which is the same as the dimension of its projective span $Span_{\mathbb{R}}(P)$.

A *face* of P is a subset of P which is given by converting some of these non-strict inequalities to equalities. Define the set $Faces(P)$ to be the set of faces of P. The *interior* $Int(P)$ of P is the topological interior of P in $Span_{\mathbb{R}}(P)$. Again, $Int(P)$ is a hyperbolic polyhedron. We will refer to $Int(P)$ as an *open polyhedron*.

In the paper we will be also using *complex span* $Span_{\mathbb{C}}(P)$ of convex projective and hyperbolic polytopes P; the complex-projective space $Span_{\mathbb{C}}(P)$ is the complexification of $Span_{\mathbb{R}}(P)$. To simplify the notation, we set $Span(P) := Span_{\mathbb{C}}(P)$.

An (isometric) *morphism* of two hyperbolic polyhedra is an isometric map $f : P \to Q$ so that $f(P)$ is a face of Q. Similarly, a projective morphism $f : P \to Q$ of two convex projective polytopes is a restriction of an (invertible) projective transformation which sends P to a face of Q.

Definition 3.1 A *hyperbolic (resp. projective) polyhedral complex* is a small category $\mathcal{C}$ whose objects are convex hyperbolic (resp. projective) polyhedra and morphisms are their isometric morphisms (resp. projective morphisms) satisfying the following axioms:

Axiom 1 For every $c_1 \in Ob(\mathcal{C})$ and every face c_2 of c_1, $c_2 \in Ob(\mathcal{C})$, the inclusion map $\iota : c_1 \to c_2$ is a morphism of $\mathcal{C}$.

Axiom 2 For every $c_1, c_2 \in Ob(\mathcal{C})$ there exists at most one morphism $f = f_{c_2,c_1} \in Mor(\mathcal{C})$ so that $f(c_1) \subset c_2$.

Objects of a polyhedral complex $\mathcal{C}$ are called *faces* of $\mathcal{C}$ and the morphisms of $\mathcal{C}$ are called *incidence maps* of $\mathcal{C}$. A *facet* of $\mathcal{C}$ is a face P of $\mathcal{C}$ so that for every morphism $f : P \to Q$ in $\mathcal{C}$, $f(P) = Q$. A *vertex* of $\mathcal{C}$ is a zero-dimensional face. The *dimension* $\dim(\mathcal{C})$ of $\mathcal{C}$ is the supremum of dimensions of faces of $\mathcal{C}$. A polyhedral complex $\mathcal{C}$ is called *pure* if the dimension function is constant on the set of facets of $\mathcal{C}$; the constant value in this case is the dimension of $\mathcal{C}$. A *subcomplex* of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$. If c is a face of a complex $\mathcal{C}$ then $Res_{\mathcal{C}}(c)$, the *residue* of c in $\mathcal{C}$, is the minimal subcomplex of $\mathcal{C}$ containing all faces c' such that there exists an incidence map $c \to c'$. For instance, if c is a vertex of $\mathcal{C}$ then its residue is the same as the *star* of c in $\mathcal{C}$; however, in general these are different concepts.

Example 3.2 Consider $\mathcal{C}$ which consists of two edges e_1, e_2 and three vertices v_1, v_2, v_3, so that $e_1 = [v_1, v_2], e_2 = [v_2, v_3]$. Then star of the edge e_2 in $\mathcal{C}$ is the entire complex $\mathcal{C}$ (since every facet of $\mathcal{C}$ has nonempty intersection with e_2), while the residue of e_2 is just the edge e_2 (and its vertices, of course).

We generate the equivalence relation $\sim$ on a polyhedral complex $\mathcal{C}$ by declaring that $c \sim f(c)$, where $c \in Ob(\mathcal{C})$ and $f \in Mor(\mathcal{C})$. This equivalence relation also induces the equivalence relation $\sim$ on points of faces of $\mathcal{C}$.

If $\mathcal{C}$ is a polyhedral complex, its *poset* $\mathrm{Pos}(\mathcal{C})$ is the partially ordered set $Ob(\mathcal{C})$ with the relation $c_1 \le c_2$ iff $c_1 \sim c_0$ so that $\exists f \in Mor(\mathcal{C})$, $f : c_0 \to c_2$.

We can associate to a hyperbolic polyhedral complex $\mathcal{C}$, a projective polyhedral complex $\check{\mathcal{C}}$ as follows. Given a face c of $\mathcal{C}$, we define the convex projective polytope $\check{c}$ as above; complex-projective spans of the polytopes c and $\check{c}$ are, of course, the same. In particular, (isometric) morphisms $c \to c'$ extend uniquely to (projective) morphisms $\check{c} \to \check{c}'$; thus $\mathcal{C}$ yields a projective polyhedral complex $\check{\mathcal{C}}$ whose objects are convex projective polytopes $\check{c}$. It is clear that $\mathcal{C}$ and $\check{\mathcal{C}}$ have isomorphic posets.

We define the *topological pushout* (also known as the *underlying space* or *amalgamation*) $C = |\mathcal{C}|$ of a polyhedral complex $\mathcal{C}$ as the topological space which is obtained from the disjoint union

$$\amalg_{c \in Ob(\mathcal{C})}\ c$$

by identifying points using the equivalence relation: $\sim$. We equip $|\mathcal{C}|$ with the quotient topology.

Definition 3.3 If $\mathcal{C}$ is a polyhedral complex and $\mathcal{B}$ is its subcomplex. For $c \in Ob(\mathcal{C})$ define the polyhedron

$$c^{\mathcal{B}} := c \setminus \bigcup_{b \le c, b \in \mathcal{B}} f(b), \quad \text{where } f : b \to c,\ f \in Mor(\mathcal{C}).$$

For a morphism $f \in Mor(\mathcal{C})$, $f : c_1 \to c_2$, we set $f^{\mathcal{B}} : c_1^{\mathcal{B}} \to c_2^{\mathcal{B}}$ be the restriction of f. We define the *difference complex* $\mathcal{C} - \mathcal{B}$ as the following polyhedral complex:

$$Ob(\mathcal{C} - \mathcal{B}) = \{c^{\mathcal{B}} : c \in Ob(\mathcal{C})\},$$
$$Mor(\mathcal{C} - \mathcal{B}) = \{f^{\mathcal{B}} : c_1^{\mathcal{B}} \to c_2^{\mathcal{B}},\ \text{where } f \in Mor(\mathcal{C}),\ f : c_1 \to c_2\}.$$

A complex $\mathcal{C}$ is said to be *finite* it has only finitely many objects and morphisms. A complex $\mathcal{C}$ is *locally finite* if for every face $a \in Ob(\mathcal{C})$ the sets of morphisms

$$\{f : a \to b, b \in Ob(\mathcal{C})\}$$

is finite. The key example of a hyperbolic polyhedral complex used in this paper (the Dirichlet tiling of $\mathbb{H}^n$) will be infinite but locally finite. In this paper we will be exclusively interested in locally finite complexes.

Definition 3.4 Let $\mathcal{C}$ be a pure n-dimensional polyhedral complex. The *nerve* $Nerve(\mathcal{C})$ of $\mathcal{C}$ is the simplicial complex whose vertices are facets of $\mathcal{C}$ (the notation is $v = c^*$, where c is a facet of $\mathcal{C}$); distinct vertices $v_0 = c_0^*, \ldots, v_k = c_k^*$ or $Nerve(\mathcal{C})$ span a k-simplex if there exists an $n-k$-face c of $\mathcal{C}$ and incidence maps $c \to c_i$, $i = 0, \ldots, k$. The simplex $\sigma = [v_0, \ldots, v_k]$ then is said to be *dual* to the face c.

Similarly to [17] we have:

Lemma 3.5 *If $\mathcal{C}$ is locally finite then $|\mathcal{C}|$ is homotopy-equivalent to $|Nerve(\mathcal{C})|$.*

Definition 3.6 A polyhedral complex $\mathcal{C}$ is *simple* if:

(1) $\mathcal{C}$ is pure and $\dim(\mathcal{C}) = n$,
(2) For $k = 0, \ldots, n$ and every k-face c of $\mathcal{C}$, $Nerve(Res_{\mathcal{C}}(c))$ is isomorphic to the complex $\mathcal{C}(\Delta^{n-k})$.

For a polyhedral complex $\mathcal{C}$ we define its *k-skeleton*, to be the subcomplex $\mathcal{C}^{(k)}$ consisting of faces of dimension $\le k$. For a pure n-dimensional complex $\mathcal{C}$ we define its *punctured* complex $\mathcal{C}'$ by: $\mathcal{C}' := \mathcal{C} - \mathcal{C}^{(n-3)}$. (In this paper we will only be using this construction for $n = 3$, when $\mathcal{C}'$ is obtained from $\mathcal{C}$ by removing vertices.) We say that $\mathcal{C}$ is *weakly simple* if the punctured complex $\mathcal{C}'$ is simple. In other words, every $n-2$-dimensional face is incident to exactly 3 facets and every $n-1$-dimensional face is incident to exactly 2 facets.

The point of considering punctured complexes is that if $|\mathcal{C}|$ is a manifold at every point of $\mathcal{C}^{(n-3)}$, then

$$\pi_1(|\mathcal{C}|) \cong \pi_1(|\mathcal{C}'|).$$

Thus, in this situation, passing to the punctured complex does not change the fundamental group, while proving simplicity for the punctured complex is much easier than for the original one.

Voronoi tiling of $\mathbb{H}^n$

Definition 3.7 Let $Y \subset \mathbb{H}^n$ be a locally finite subset (i.e., every compact in $\mathbb{H}^n$ contains only finitely many points of Y). The *Voronoi tiling* $\mathcal{V}(Y)$ of $\mathbb{H}^n$ associated with Y is defined by: For each $y \in Y$ take the *Voronoi cell*

$$V(y) := \{x \in \mathbb{H}^n : d(x, y) \le d(x, y'), \forall y' \in Y\}.$$

Thus, each cell $V(y)$ is given by the collection of non-strict linear inequalities $d(x, y) \le d(x, y')$. Then each cell $V(y)$ is a closed (possibly unbounded) polyhedron in $\mathbb{H}^n$. The union of Voronoi cells is the entire $\mathbb{H}^n$. Assuming that each $V(y)$ has only finitely many faces, we thus obtain the polyhedral complex, called the *Voronoi complex*, $\mathcal{V}(Y)$ using the polyhedra $V(y)$ as facets and faces of facets as faces of $\mathcal{V}(Y)$.

A special case of this construction is given by orbits of a discrete convex-cocompact subgroup $\Gamma < PO(n, 1)$: If $x \in \mathbb{H}^n$ is a point not fixed by any $\gamma \in \Gamma \setminus \{1\}$, then the Dirichlet tiling $\mathcal{D}_x(\Gamma)$ is the same as Voronoi tiling with respect to the set $Y := \Gamma \cdot x$. We will use the same notation $\mathcal{D}_x(\Gamma)$ for the associated hyperbolic polyhedral complex, the *Dirichlet complex*. (Recall that, since Γ is convex-cocompact, every $D_{\gamma x}$ is a convex hyperbolic polyhedron in our sense since it has only finitely many faces.)

We note that if c is a face of $D_x = D_x(\Gamma)$, then its stabilizer in Γ has to be finite, otherwise D_x would fail to be a fundamental domain. In particular, the stabilizer of c consists entirely of elliptic elements.

Lemma 3.8 *Let $\mathcal{D}_x(\Gamma)$ be the Dirichlet complex of a convex-cocompact group $\Gamma < PO(n, 1)$. Then Γ contains a finite-index torsion-free normal subgroup Γ' so that $(\mathcal{D}_x(\Gamma))/\Gamma'$ is a hyperbolic polyhedral complex.*

Proof First, since Γ is convex-cocompact, it is also finitely-generated, see e.g. [6]. Hence, by Selberg's Lemma [22], Γ contains a torsion-free subgroup Γ_1 of finite index. One could now take the quotient $(\mathcal{D}_x(\Gamma))/\Gamma_1$: It satisfies all properties of a polyhedral complex, except Axiom 2 could fail: If $\tilde{c}_1 \le \tilde{c}_2$ are incident faces of $\mathcal{D}_x(\Gamma)$, we could have some $\gamma \in \Gamma_1 \setminus \{1\}$ such that $\gamma(\tilde{c}_1) \le \tilde{c}_2$. Dividing by Γ_1 we then would have more than one morphism $c_1 \to c_2$, where c_i is the projection of $\tilde{c}_i$, $i = 1, 2$. We will see below how to eliminate such elements γ by passing to a further finite index subgroup in Γ_1.

Since D_x has only finitely many faces, there are only finitely many nontrivial elements $\gamma_i \in \Gamma_1$, $i = 1, \dots, m$, so that $\gamma_i D_x \cap D_x \ne \emptyset$. Since Γ_1 is residually finite, it contains a finite-index subgroup Γ_2 so that $\gamma_i \notin \Gamma_2$, $i = 1, \dots, m$. Lastly, we take $\Gamma' < \Gamma_2$ a finite index subgroup which is normal in Γ. Then for each $\gamma \in \Gamma' \setminus \{1\}$, $\gamma D_x \cap D_x = \emptyset$. By normality of Γ' in Γ, we also have

$$\gamma D_{\alpha x} \cap D_{\alpha x} = \emptyset$$

for all $\alpha \in \Gamma$. This implies that Axiom 2 holds for the quotient complex $(\mathcal{D}_x(\Gamma))/\Gamma'$. $\square$

Weak simplicity criterion for Dirichlet complexes

Lemma 3.9 *The Dirichlet tiling $\mathcal{D}_x(\Gamma)$ is weakly simple provided that for every $y \in \partial D_x \subset \mathbb{H}^n$ and every collection of elements $\gamma_1, \dots, \gamma_k \in \Gamma$ so that*

$$\dim\left(\bigcap_{i=1}^{k} Bis(x, \gamma_i x)\right) = n - 2,$$

and

$$y \in \bigcap_{i=1}^{k} Bis(x, \gamma_i x),$$

we have $k = 2$.

Proof Since Γ acts transitively on the facets of the tiling $\mathcal{D}_x(\Gamma)$, it suffices to prove weak simplicity of $\mathcal{D}_x(\Gamma)$ along codimension 2 cells E contained in the boundary of D_x. Let $\gamma_0 := 1, \gamma_1, \dots, \gamma_k \in \Gamma$ be the elements of Γ such that E is contained in $\gamma_i(D_x)$, $i = 0, \dots, k$. Weak simplicity of $\mathcal{D}_x$ then means that $k = 2$. We relabel the elements γ_i above so that

$$\gamma_{i+1} D_x \cap \gamma_i D_x$$

is a codimension 1 face F_i for $i = 0, \dots, k$, where i is taken modulo k. Then F_i is contained in the bisector $Bis(\gamma_i x, \gamma_{i+1} x)$, $i = 0, \dots, k$. Therefore, for every $y \in E$,

$$d(y, \gamma_i x) = d(y, x),$$

that is,

$$y \in \bigcap_{i=1}^{k} Bis(x, \gamma_i x)$$

and, hence,

$$E \subset \bigcap_{i=1}^{k} Bis(x, \gamma_i x).$$

□

Remark 3.10 The same proof, of course, yields the simplicity criterion for $\mathcal{D}_x(\Gamma)$: It is simple if and only if for every $y \in \partial D_x$, the bisectors $Bis(x, \gamma_i(x))$ passing through y are have transversal intersection in $\mathbb{H}^n$.

Linear algebra problems Let F be a field. For a subset $\underline{A} = \{A_1, \ldots, A_k\} \subset Mat_{n,n}(F)$, $k \le n$, we define the map

$$B = B_{\underline{A}} : F^n \to Mat_{n,k}(F) \tag{2}$$

by

$$x \mapsto (B_1 x, \ldots, B_k x)$$

where $B_i = A_i - I$, $i = 1, \ldots, k$ and we regard vectors $B_i x$ as columns of the matrix $B_{\underline{A}}$. We say that the map B and the set $\underline{A}$ are *singular* if for every $x \in F^n$, $\mathrm{rank}(B(x)) < k$. We note that the image of the map B is a linear subspace of $Mat_{n,k}(F)$. The problem of describing linear subspaces of $Mat_{n,k}(F)$ consisting of matrices of rank $< k \le n$ has a long history, see [18] for a survey.

If we do not make any restrictions on the matrices A_i, then the problem of describing singular k-tuples $\underline{A}$ is essentially equivalent to the problem of describing linear subspaces of $Mat_{n,k}(F)$ and is hopelessly complicated. Suppose, however, one takes A_i from an algebraic subgroup $G < GL(n, F)$, e.g., $G = O(n, F)$.

Problem 3.11 Let $F = \mathbb{R}$. Describe singular k-tuples $\underline{A}$ of matrices $A_i \in O(n, 1)^{\uparrow}$. In particular, suppose that no matrices in A_i share a common eigenvector. Is it true that in this case $\underline{A}$ is nonsingular?

A positive answer would be a key step towards proving

Conjecture 3.12 *Suppose that $\Gamma < PO(n, 1)$ is a discrete subgroup of the class $\mathcal{K}$. Then for generic $x \in \mathbb{H}^n$ the Dirichlet complex $\mathcal{D}_x(\Gamma)$ is simple.*

Note that the linear maps $B = B_{\underline{A}} : F^n \to Mat_{n,k}(F)$ are injective provided that the linear transformations A_i do not have a common fixed vector. In this case, Problem 3.11 becomes a special case of the problem of describing k-dimensional linear subspaces of $Mat_{n,k}(F)$ (with $k \le n$) consisting of matrices of rank $< k$. *Rank* of such a subspace is the maximal rank of a matrix which belongs to this subspace.

Linear subspaces of $Mat_{n,k}(F)$ of rank 1 are easy to describe. Classification of subspaces of ranks 2 and 3 was given in [3, 11]. It is easy to see that the classes of *primitive* subspaces of rank ≤ 3 described in [3, 11] (with $F = \mathbb{C}$) do not appear as images of maps $B_{\underline{A}}$, where $\underline{A} = \{A_1, A_2, A_3, A_4\}$ are in $O(4, \mathbb{C})$. However, it is unclear how to deal with the non-primitive subspaces. For instance, it is unclear if there are (pairwise noncommuting) elements A_i of $O(4, \mathbb{C})$ so that the matrices B_i are linearly dependent as elements of $Mat_{4,4}(\mathbb{C})$. Such quadruples would correspond to the case when

$$\bigcap_{x \in F^4} Ker\big(B(x)\big) \neq 0.$$

4 Complexes of varieties

Our discussion here closely follows [17].

Definition 4.1 Let $\mathbf{V}$ denote either the category of varieties (over a fixed field k) or the category of topological spaces.

Let $\mathcal{C}$ be a finite hyperbolic polyhedral complex. A $\mathbf{V}$-*complex based on* $\mathcal{C}$ is a functor Φ from $\mathcal{C}$ to $\mathbf{V}$ so that morphisms $c_i \to c_j$ go to closed embeddings $\phi_{ij} : \Phi(c_i) \to \Phi(c_j)$. By abuse of terminology, we will sometimes refer to the image category $im(\Phi)$ as a $\mathbf{V}$-complex based on $\mathcal{C}$. We will use the notation X_i for $\Phi(c_i)$. The varieties X_i will be called *strata* of the complex of varieties $im(\Phi)$.

We call the functor Φ *strictly faithful* if the following holds:

If $x_i \in \Phi(c_i)$, $x_j \in \Phi(c_j)$ and $\phi_{ik}(x_i) = \phi_{jk}(x_j)$ for some k then there is an ℓ and $x_\ell \in \Phi(c_\ell)$ such that $\phi_{\ell i}(x_\ell) = x_i$ and $\phi_{\ell j}(x_\ell) = x_j$.

The relation $x_i \sim \phi_{ij}(x_i)$ for every i, j and $x_i \in X_i$ generates an equivalence relation on the points of $\amalg_{i \in I} \Phi(c_i)$, also denoted by $\sim$.

In the category of topological spaces, the direct limit (or push-out) $\lim \Phi(\mathcal{C})$ of the diagram $\Phi(\mathcal{C})$ exists and its points are identified with $(\amalg_{i \in I} \Phi(c_i))/\sim$.

For example, suppose that Φ_{taut} is the tautological functor which identifies each face of $\mathcal{C}$ with the corresponding underlying topological space. Then $\lim \Phi_{taut}(\mathcal{C})$ is nothing but $|\mathcal{C}|$.

As in [17] we have:

Lemma 4.2 *Suppose that Φ is strictly faithful and $\Phi(\mathcal{C})$ consists of cell complexes and cellular maps of such complexes. Then $\pi_1(\lim \Phi(\mathcal{C})) \cong \pi_1(|\mathcal{C}|)$ provided that each $\Phi(c), c \in Ob(\mathcal{C})$ is* 1*-connected.*

The following result was proven in [17], Proposition 31 for Euclidean polyhedral complexes, but the same proof applies to hyperbolic complexes:

Proposition 4.3 *Let $\Phi : \mathcal{C} \to \mathbf{V}$ be a complex of varieties based on a finite hyperbolic polyhedral complex. Assume that for each k and each $J \subset I$ the subvariety $\bigcup_{j \in J} im(\phi_{jk}) \subset X_k$ is seminormal. For instance, this assumption holds if this union is a divisor with normal crossings. Then the direct limit* $\lim \Phi(\mathcal{C})$ *exists in the category of varieties. Furthermore, as a topological space,* $\lim \Phi(\mathcal{C})$ *is homeomorphic to the topological push-out* $\lim \Phi_{top}(\mathcal{C})$.

For convenience of the reader, here is the definition of a seminormal variety:

Definition 4.4 Recall that a complex space X is called *normal* if for every open subset $U \subset X$, every bounded meromorphic function on U is holomorphic. A complex space X is called *seminormal* if for every open subset $U \subset X$, every continuous meromorphic function on U is holomorphic.

Example 4.5 The subvarieties $(x^2 = y^3) \subset \mathbb{C}^2$ and $(x^3 = y^3) \subset \mathbb{C}^2$ are normal but not seminormal (by taking functions x/y and x^2/y respectively). On the other hand, the subvariety $(x^2 = y^2) \subset \mathbb{C}^2$ is seminormal.

Complex of projective spaces Let $\mathcal{C}$ be a finite simple projective polyhedral complex; we set $C := |\mathcal{C}|$. As in [17], we define the functor $\Phi : \mathcal{C} \to \mathbf{V}$ which sends each c to the complex-projective space $\mathcal{P}_c := Span(c) \times \{c\}$ and each morphism $c_i \to c_j$ to the linear map of the complex-projective spaces $F_{c_j, c_i} : \mathcal{P}_{c_i} \to \mathcal{P}_{c_j}$ which restricts to the original morphism $c_i \to c_j$. We let $\mathcal{P} = \mathcal{P}_\mathcal{C}$ denote $im(\Phi)$.

Remark 4.6 The point of using $\mathcal{P}_c = Span(c) \times \{c\}$ rather than $\mathcal{P}_c = Span(c)$, is that we want to stress the fact that for different c_1, c_2, the spaces $\mathcal{P}_{c_i}$ are regarded as disjoint projective spaces, even in the case of faces c_i of the same cell c.

The following definition is taken from [17]:

Definition 4.7 (Parasitic subspaces) Let $\sigma := (c_1, c_2, \dots, c_k)$ be a tuple of faces incident to a face c of $\mathcal{C}$. Consider the intersections

$$I_{c,\sigma} := \bigcap_{i=1}^{k} F_{c,c_i}(\mathcal{P}_{c_i}) \subset \mathcal{P}_c$$

such that there is no face c_0 such that $I_{c,\sigma} = F_{c,c_0}(\mathcal{P}_{c_0})$ and c_0 is incident to all the $c_1, c_2, \dots, c_k$. Then the subspace $I_{c,\sigma} \subset \mathcal{P}_c$ is called a *parasitic intersection* in $\mathcal{P}_c$.

Remark 4.8 1. Instead of projective polytopes one can work directly with hyperbolic polyhedra $c = \mathbb{H}^n \cap \check{c}$ (see beginning of Sect. 3 for the definition of projective polytope $\check{c}$ associated with the hyperbolic polyhedron c); here we embed $\mathbb{H}^n$ in $\mathbb{RP}^n$ using Klein model of the projective space. The notion of parasitic intersection will be the same. We have chosen to work with the projective polytopes to keep the discussion as close to [17] as possible.

2. Here is an explanation why parasitic intersections cause problems. Our goal is to glue projective spaces $\mathcal{P}_c$ using the morphisms F_{c_j,c_i}. Let P be the result of gluing (which, at the moment, we treat just as a topological space). We would like $\pi_1(P)$ to be isomorphic to $\pi_1(C)$. If parasitic intersections do not exist (this happens, for instance, if $\mathcal{C}$ is a simplicial complex), then nerve of the covering of P by the maximal spaces $\mathcal{P}_c, c \in Ob(\mathcal{C})$, is isomorphic to the nerve of the covering of $\mathcal{C}$ by the facets. In this case, $\pi_1(P) \cong \pi_1(C)$. However, in presence of parasitic intersections, the two nerve are isomorphic and fundamental groups could be non-isomorphic as well. For instance, in the example of Fig. 1, the complex $\mathcal{C}$ consists of 3 facets c_i which are squares each of which misses two edges and all vertices. The three non-missing edges e_i are pairwise disjoint. However, their projective spans intersect in the common point v which is the parasitic intersection. Then $\pi_1(C) \cong \mathbb{Z}$, while $\pi_1(P) = 1$.

The way to deal with parasitic intersections is to below them up. However, we have to ensure that as the result of blow-up we still have a complex of varieties. In general, this may not be the case, since this collection of parasitic intersections in spaces $\mathcal{P}_c, c \in Ob(\mathcal{C})$, is not stable under applying morphisms $F_{c',c}$ and taking preimages under these morphisms. We thus have to *saturate* the collection of parasitic intersections using the morphisms $F_{c,c'}$. This is done as follows. Let T denote the push-out of the category $im(\mathcal{P}_{top})$, where we regard each $\mathcal{P}_b, b \in Ob(\mathcal{C})$, as a topological space, so the push-out exists. Then for each $a \in Ob(\mathcal{C})$ we have the (injective) projection map $\rho_a : \mathcal{P}_a \to T$. For each parasitic intersection $I_{c,\sigma} \subset \mathcal{P}_c$, we define

$$I_{c,\sigma,a} := \rho_a^{-1} \rho_c(I_{c,\sigma}).$$

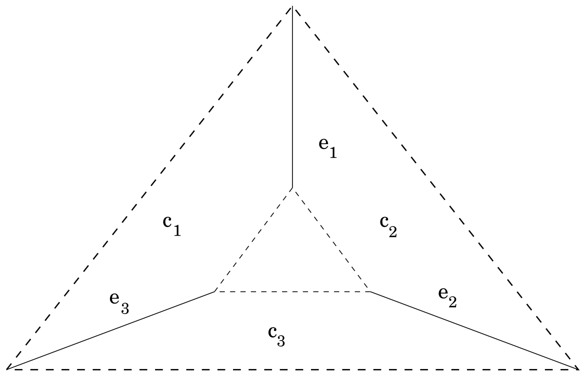

Fig. 1 The nerve $\mathcal{N}$ of push-out P of the complex $\mathcal{P}$ consisting of projective spaces $\mathcal{P}_{e_i}, \mathcal{P}_{c_i}$ $(i = 1, 2, 3)$ and v, is isomorphic to the face-complex of the 2-simplex and, hence, is simply-connected

We call such $I_{c,\sigma,a}$ a *primary parasitic subspace* in $\mathcal{P}_a$. It is immediate that each primary parasitic subspace in $\mathcal{P}_a$ is a projective space linearly embedded in $\mathcal{P}_a$. With this definition, the collection of parasitic subspaces $I_{c,\sigma,a}$ is stable under taking images and preimages of the morphisms $F_{c,c'}$.

For the purposes of this paper, we will need an equivariant version of the above definition. Each $\mathcal{P}_c$ embeds in $|\mathcal{P}|$, the push-out of $\Phi_{top}(\mathcal{P})$. By abusing the notation we retain the notation $\mathcal{P}_c$ for the image in $|\mathcal{P}|$. Let Θ be a group acting faithfully and isometrically on $\mathcal{C}$. This action extends to a faithful (linear) action $\Theta \curvearrowright \mathcal{P}$. For $\theta \in \Theta \setminus \{1\}$, consider the fixed-point set $Fix(\theta)$ of θ in $|\mathcal{P}|$. For each point $p \in Fix(\theta)$ we take the smallest face $c = c(p)$ such that $p \in \mathcal{P}_c$. By minimality, $\theta(c) = c$. We then let $Fix_c(\theta) := Fix(\theta) \cap \mathcal{P}_c$, it is a finite union of disjoint projective subspaces in $\mathcal{P}_c$. We obtain a set of projective spaces $\mathcal{P}_{c_i}$ (in the example we are mostly interested in this is a single projective space) so that

$$\forall i, \quad \theta(c_i) = c_i, \quad \text{and} \quad Fix(\theta) \subset \bigcup_i Fix_{c_i}(\theta).$$

Assumption 4.9 For every $\theta \in \Theta \setminus \{1\}$ which does not act freely on C, $\theta^2 = 1$ and, moreover, for every c_i as above, $Fix_{c_i}(\theta) = \{p_i\} \sqcup p_i^{\perp}$, where $p_i^{\perp}$ is a codimension 1 projective subspace in $\mathcal{P}_{c_i}$ so that

$$c_i \cap p_i^{\perp} = \emptyset.$$

Furthermore, each p_i belongs to exactly three faces of $\mathcal{C}$: Two facets a_i, b_i and one θ-invariant codimension 1 face c_i incident to a_i, b_i. Note that we do not assume that $\mathcal{C}$ is finite here.

Thus, $p_i^{\perp} \subset \mathcal{P}_{c_i} = \mathcal{P}_{a_i} \cap \mathcal{P}_{b_i}$. However, a priori, $p_i^{\perp} \subset |\mathcal{C}|$ could intersect other strata as well. We would like to eliminate these intersections. (Notice that we will be ignoring parasitic subspaces of $\mathcal{P}_{c_i}$ which could cross $p_i^{\perp}$.)

Namely, for each face c_i we consider strata $\mathcal{P}_{e_i} \subset \mathcal{P}_{c_i}$, where e_i's are proper faces of c_i. For each e_i define $Q_{e_i,p_i} := \mathcal{P}_{e_i} \cap p_i^{\perp}$ provided that this intersection is nonempty. We would like to get rid of the subspaces Q_{e_i,p_i}, so we declare the intersections $Q_{e_i,p_i} \in \mathcal{P}_{c_i}$ to be *secondary parasitic subspaces*. We saturate the collection of secondary parasitic subspaces as we did in the primary case.

Remark 4.10 The case we are mostly interested in is when $\dim(\mathcal{C}) = 3$, so each e_i is an edge and each secondary parasitic subspace Q_{e_i,p_i} is just a point.

We now assume that $\mathcal{C}$ and Θ are both finite. We proceed as in [17] and blow up all the parasitic subspaces: We first blow up all primary parasitic subspaces (by induction on dimension) and then blow up all secondary parasitic subspaces (again, by induction on dimension). The construction is canonical, so the group Θ continues to act on the blow-up $b\mathcal{P}$. By applying Proposition 4.3, we conclude that the Θ-equivariant push-out $X := |b\mathcal{P}|$ exists in the category of projective varieties and is equivariantly homeomorphic to the topological push-out. The same arguments as in [17, Sect. 5] show that the variety X is projective. As in [17], the variety X has only normal crossing singularities.

Furthermore, by the construction, in view of the Assumption 4.9:

Lemma 4.11 1. *$\theta \in \Theta$ has a fixed point in C if and only if θ has a fixed point in X. Such θ has order* 2.

2. *For every $\theta \in \Theta \setminus \{1\}$ which does not act freely on X, every component of $Fix(\theta) \subset X$ is contained in the intersection of exactly two top-dimensional strata* (*of dimension n*) *intersecting normally. The local models for the action of θ are described below.*

Let $y_1, \dots, y_{n+1}$ be coordinates on $\mathbb{C}^{n+1}$. Then:

1. Near an isolated fixed point p_i:

$$y_1 y_2 = 0, \qquad \theta(y_1, y_2, \dots, y_{n+1}) = (y_2, y_1, -y_3, -y_4, \dots, -y_{n+1}).$$

2. Along the $n-2$-dimensional component $bFix_c(\theta)$, (the blow-up of $Fix_c(\theta)$), where $\dim(c) = n-1$:

$$y_1 y_2 = 0, \qquad \theta(y_1, y_2, y_3, \dots, y_{n+1}) = (y_2, y_1, -y_3, y_4, \dots, y_n, y_{n+1}).$$

The case we are mostly interested in is when $n = 3$, so the latter action becomes:

$$y_1 y_2 = 0, \qquad \theta(y_1, y_2, y_3, y_4) = (y_2, y_1, -y_3, y_4).$$

We will refer to these singularities together with the $\mathbb{Z}_2$-actions as $(Y_1, 0)$ and $(Y_2, 0)$ respectively. Notice that if we blow up the origin in $(Y_1, 0)$, then we obtain singularity of the 2nd type.

Notice that Y_2 splits equivariantly as the product $Y \times \mathbb{C}$, where

$$Y = \{(y_1, y_2, y_3) \in \mathbb{C}^3 : y_1 y_2 = 0\}, \qquad \theta(y_1, y_2, y_3) = (y_2, y_1, -y_3)$$

and the action of θ on the remaining factor $\mathbb{C}$ is by the identity. Hence, $Y_2/\mathbb{Z}_2 \cong Y/\mathbb{Z}_2 \times \mathbb{C}$. The variety $Y/\mathbb{Z}_2$ is a normal crossing along the line $y_1 = y_2 = 0$ away from the origin. I am grateful to János Kollár for providing the proof of the following:

Lemma 4.12 *The germ of $Y/\mathbb{Z}_2$ at the origin is isomorphic to the Whitney umbrella $u^2 = wv^2$.*

Proof The ring of invariants $\mathbb{C}[y_1, y_2, y_3]^\Theta$ is generated by the polynomials $y_1 y_2, y_1 + y_2, y_3(y_1 - y_2), y_3^2$ subject to the equation

$$\big(y_3(y_1 - y_2)\big)^2 = y_3^2\big((y_1 + y_2)^2 - 4y_1 y_2\big).$$

Dividing this ring by the ideal generated by $y_1 y_2$ we obtain the ring Q with the generators $u := y_1 + y_2$, $v := y_3(y_1 - y_2)$, $w := y_3^2$ subject to the equation

$$\big(y_3(y_1 - y_2)\big)^2 = y_3^2(y_1 + y_2)^2.$$

Equivalently, Q is generated by u, v, w subject to the equation

$$v^2 = wu^2.$$

However, this is the quotient ring of the Whitney umbrella. □

As in [17], the blow-up $b\mathcal{P}$ is strictly faithful and, hence, $\pi_1(X) \cong \pi_1(C)$, where $C = |\mathcal{C}|$. We let N denote the group $\pi_1(X) = \pi_1(C)$.

Proposition 4.13 *Suppose that N has trivial center. Then $\pi_1(X/\Theta) \cong \pi_1(C/\Theta)$.*

Proof We have a Θ-equivariant isomorphism of fundamental groups $\pi_1(C) \to \pi_1(X)$. Thus, considering the quotient–orbihedra $\mathcal{O}_C := C/\Theta$ and $\mathcal{O}_X := X/\Theta$, we obtain group extensions

$$1 \to N \to \pi_1(\mathcal{O}_C) \to \Theta \to 1, \qquad 1 \to N \to \pi_1(\mathcal{O}_X) \to \Theta \to 1$$

where the homomorphisms $\psi_i : \Theta \to Out(N)$ associated with the actions of Θ on $\pi_1(C)$ and $\pi_1(X)$ are the same. Since N has trivial center, by Corollary 6.8 in [7, Chap. IV], the group extensions above are naturally isomorphic. Define normal subgroups F_C, F_X of the groups $\pi_1(\mathcal{O}_C), \pi_1(\mathcal{O}_X)$ to be the normal closures of the elements of the respective groups, which do not act freely on the universal covers of the orbihedra $\mathcal{O}_C$ and $\mathcal{O}_X$. By Armstrong's theorem [2], the fundamental groups of C and X are obtained from the orbihedral fundamental groups $\pi_1(\mathcal{O}_C), \pi_1(\mathcal{O}_X)$ by dividing by the subgroups F_C, F_X. We claim that the isomorphism $\pi_1(\mathcal{O}_C) \to \pi_1(\mathcal{O}_X)$ carries F_C to F_X isomorphically.

Indeed, let $\tilde{C} \to C$ and $\tilde{X} \to X$ denote the universal covers of C and X respectively. The space $\tilde{C}$ has a natural structure push-out of a polyhedral complex $\tilde{\mathcal{C}}$, while $\tilde{X}$ has a natural structure of push-out of a complex of varieties $\tilde{\mathcal{P}}$ based on $\tilde{\mathcal{C}}$. The strata of $\tilde{X}$ project isomorphically to the strata of X since the latter are simply-connected.

Suppose now that, say, $\tilde{\theta} \in \pi_1(\mathcal{O}_C) \setminus \{1\}$ is a lift of $\theta \in \Theta$ has a fixed point $\tilde{p}$ in the universal cover of C. The isomorphism $\pi_1(C) \to \pi_1(X)$ is induced by the natural embedding of the universal covers $\iota : \tilde{C} \to \tilde{X}$. Therefore, such $\tilde{\theta}$ also fixes the point $\iota(p) \in X$. Conversely, if $\tilde{\theta} \in \pi_1(\mathcal{O}_X) \setminus \{1\}$ fixes a point $\tilde{q}$ in the universal cover of X, then $\tilde{q}$ belongs to a minimal stratum $\tilde{X}_i$ of $\tilde{X}$, which corresponds to a face $\tilde{c}_i$ of $\tilde{\mathcal{C}}$. Then $\tilde{\theta}$ has to preserve $\tilde{X}_i$ and, hence, $\tilde{c}_i$. The projection $\tilde{X}_i \to X_i \subset X$ is an isomorphism conjugating the action of $\tilde{\theta}$ to the action of θ. Since θ was fixing a point in $c_i \in Ob(\mathcal{C})$ (where c_i is the image of $\tilde{c}_i$ under the projection $\tilde{C} \to C$), we conclude that θ also fixes a point in $\tilde{c}_i$. Proposition follows. $\square$

Dimension reduction Let V be the variety obtained from X/Θ as follows: We first equivariantly blow up in X all isolated fixed points of involutions $\theta \in \Theta$ and then divide the resulting variety by Θ. The quotient has only normal crossing singularities and singularities of the 2nd type, more precisely, of the type $Y_2/\mathbb{Z}_2$. These singularities split as the product $Y/\mathbb{Z}_2 \times \mathbb{C}$, where $Y/\mathbb{Z}_2$ is a Whitney umbrella. We now embed V in the projective space and intersect it with a generic hyperplane. The result is a projective surface V whose singularities are only normal crossings and Whitney umbrellas. Furthermore, by Lefschetz Hyperplane section theorem $\pi_1(V) \cong \pi_1(W)$, see [12, p. 27]. Since V was irreducible, so is W: Take an open dense nonsingular subvariety $W^\circ \subset W$; by Lefschetz Hyperplane section theorem W° is again connected. Thus, V is also irreducible since it contains an irreducible open dense subvariety W°.

5 Generic transversality of triples of bisectors in $\mathbb{H}^n$

The main result of this section is

Theorem 5.1 *Let $A_1, A_2, A_3 \in O(n,1)^{\uparrow}$ be distinct nontrivial elements of a group $\Gamma < O(n,1)$ of the class $\mathcal{K}$, $n \geq 2$. Assume also that A_1, A_2, A_3 do not generate a cyclic group. Then for generic $x \in H$, the vectors*

$$B_i(x) = A_i(x) - x, \quad i = 1, 2, 3$$

are linearly independent.

Proof Recall that in (2), we defined the matrix-valued map $x \mapsto B_{\underline{A}}(x)$. Linear dependence of the vectors $B_i(x)$ is equivalent to the condition that $\text{rank}(B_{\underline{A}}(x)) \leq 2$, which, in turn, is expressed in terms of vanishing of determinants of 3×3 minors of the $(n+1) \times 3$ matrix $B_{\underline{A}}(x)$. Therefore, the set of $x \in H$ such that $\text{rank}(B_{\underline{A}}(x)) \leq 2$ is an algebraic subset. Hence, this set is either the entire H or it is a closed set with empty interior.

We suppose therefore, that for every $x \in H$ the vectors $B_i(x)$, $i = 1, 2, 3$ are linearly dependent. Then, by linearity, the same is true for all $x \in \mathcal{L}^{\uparrow}$. Since $\mathcal{L}^{\uparrow}$ is Zariski dense in $\mathbb{R}^{n+1}$, the same conclusion holds for all $x \in \mathbb{R}^{n+1}$.

We let $\Omega \subset C^+ \subset \mathbb{R}^{n+1}$ denote the set of $x \in C^+$ such that $\text{rank}(B_{\underline{A}}(x)) = 2$. Our first goal is to understand the complement of Ω, i.e., the set of $x \in C^+$ such that all the three vectors $B_i(x)$ are multiples of each other. We will consider a (seemingly) larger set

$$\Sigma = \Sigma_{12} \cup \Sigma_{23} \cup \Sigma_{31} \subset C^+$$

where $\Sigma_{ij} = \{x \in C^+ : \dim(Span(B_i(x), B_j(x))) \leq 1\}$.

Lemma 5.2 *Let $x \in C^+$ be a nonzero vector. Then $x \in \Sigma_{12}$ iff one of the following holds*:

1. *$A_1x = x$ or $A_2x = x$ or $A_1x = A_2x$, i.e., x is fixed by $A_2^{-1}A_1$. This can happen only if $x \notin C$.*
2. *x is a common eigenvector of A_1, A_2. This can happen only if $x \in C$.*

Proof If $B_i(x) = 0$ then $A_i(x) = x$. We, thus, will assume that $B_i(x) \neq 0$, $i = 1, 2$.

The condition $x \in \Sigma_{12}$ then is equivalent to

$$B_1(x) = \mu B_2(x), \quad \mu \neq 0.$$

In other words,

$$A_1x - \mu A_2x = (1-\mu)x.$$

1. If $\mu = 1$ then $A_1x = A_2x$, $A_2^{-1}A_1x = x$. Furthermore, every x satisfying these properties belongs to Σ_{12} (by taking $\mu = 1$).

2. Suppose now that $\mu \neq 1$.
   a. If $x \cdot x \neq 0$, Lemma 2.2 then implies that $A_i x = x$, $i = 1, 2$, contradicting our assumption $B_i(x) \neq 0$, $i = 1, 2$.
   b. Suppose that $x \cdot x = 0$. Then linear dependence of the vectors x, $A_1 x$, $A_2 x$ (which belong to the conic C) implies that they belong to a common line in $\mathbb{R}^{n+1}$. In particular, x is a common eigenvector of A_1, A_2. $\square$

Corollary 5.3 *The set Σ_{ij} is a finite union of lines. In particular, Σ does not separate C^+ and, thus, the open set Ω is connected.*

Proof We need to observe two things: First, Γ does not contain elliptic elements besides Cartan involutions. Hence, fixed-point sets and eigenspaces of $A_1, A_2, A_2^{-1}A_1$ in C^+ are at most lines. Secondly, since $n \geq 2$, no line can separate C. Now, the statement follows from Lemma 5.2. $\square$

Recall that we are assuming that for all $x \in C^+$, the vectors $B_i(x)$, $i = 1, 2, 3$ are linearly dependent. Thus, there exist (possibly multivalued and discontinuous) functions $\alpha_i(x)$, $i = 1, 2, 3$ so that for all $x \in C^+$

$$\sum_{i=1}^{3} \alpha_i(x) B_i(x) = 0. \tag{3}$$

If for $x \in C^+$ one can take $\alpha_k(x) = 0$, then $x \in \Sigma_{ij}$, $\{i, j, k\} = \{1, 2, 3\}$. In particular, for each $x \in \Omega$ all the quantities $\alpha_i(x)$ are nonzero. Hence, we can select (say, by setting $\alpha_1(x) \equiv 1$) nonvanishing continuous functions $\alpha_i(x)$, $i = 1, 2, 3$, $x \in \Omega$, so that (3) holds. We fix these functions from now on.

Lemma 5.4 $\Sigma_{12} = \Sigma_{23} = \Sigma_{31} = \Sigma = C^+ \setminus \Omega$.

Proof The following argument is borrowed from [9]. Set $V = \mathbb{R}^{n+1}$. For $i \neq j$ consider the rational maps $\Phi_{ij} : [x] \in \mathbb{P}V \to \mathbb{P}(\Lambda^2 V)$, given by projectivization of the correspondence

$$x \mapsto B_i(x) \wedge B_j(x)$$

It is clear that the domain of Φ_{ij} in C^+ is $C^+ \setminus \Sigma_{ij}$ and Φ_{ij} does not extend continuously to any point of Σ_{ij}. The assumption that the vectors $B_i(x)$, $i = 1, 2, 3$ are linearly dependent for all x implies that $\Phi_{12} = \Phi_{23} = \Phi_{31}$. Therefore, their sets of indeterminacy $\Sigma_{12}, \Sigma_{23}, \Sigma_{31}$ are also equal. In particular, for every $x \in \Sigma$, $\mathrm{rank}(B_{\underline{A}}(x)) = 1$. $\square$

We now begin the actual proof of Theorem 5.1.

Case 1 (The generic case) A_i, $i=1,2,3$ are all loxodromic and no two of them have a common eigenvector in C. In particular, in view of Lemma 5.2, every eigenvector $x \in C$ of A_i, $i=1,2,3$, belongs to Ω.

Let $x_1 \in C$ be an eigenvector of A_1 with the eigenvalue $\lambda_1 > 1$. Then for $x = x_1$, (3) implies:

$$\big[\alpha_1(x)(\lambda_1 - 1) - \alpha_2(x) - \alpha_3(x)\big]x + \alpha_2(x)A_2(x) + \alpha_3(x)A_3(x) = 0.$$

Note that all three vectors $x, A_2(x), A_3(x)$ belong to the cone C. If these vectors were to span a plane P, then P would intersect the quadric C along three distinct lines, which is absurd. Thus, the vectors

$$\big[\alpha_1(x)(\lambda_1 - 1) - \alpha_2(x) - \alpha_3(x)\big]x, \qquad \alpha_2(x)A_2(x), \qquad \alpha_3(x)A_3(x)$$

belong to the same line. Since $x \in \Omega$, the last two vectors are nonzero.

a. Suppose that $[\alpha_1(x)(\lambda_1 - 1) - \alpha_2(x) - \alpha_3(x)] \neq 0$. Then x is a common eigenvector for A_2, A_3 contradicting the assumptions of Case 1.
b. Thus, $[\alpha_1(x)(\lambda_1 - 1) - \alpha_2(x) - \alpha_3(x)] = 0$ and

$$\alpha_2(x)A_2(x) + \alpha_3(x)A_3(x) = 0.$$

Then, since $A_2(x), A_3(x) \in C^+$ (the future light cone), it follows that $\alpha_2(x)$, $\alpha_3(x)$ have to have opposite signs. By applying the same argument to the eigenvectors x_i of A_i, $i=2,3$, we obtain:

$$\alpha_i(x_k)\alpha_j(x_k) < 0,$$

for all choices of pairwise distinct $i, j, k \in \{1,2,3\}$.

It immediately follows that there is an index $i \in \{1,2,3\}$ such that the function $\alpha_i(x)$ changes its sign on the set $\Omega \cap C$. However, Ω is connected and $\alpha_i(x) \neq 0$ on Ω. Contradiction.

Case 2 Suppose that A_i, $i=1,2,3$ are all loxodromic and A_1, A_2 share a common eigenvector in C.

Then discreteness of Γ implies that A_1, A_2 share both eigenvectors in C (see Sect. 2). Let P_{12} be the plane spanned by these eigenvectors. If $\alpha_3(x) = 0$ for some $x \in H \cap P_{12}$, then (by (3)) we get

$$\alpha_1(x)A_1(x) + \alpha_2(x)A_2(x) = \big(\alpha_1(x) + \alpha_2(x)\big)x.$$

By Lemma 2.2, it follows that $A_1(x) = A_2(x) = x$, contradicting the assumption that A_1, A_2 are loxodromic. Thus, $\alpha_3(x) \neq 0$ for all $x \in H \cap P_{12}$. Then for all $x \in H \cap P_{12}$, (3) implies that $A_3(x)$ is a linear combination of $A_1(x), A_2(x), x$ which all belong to P_{12}. It then follows that A_3 preserves

$L = H \cap P_{12}$, i.e., A_1, A_2, A_3 have a common axis in the hyperbolic plane. Consider the group $\langle A_1, A_2, A_3\rangle$ generated by A_1, A_2, A_3. This group acts discretely on L (since is a subgroup of the discrete group $\Gamma < PO(3,1)$). If the action of $\langle A_1, A_2, A_3\rangle$ on L were not faithful, this group would contain an elliptic element fixing L pointwise. This contradicts our assumption that all elliptic elements of Γ are Cartan involutions. Hence, the group $\langle A_1, A_2, A_3\rangle$ acts faithfully on L as a discrete group of translations. Therefore, $\langle A_1, A_2, A_3\rangle \cong \mathbb{Z}$, contradicting the hypothesis of Theorem 5.1.

Case 3 A_1, A_2 are loxodromic and $A_3 = J$ is elliptic (a Cartan involution). Let $p \in H$ be the unique fixed vector of A_3. Then $p = p_3 \in \Sigma_{13}$ (see Lemma 5.2). By Lemma 5.4, it follows that $p \in \Sigma_{12}$. Since p is not an eigenvector of A_i, $i = 1, 2$, it follows (by Lemma 5.2) that $A_1(p) = A_2(p)$, i.e., $A_2^{-1}A_1(p) = p$. Thus, $A_2^{-1}A_1$ is elliptic fixing p. Since Γ belongs to the class $\mathcal{K}$, $A_2^{-1}A_1$ is a Cartan involution. Therefore, $A_2^{-1}A_1 = J = A_3$ (since a Cartan involution is determined by its fixed point).

Our goal is to obtain a contradiction. Let $x_i^{\pm} \in C$ be the eigenvectors of A_i, $i = 1, 2$ with eigenvalues $\lambda_i^{\pm 1}$, $i = 1, 2$. Let $x = x_1^{\pm}$. Note that x cannot be an eigenvector of J.

Lemma 5.5 *Either x is an eigenvector of A_2, or $J(x)$ is a multiple of $A_2(x)$.*

Proof Our proof is similar to the argument in Case 1. We will assume that x is not an eigenvector of A_2. Then, by Lemma 5.2, $x \in \Omega$. In particular, $\alpha_i(x) \neq 0$, $i = 1, 2, 3$. We have:

$$\bigl[\alpha_1(x)\bigl(\lambda_1^{\pm 1} - 1\bigr) - \alpha_2(x) - \alpha_3(x)\bigr]x + \alpha_2(x)A_2(x) + \alpha_3(x)J(x) = 0.$$

As before, the vectors x, $A_2(x)$, $A_3(x)$ belong to the cone C. Thus, the vectors

$$\bigl[\alpha_1(x)(\lambda_1 - 1) - \alpha_2(x) - \alpha_3(x)\bigr]x, \qquad \alpha_2(x)A_2(x), \qquad \alpha_3(x)A_3(x)$$

belong to the same line and the last two vectors are nonzero.

a. Suppose that $[\alpha_1(x)(\lambda_1^{\pm 1} - 1) - \alpha_2(x) - \alpha_3(x)] \neq 0$. Then x is a common eigenvector of A_2, A_3 contradicting our assumption that $A_3 = J$ is a Cartan involution.
b. Thus, $\alpha_2(x)\alpha_3(x) \neq 0$ and $[\alpha_1(x)(\lambda_1^{\pm 1} - 1) - \alpha_2(x) - \alpha_3(x)] = 0$. Hence, $A_2(x)$, $J(x)$ are multiples of each other. □

The same argument, of course, applies to the eigenvectors of A_2.

Corollary 5.6 *One of the following holds*:

a. A_1, A_2 *have a common axis and commute*.
b. *For each* $i = 1, 2$, $C_i = A_{i+1}^{-1} J$ *and* A_i *generate a cyclic group* (i *is taken modulo* 2).

Proof If A_1, A_2 has a common eigenvector in C, then, by discreteness of Γ, they share both eigenvectors in C and, hence, have a common axis in $\mathbb{H}^3$. Since $A_1, A_2 \in \Gamma$ and Γ is in the class $\mathcal{K}$, it follows that A_1, A_2 commute. Thus, suppose that $[A_1, A_2] \neq 1$. Then, by Lemma 5.5, for $x = x_i^{\pm}$, $C_i = A_{i+1}^{-1} J(x)$ is a multiple of x. Hence, the elements A_i, C_i share both eigenvectors in C. Therefore, they have the same axis in $\mathbb{H}^n$ and, since Γ is in the class $\mathcal{K}$, these elements have to generate a cyclic group. $\square$

Note that, since $A_3 = J = A_2^{-1} A_1$, it follows that in the case (a) of this corollary, all three elements A_1, A_2, A_3 commute. This is impossible since A_1, A_2 are loxodromic and J is a Cartan involution. Thus, (b) holds for both $i = 1, 2$ and $A_i, C_i = A_{i+1}^{-1} J$ generate a cyclic group.

Combining the equations

$$A_2 = A_1 J, \qquad A_2 = J C_1^{-1},$$

we obtain

$$J A_1 J = C_1^{-1}. \tag{4}$$

Therefore, J preserves the axis L_1 of A_1 in $\mathbb{H}^n$. Since J is a Cartan involution, it has to reverse orientation on L_1. We write $A_1 = AR$, where A is a hyperbolic element with the axis L_1 and R is an elliptic element fixing L_1 pointwise. In particular, $RJ = JR$. Then, $C_1^{-1} = A^{-1} R$ and $C_1 = AR^{-1} = A_1 R^{-2}$ and

$$A_1 C_1^{-1} = R^2.$$

Since $A_1, C_1 \in \Gamma$, we also have $R^2 \in \Gamma$. By our assumptions on elliptic elements of Γ, $R^2 = 1$. Thus, $C_1 = A_1$. For the same reason, $C_2 = A_2$. Hence, by (4), J anticommutes with both A_1, A_2. In particular, the fixed point of J belongs to $L_1 \cap L_2$ and J preserves both L_1 and L_2.

However, $A_2 = A_1 J$ and, since A_1, J preserve L_1, it follows that A_2 also preserves L_1 as well, i.e., A_1, A_2 are loxodromic isometries with the common axis $L = L_1 = L_2$. But then the composition $J = A_2^{-1} A_1$ has to be either loxodromic or elliptic fixing L or the identity. This contradicts the assumption that J is a Cartan involution.

Case 4 A_2, A_3 are (distinct) elliptic of order 2, so $A_i = A_i^{-1}$, $i = 2, 3$. (We make no assumptions about A_1 apart from $A_1 \neq A_2$, $A_1 \neq A_3$.) The same arguments as in Case 3 (considering fixed points p_2, p_3 of A_2, A_3) show that $A_2A_1 = A_2^{-1}A_1 = A_3$, $A_3A_1 = A_3^{-1}A_1 = A_2$. Thus,

$$A_2 = A_1A_3 = A_3A_1$$

and, hence A_1, A_3 commute. Since A_3 is a Cartan involution and A_1 is loxodromic or Cartan, it follows that $A_1 = A_3$, which contradicts the assumption that the elements A_1, A_2, A_3 are all distinct.

This concludes the proof of Theorem 5.1. □

Remark 5.7 After completing this paper I received the preprint [24] by A. Ushijima where Theorem 5.1 is proven for triples of loxodromic elements of the group $SO(3,1)^{\uparrow} \cong PSL(2,\mathbb{C})$. The arguments in [24] are different from the ones used in our proof.

6 Dirichlet domains of cyclic loxodromic groups

In this section we discuss the exceptional case of triples of elements of cyclic loxodromic groups. The following result is implicit in the work of T. Drumm and J. Poritz [10, Sects. 5, 7], who analyzed Dirichlet fundamental domains of cyclic subgroups of $SO(3,1)^{\uparrow}$ in great detail:

Theorem 6.1 *Let $\langle A\rangle < SO(3,1)^{\uparrow} \cong PSL(2,\mathbb{C})$ be a cyclic loxodromic group. Then the Dirichlet tiling $\mathcal{D}_x$ of $\langle A\rangle$ in $\mathbb{H}^3$ is simple for every choice of $x \in \mathbb{H}^3$.*

Conjecture 6.2 *The same conclusion holds for all cyclic loxodromic subgroups of $PO(n,1)$, $n \geq 3$.*

Proof of Theorem 6.1 We will be using notation and terminology of [10]. In particular, we will use the notation $X_n = Bis(x, A^n x)$, $n \neq 0$, for the bisectors. We will be using the notation $\Delta := D_x$ for a fixed Dirichlet domain, and F_n for $X_n \cap \Delta$, provided that this intersection is 2-dimensional. We also use the notation S_n for the intersections of the ideal boundaries of X_n and Δ, provided that this intersection is 1-dimensional (an edge of the circular polygon $\partial_\infty \Delta$).

1. Let $v \in \mathbb{H}^3$ be a vertex of $\Delta = D_x$. According to the conclusion on the bottom of page 177 of [10], the vertex v is *splendid*, i.e., it belongs to exactly three faces F_i, F_j, F_{i+j} of Δ. If $\mathcal{D}_x$ is not simple at v, by Lemma 3.9 and the following remark, v belongs to a bisector X_n so that $X_n \cap \Delta$ is not a 2-face. By Proposition 7.6 of [10], $X_n \cap \Delta \neq v$. By Proposition 7.7 of [10], X_n cannot contain a finite edge E of Δ incident to v. By Proposition 7.8 of [10],

X_n cannot contain an infinite edge E of Δ incident to v. Thus, $\mathcal{D}_x$ is simple at v.

2. Let E be a bi-infinite edge of Δ such that $\mathcal{D}_x$ is not simple along E. By Lemma 3.9, $E = X_n \cap \Delta$ for some bisector X_n. Let $v, w \in S^2$ denote the ideal boundary points of E: $\partial_\infty E = \{v, w\}$. Then the ideal boundary circle of X_n passes through v, w. By Corollary 5.7 of [10], it follows that the ideal boundary of Δ has exactly four sides and, by Lemma 5.5 of [10] these sides are: S_i, S_j, S_{-i}, S_{-j}. Without loss of generality, we can assume that $v = S_i \cap S_j$. Then, by Corollary 5.7 of [10], $n = i + j$. Up to relabeling, there are two options for the vertex w:

a. $w = S_{-i} \cap S_{-j}$. However, by Lemma 5.5 of [10], $w \in \partial_\infty X_{-i-j} \cap \partial_\infty \Delta$. Since the involution ϕ defined in [10] swaps $\partial_\infty X_{i+j} \cap \partial_\infty \Delta$ and $\partial_\infty X_{-i-j} \cap \partial_\infty \Delta$, it follows that

$$\partial_\infty X_{-i-j} \cap \partial_\infty \Delta = \{v, w\}.$$

By repeating the arguments in the proof of Proposition 4.4 of [10], we see that $-i - j = n$. Hence, $-i - j = n = i + j$ and $n = 0$, contradiction.

b. $w = S_{-j} \cap S_i$. Then $w \in \partial_\infty X_{i-j} \cap \partial_\infty \Delta$. In this case there is no reason to expect that $\{v, w\} = \partial_\infty X_{i-j} \cap \partial_\infty \Delta$. Nevertheless, by Proposition 4.5 of [10], we get:

$$A^{j-i}(w) \in A^{-n}\big(\{v, w\}\big).$$

If $A^{j-i}(w) = A^{-n}(w)$ then (as in the proof of Proposition 4.4 of [10]) $i + j = n = j - i$ and, hence, $i = 0$, contradiction. If $A^{j-i}(w) = A^{-n}(v) = A^{-i-j}(v)$ then

$$A^{-2j}(v) = w.$$

However, by looking at the fundamental domain $\partial_\infty \Delta$ we also see that

$$A^{-j}(v) = w,$$

thus $-2j = -j$ and $j = 0$. Contradiction. $\square$

In view of Theorem 6.1, it remains to consider cyclic subgroups of $O(3,1)^\uparrow$ generated by orientation-reversing loxodromic isometries A. Such isometries A are called *glide-reflections*: A is the composition $A_0 R$, where A_0 is hyperbolic and R is a reflection in a hyperplane containing the axis L of A_0.

Proposition 6.3 *For every A as above, the Dirichlet tiling $\mathcal{D}_x$ of $\langle A \rangle$ in $\mathbb{H}^3$ is simple for every choice of $x \in \mathbb{H}^3$.*

Proof One can, in principle, go through the proofs given in [10] and modify them when necessary in order to allow orientation-reversing loxodromic elements. Instead, we will give a direct argument.

Let $L \subset \mathbb{H}^3$ denote the axis of A. Since A is orientation-reversing, there exists a hyperbolic plane $P \subset \mathbb{H}^3$ containing L, invariant under A, so that A reverses orientation on P. Hence, A preserves the half-spaces bounded by P. For a point $x \in \mathbb{H}^3$ let $x_P \in P$ denote the point nearest to x. The nearest-point projection $x \to x_P$ commutes with the action of A.

Lemma 6.4 *For every $x \in \mathbb{H}^3$ and $n \in \mathbb{Z} \setminus \{0\}$, the bisector $Bis(x, A^n x)$ is orthogonal to P and*

$$Bis\big(x, A^n x\big) = Bis\big(x_P, A^n x_P\big).$$

Proof We set $y := A^n x$ and let $p \in \mathbb{R}^{3,1}$ be such that $P = p^\perp \cap H$. The extended bisector $\widetilde{Bis}(x, y) \subset \mathbb{R}^{3,1}$ equals $(x - y)^\perp$. Computing $(x - A^n x) \cdot p$ and using the fact that $Ap = p$, we obtain: $(x - A^n x) \cdot p = x \cdot p - x \cdot A^{-n} p = 0$. Thus, $p \in \widetilde{Bis}(x, y) \subset \mathbb{R}^{3,1}$. Therefore, since $\widetilde{Bis}(x, y) = (x - y)^\perp$ and $P = p^\perp \cap H$, the hyperplanes $\beta := Bis(x, y)$ and P in $\mathbb{H}^3$ are orthogonal. Let $R_\beta \in PO(3, 1)$ be the isometric reflection in the hyperplane β. Since β is orthogonal to P, R_β preserves P. In particular, R_β commutes with the projection $z \to z_P, z \in \mathbb{H}^3$. Since $R_\beta(x) = y$, it follows that $R_\beta(x_P) = y_P$. Therefore, $\beta = Bis(x, y)$ is the bisector for $x_P, y_P = A^n x_P$ as well. ☐

In view of this lemma, Dirichlet tilings $\mathcal{D}_x$ and $\mathcal{D}_{x_P}$ (with respect to the group $\langle A\rangle$) are the same. Therefore, it suffices to prove simplicity of the Dirichlet tilings $\mathcal{D}_{x_P}$ of $\langle A\rangle$ on $P = \mathbb{H}^2$. We, thus, assume that $x \in P$. The isometry A of $\mathbb{H}^2$ is the composition of the hyperbolic isometry A_0 preserving L and the reflection R in $\mathbb{H}^2$ fixing L.

Let $x_L \in L$ denote the point in L nearest to x. Again, the nearest-point projection $x \to x_L$ commutes with the action of A. For $n \in \mathbb{Z} \setminus \{0\}$ we let $m_n \in L$ denote the midpoint of $x_L, A^n x_L$. We claim that $Bis(x, A^n x)$ passes through m_n. Indeed, similarly to Lemma 6.4,

$$Bis\big(A_0^n x, x\big) = Bis\big(A^n x_L, x_L\big).$$

Hence, $d(x, m_n) = d(A_0^n x, m_n)$, while

$$d\big(m_n, A_0^n x\big) = d\big(m_n, R^n A_0^n x\big) = d\big(m_n, A^n x\big).$$

Thus, $d(x, m_n) = d(A^n x, m_n)$ and $m_n \in Bis(x, A^n x)$ proving the claim.

We now consider the bisectors $Bis(x, A^{\pm 1} x), Bis(x, A^{\pm 2} x)$. These bisectors bound a convex polygon $F \subset \mathbb{H}^2$ (of infinite area) containing x.

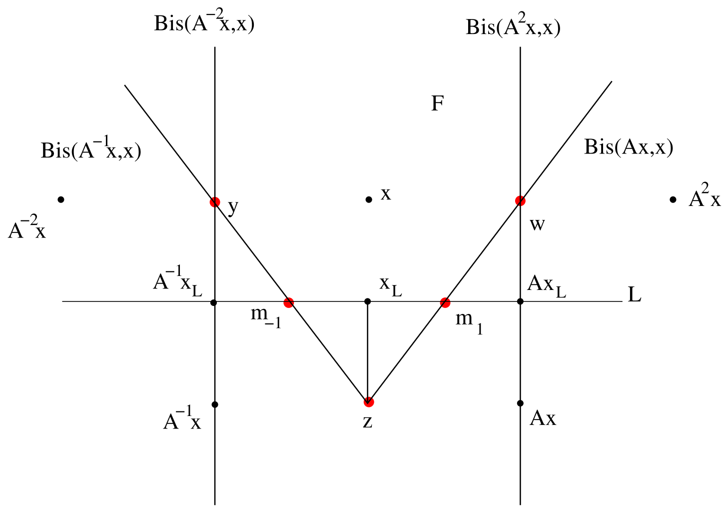

Fig. 2 Bisectors for the cyclic loxodromic group $\langle A\rangle$

The vertices of this polygon are $y = Bis(A^{-2}x, x) \cap Bis(A^{-1}x, x)$, $z = Bis(A^{-1}x, x) \cap Bis(Ax, x)$ and $w = Bis(A^2x, x) \cap Bis(Ax, x)$. We next observe that

$$d(y, L) = d(z, L) = d(w, L).$$

This follows from congruence of the triangles

$$\Delta(ym_{-2}m_{-1}), \qquad \Delta(m_{-1}x_Lz), \qquad \Delta(m_1x_Lz), \qquad \Delta(wm_2m_1).$$

Since A sends $Bis(A^{-1}x, x)$ to $Bis(Ax, x)$ and preserves the distance to L, it follows that

$$y \xrightarrow{A} z \xrightarrow{A} w$$

and $A^2 : y \to w$. Thus, the polygon F is a fundamental domain for the action of the group $\langle A\rangle$ on $\mathbb{H}^2$. Since, clearly, $D_x \subset F$, we have $D_x = F$. Furthermore, the only vertex-cycle of the fundamental domain F is

$$y \xrightarrow{A} z \xrightarrow{A} w \xrightarrow{A^{-2}} y,$$

which has length 3. Therefore, in the Dirichlet tiling $\mathcal{D}_x$ in $\mathbb{H}^2$, there are exactly three fundamental tiles adjacent to each of the y, z, w. See Fig. 2. Hence, $\mathcal{D}_x$ is simple. Proposition 6.3 follows. □

7 Two examples

Example 7.1 There exists a cyclic loxodromic subgroup $\langle A\rangle < SO(3,1)^{\uparrow}$ for which there exists an open nonempty subset $U \subset \mathbb{H}^3$ so that for all $x \in U$ the triple intersection of bisectors

$$Bis\big(A^{-1}x, x\big) \cap Bis\big(A^2x, x\big) \cap Bis\big(A^3x, x\big) \subset \mathbb{H}^3$$

is non-transversal (i.e., is a hyperbolic geodesic).

Let $A \in PO(2,1)$ be an orientation-reversing loxodromic isometry of $\mathbb{H}^2$. We extend A to an orientation-preserving isometry of $\mathbb{H}^3$ (also denoted A). We will consider a triple of distinct nontrivial elements $A_1, A_2, A_3 \in \langle A\rangle$ such that A_1, A_3 are orientation-reversing and A_2 is orientation-preserving isometries of $\mathbb{H}^2$.

Let $B : \mathbb{R}^4 \to Mat_{4,3}$ be the associated mapping $x \mapsto (B_1(x), B_2(x), B_3(x))$, where $B_i = A_i - I$, see (2).

Lemma 7.2 *B is singular if and only if $A_2 = A_1A_3$.*

Proof We choose the basis of eigenvectors e_1, e_2, e_3, e_4 of A in $\mathbb{R}^4$, where $e_1, e_2 \in C$ are eigenvectors with eigenvalues $\lambda, \lambda^{-1} \neq 1$, $e_1 \cdot e_2 = -1$, the unit vectors e_3, e_4 are orthogonal to e_1, e_2 and each other and

$$Ae_i = -e_i, \quad i = 3, 4.$$

We now consider vectors $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ so that $x_1x_2x_3x_4 \neq 0$. Let $\lambda_i, \lambda_i^{-1}$ be the eigenvalues of A_i corresponding to the eigenvectors e_1, e_2 respectively. Consider the 4×3 matrix $B(x)$, $x \in \mathbb{R}^4$. The two bottom rows of this matrix are $x_i[-2, 0, -2]$, $i = 3, 4$. Hence, $\operatorname{rank}(B(x)) = 2$ if and only if the following determinant equals zero:

$$\Delta = \begin{vmatrix} \lambda_1 - 1 & \lambda_2 - 1 & \lambda_3 - 1 \\ \lambda_1^{-1} - 1 & \lambda_2^{-1} - 1 & \lambda_3^{-1} - 1 \\ 1 & 0 & 1 \end{vmatrix}.$$

(Hence, this is independent of x.) Computing Δ we obtain:

$$\Delta = \begin{vmatrix} \lambda_1 - 1 & \lambda_2 - 1 \\ \lambda_1^{-1} - 1 & \lambda_2^{-1} - 1 \end{vmatrix} + \begin{vmatrix} \lambda_2 - 1 & \lambda_3 - 1 \\ \lambda_2^{-1} - 1 & \lambda_3^{-1} - 1 \end{vmatrix} = \begin{vmatrix} \lambda_2 - 1 & \lambda_3 - \lambda_1 \\ \lambda_2^{-1} - 1 & \lambda_3^{-1} - \lambda_1^{-1} \end{vmatrix}.$$

This determinant equals 0 iff

$$\frac{\lambda_2 - 1}{\lambda_2^{-1} - 1} = \frac{\lambda_3 - \lambda_1}{\lambda_3^{-1} - \lambda_1^{-1}} \quad \Longleftrightarrow \quad \lambda_2 = \lambda_1 \lambda_3.$$

Equivalently, $A_2 = A_1 A_3$. □

Remark 7.3 By adopting arguments from [10], one can prove that the same conclusion holds for all triples of loxodromic elements $\underline{A} = \{A_1, A_2, A_3\}$ of $PO(n, 1)$ generating a cyclic group: After reordering these elements if necessary, $A_2 = A_1 A_3$ if and only if the associated map $B_{\underline{A}}$ is singular.

To get a specific example, we will take $A_1 = A^{-1}$, $A_2 = A^2$, $A_3 = A^3$. Our next goal is to find conditions on x and λ under which the Gramm matrix $Gr(x)$ of the vectors $\{B_1(x), B_2(x)\}$ is positive-definite, i.e., when the restriction of the Lorentzian inner product to

$$Span\big(B_1(x), B_2(x)\big)^{\perp}$$

is indefinite, that is, $Bis(A_1 x, x) \cap Bis(A_2 x, x) \cap H \neq \emptyset$. We have:

$$Gr(x) = \begin{bmatrix} -2(\lambda - 1)(\lambda^{-1} - 1)x_1 x_2 + 4x_3^2 & -\mu x_1 x_2 \\ -\mu x_1 x_2 & -2(\lambda^2 - 1)(\lambda^{-2} - 1)x_1 x_2 \end{bmatrix}$$

where

$$\mu = (\lambda - 1)\big(\lambda^2 - 1\big) + \big(\lambda^{-1} - 1\big)\big(\lambda^{-2} - 1\big) = (\lambda - 1)\big(\lambda^2 - 1\big)\big(1 + \lambda^{-3}\big).$$

Then

$$\det\big(Gr(x)\big) = -\nu^2 x_1^2 x_2^2 + 8\big(\lambda^2 - 1\big)^2 \lambda^{-2} x_1 x_2 x_3^2,$$

where

$$\nu = (\lambda - 1)\big(\lambda^2 - 1\big) - \big(\lambda^{-1} - 1\big)\big(\lambda^{-2} - 1\big) = (\lambda - 1)\big(\lambda^2 - 1\big)\big(1 - \lambda^{-3}\big).$$

In addition, we have the condition $x \in \mathcal{L}^{\uparrow}$, i.e., $x_3^2 < 2x_1 x_2$. We now fix $x \in H$ such that $x_3 \neq 0$ and let $\lambda \to 1+$. Then,

$$\big((\lambda - 1)\big(\lambda^2 - 1\big)\big(1 - \lambda^{-3}\big)\big)^2 \sim (\lambda - 1)^6 = o\big(\big(\lambda^2 - 1\big)^2 \lambda^{-2}\big)$$

as $\lambda \to 1+$. This means that each all $\lambda > 1$ sufficiently close to 1, $\det(Gr(x)) > 0$ and, hence, the open set

$$U_\lambda = \big\{x \in H : \det\big(Gr(x)\big) > 0\big\}$$

is nonempty. Hence, $Bis(A_1x, x) \cap Bis(A_2x, x) \cap Bis(A_3x, x) \cap H \neq \emptyset$ for all $x \in U_\lambda$. The reader who enjoys computations will verify that for every λ with

$$1 < \lambda < \frac{3+\sqrt{5}}{2} \approx 2.6,$$

the set U_λ is nonempty. Therefore, for all such λ, there exists an open nonempty set $U_\lambda \subset \mathbb{H}^3$ so that for all $x \in U_\lambda$

$$Bis(A_1x, x) \cap Bis(A_2x, x) \cap Bis(A_3x, x) \subset \mathbb{H}^3$$

is a complete hyperbolic geodesic, i.e., the triple intersection of bisectors in $\mathbb{H}^3$ is nontransversal. Furthermore, the loxodromic elements A_i belong to a cyclic group $\langle A \rangle$ of orientation-preserving isometries of $\mathbb{H}^3$ that stabilize a hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. (The group $\langle A \rangle$ does not preserve the orientation on $\mathbb{H}^2$.)

Note that the example constructed above does not contradict Theorem 6.1: The nontransversal triple intersections do not occur on the boundary of the Dirichlet domain D_x. In our second example of a discrete abelian subgroup of $SO(3, 1)$, such nontransversal intersections occur on the boundary of D_x for an open nonempty set of $x \in \mathbb{H}^3$.

Example 7.4 Let A be a hyperbolic isometry of $\mathbb{H}^3$ with the axis L and let $R \in PO(3, 1)$ be the order 2 elliptic rotation around L. We consider the abelian group $\Gamma := \langle A, R \rangle$ generated by these isometries. Let $A_1 := R$, $A_2 := A$, $A_3 := RA$. Then for all $x \in \mathbb{H}^3 \setminus L$ the triple intersection of bisectors

$$I_x = Bis(A_1x, x) \cap Bis(A_2x, x) \cap Bis(A_3x, x) \subset \mathbb{H}^3$$

is a geodesic. Furthermore, the geodesic I_x is contained in the boundary of the Dirichlet domain D_x of Γ.

Note that in this example, Γ preserves hyperbolic planes $P \subset \mathbb{H}^3$ containing L and x (Γ reverses the orientation on P). We first compute the fundamental domain D_x for Γ: It is the solid S in $\mathbb{H}^3$ containing x and bounded by the bisectors $Bis(Ax, x), Bis(A^{-1}x), Bis(x, Rx)$. Indeed, clearly, S contains D_x. On the other hand, S is a fundamental polyhedron for Γ which can be easily verified using the Poincaré's fundamental domain theorem, see e.g. [19], [21, Sect. 6.6]. Since D_x is also a fundamental domain of Γ, it follows that $S = D_x$. Next, the intersection

$$I_x := Bis(Ax, x) \cap Bis(Rx, x) \subset \mathbb{H}^3$$

is a hyperbolic geodesic in $\mathbb{H}^3$ contained in the boundary of D_x (since the bisectors $Bis(Ax, x), Bis(A^{-1}x)$ are disjoint). Set $Q := Bis(Rx, x)$. Then the reflection R_Q in the hyperplane Q sends x to Rx and Ax to RAx and fixes I_x. Therefore, for every $y \in I_x$, $d(y, x) = d(y, Ax) = d(y, RAx)$. Hence,

$$I_x = Bis(Ax, x) \cap Bis(Rx, x) \cap Bis(RAx, x). \qquad \square$$

8 Proof of Theorem 1.6

We now can prove Theorem 1.6. In view of Lemma 3.9, we need to prove that for generic choice of $x \in \mathbb{H}^3$, for every edge $E \subset D_x = D_x(\Gamma)$, E is the intersection of exactly two bisectors $Bis(A_1 x, x), Bis(A_2, x, x)$, where $A_1, A_2 \in \Gamma$. First, for every triple $\underline{A} = \{A_1, A_2, A_3\}$ of nontrivial distinct elements of Γ which do not belong to a common cyclic subgroup, we define the set $\mathcal{E}(\underline{A})$ consisting of those $x \in \mathbb{R}^4$ for which the intersection

$$\bigcap_{i=1}^{3} \widetilde{Bis}(A_i x, x) \subset \mathbb{R}^4$$

has dimension ≥ 2. In other words,

$$\mathcal{E}(\underline{A}) = \left\{x \in \mathbb{R}^4 : \operatorname{rank}\left(B_{\underline{A}}(x)\right) \leq 2\right\},$$

see Sect. 3 for the notation.

The set $\mathcal{E}(\underline{A})$ is clearly algebraic in $\mathbb{R}^4$ and is stable under multiplication of x by scalars. According to Theorem 5.1, $\mathcal{E}(\underline{A})$ is a proper algebraic subset of $\mathbb{R}^4$. In particular, its intersection with $\mathcal{L}^{\uparrow}$ is closed and has topological dimension ≤ 3. Since $\mathcal{E}(\underline{A})$ is stable under scaling, the intersection $\mathcal{E}_H(\underline{A}) := \mathcal{E}(\underline{A}) \cap H$ is nowhere dense. Since Γ is countable, the union $\mathcal{E}_H$ of the subsets $\mathcal{E}_H(\underline{A})$ (taken over all triples $\underline{A}$ of distinct nontrivial elements of Γ generating non-cyclic groups) is nowhere dense in $H = \mathbb{H}^3$. Next, consider the triples $\{A_1, A_2, A_3\}$ of distinct nontrivial elements of Γ generating cyclic subgroups. For such a triple, by Theorem 6.1 and Proposition 6.3,

$$\bigcap_{i=1}^{3} Bis(A_i x, x) \subset \mathbb{H}^3$$

is disjoint from $D_x(\langle A \rangle)$ for every choice of $x \in \mathbb{H}^3$. Since $D_x(\Gamma) \subset D_x(\langle A \rangle)$, it follows that such nontransversal triple intersection is disjoint from $D_x(\Gamma)$ as well, so we can ignore such triples. Thus, we conclude that $\mathcal{D}_x$ is weakly simple for all $x \in \mathbb{H}^3 \setminus \mathcal{E}_H$.

It remains now to show that for generic x, fixed points of Cartan involutions in Γ do not belong to any edge of $\mathcal{D}_x$.

Lemma 8.1 *Let $p \in H$, $A \in O(3,1)^{\uparrow}$ is an element not fixing p. Then there exists a hyperbolic plane $Q_p \subset H$ such that for all $x \in H \setminus Q_p$, $p \notin Bis(Ax, x)$.*

Proof $p \in Bis(Ax, x)$ if and only if:

$$p \cdot (Ax - x) = 0 \quad \Longleftrightarrow \quad (Ap - p) \cdot x = 0$$

Since $Ap \neq p$, the orthogonal complement to the vector $Ap - p$ is a hyperplane in $\mathbb{R}^4$. For every x away from this hyperplane, $p \notin Bis(Ax, x)$. □

Since Γ is countable, $\mathbb{H}^3$ contains only countably many fixed points p_i, $i \in \mathbb{N}$, of Cartan involutions that belong to Γ. Therefore, for every

$$x \in \mathbb{H}^3 \Big\backslash \left(\mathcal{E}_H \cup \bigcup_{i \in \mathbb{N}} Q_{p_i} \right),$$

for every Cartan involution $J \in \Gamma$, the fixed point p of J in H belongs to the unique bisector $Bis(Ax, x)$, $A \in \Gamma$, namely, $Bis(Jx, x)$. Hence, p cannot belong to an edge of $\mathcal{D}_x(\Gamma)$. Theorem 1.6 follows. □

9 3-dimensional hyperbolic orbifolds

The goal of this section is to prove Theorem 1.4. Our proof is a minor variation of the one in [20].

We recall that an *orbihedron* O is a topological space $|O|$ (the *underlying space* of O) together with an atlas where each chart is the quotient U_α / G_α of a polyhedral complex U_α by a finite PL group action $G_\alpha \curvearrowright U_\alpha$, satisfying certain compatibility conditions, see e.g. [13]. An orbihedron is called an *orbifold* if the polyhedral complexes above are PL manifolds. The *singular set* Σ_O of an orbihedron is the subset of $|O|$ consisting of points x which are covered by $\tilde{x} \in U_\alpha$ with nontrivial stabilizer in G_α. The *order* of a singular point x is the order of the stabilizer of $\tilde{x}$ in G_α. An orbifold is called a DISK, an ANNULUS or a TORUS, if it is the quotient of a disk or an annulus or a torus by a finite group action. (See [16].) For instance, the Moebius band is an ANNULUS.

Notation 9.1 Suppose that S is a surface. We let $S(m_1, \ldots, m_k)$, where $m_i = 2, 3, \ldots, \infty$ denote the 2-dimensional orbifold with boundary obtained from S as follows:

1. For each i with $m_i = \infty$, we remove the interior of a closed disk from S, so that the disks are pairwise disjoint.

2. For each i so that $m_i < \infty$, we introduce a singular point of order m_i on S (away from the disks removed and from the boundary of S).

In order to shorten the notation, if $m_1 = m_2 = \cdots = m_\ell = m$, we replace the repeating sequence $(m_1, \ldots, m_\ell)$ in our notation with $\ell \times m$. For instance,

$$S^2(\infty, \infty, 2, 2, 2, 2) = S^2(2 \times \infty, 4 \times 2),$$

the annulus with four singular points of order 2.

Below we will use the notation I for the interval $[-1, 1]$.

We now review the construction given in [20]. Define a regular 2-dimensional cell complex X obtained from $\mathbb{RP}^2$ by attaching 2-cells D_1, D_2 to two distinct projective lines L_1, L_2 in $\mathbb{RP}^2$. The lines L_1, L_2 cut $\mathbb{RP}^2$ in two 2-cells D_3, D_4. Next, as in [20], define the 2-dimensional orbihedron Y by introducing 3 singular points of order 2 in the interior of each of the disks D_i, $i = 1, \ldots, 4$. It is proven in [20] that for every finitely-presented group G there exists a finite orbi-cover $\tilde{Y} \to Y$, such that $\pi_1(|\tilde{Y}|) \cong G$.

Panov and Petrunin in [20] then "thicken" Y to a compact 3-dimensional hyperbolic orbifold Y_3 with convex boundary, where each singular point p_j of Y corresponds to a singular segment $p_j \times I$ (of the order 2) in Y_3; in addition, Y_3 constructed in [20] has an extra order 2 singular point q_i for each thickened disk D_i, $i = 1, 2$. Then, Y_3 constructed in [20] is the quotient of a closed convex subset $C \subset \mathbb{H}^3$ by a discrete convex-cocompact group of isometries $\Gamma_3 < PO(3, 1)$, which contains both orientation-preserving elliptic involutions (corresponding to the singular segments $p_j \times I$) and Cartan involutions (corresponding to the isolated singular points q_i).

We now observe that instead of thickening the orbihedron Y described above, we can thicken a slightly different one: Let Y^+ be the orbihedron obtained from Y by adding an extra order 2 singular point q_i to each cell D_i, $i = 3, 4$. (Nothing changes as far as the 2-cells D_1, D_2 are concerned, they still have three order 2 singular points each.) Now, if $f : \tilde{Y} \to Y$ is an orbi-cover, it induces an orbi-cover $f^+ : \tilde{Y}^+ \to Y^+$, which is unramified over the points q_3, q_4: The orbihedron $\tilde{Y}^+$ is obtained from $\tilde{Y}$ by declaring the points in $f^{-1}(q_3) \cup f^{-1}(q_4)$ to be singular points of order 2. Clearly, $\pi_1(|\tilde{Y}^+|) = \pi_1(|\tilde{Y}|)$. Therefore, as in [20], for every finitely-presented group G there exists a finite orbi-cover $\tilde{Y}^+ \to Y^+$, such that $\pi_1(|\tilde{Y}^+|) \cong G$. We will see below why thickening the orbihedron Y^+ is better than thickening Y.

Before thickening Y^+, we describe its double cover (as in [20]), which will, hopefully, clarify the construction. Let $\Pi : S^2 \to \mathbb{RP}^2$ be the 2-fold cover, quotient by the antipodal involution τ. We let $\alpha_i := \tilde{L}_i$, $i = 1, 2$, $\tilde{D}_j$, $j = 3, 4$ denote the complete preimages of the lines and the disks under Π. Note that each $\tilde{D}_3$ and $\tilde{D}_4$ consists of two copies of D_3 and D_4 respectively, while

α_1, α_2 are circles. We lift the orbifold data accordingly, so we obtain the orbifold O^2 (sphere with 16 singular points of order 2) which is the 2-fold cover of $\mathbb{RP}^2(8 \times 2)$. Now, thickening $\alpha_1 \cup \alpha_2$ in S^2 yields a quadruply-punctured sphere F. Of course, the restriction $\Pi : \alpha_i \to L_i$ is again a 2-fold cover. The 2-fold cover $S^1 \to S^1$ extends to a 2-fold orbifold-cover $T^2(\infty) \to D^2(3 \times 2)$ (see below). Thus, attaching two copies of $T^2(\infty)$ (the one-holed torus) to O^2 along the loops α_1, α_2, we obtain an orbihedron $\hat{Y}^+$ which is a 2-fold orbicover of Y^+.

Instead of thickening Y^+ we will equivariantly thicken its 2-fold cover $\hat{Y}^+$: The surface F is thickened to $F \times I$, while both copies of $T^2(\infty)$ are thickened to $T^2(\infty) \times I$. The 3-manifolds $Z_i := T^2(\infty) \times I$ are then attached to $F \times I$ along the appropriate annuli A_{α_i}, $i = 1, 2$, in $F \times \{\pm 1\}$ (thickenings of the loops $\alpha_1 \times \{1\}$ and $\alpha_2 \times \{2\}$), which are identified with the annuli $\partial T^2(\infty) \times I$. Lastly, the four orbi-disks in

$$\tilde{D}_3 \cup \tilde{D}_4$$

will be thickened to the appropriate orbifold I-bundles W_i, $i = 1, 2, 3, 4$, over $D^2(4 \times 2)$ and attached to $F \times I$ along $\partial F \times I$. (The precise construction of W_i will be given below.) It is then clear (e.g., from Van Kampen's theorem) that the fundamental group of the resulting orbifold $\hat{O}$ is isomorphic to $\pi_1(\hat{Y}^+)$. Assuming that τ extends to an involution of $\hat{O}$ (with isolated fixed-points only), we obtain the 3-dimensional orbifold $\mathcal{O} = \hat{O}/\tau$. By the construction, $\mathcal{O}$ and Y^+ have isomorphic fundamental groups.

We now explain how to construct W_i's and how to extend the involution τ. Begin with the 2-torus T^2 and its elliptic involution $\sigma : T^2 \to T^2$: It has 4 fixed points and the quotient T^2/σ, as an orbifold, is $S^2(4 \times 2)$. We extend σ to the orientation–reversing involution

$$\sigma : T^2 \times I \to T^2 \times I, \qquad \sigma(z, t) = \big(\sigma(z), -t\big).$$

The orbifold $(T^2 \times I)/\sigma$ has only isolated singular points (four of them). Then the projection $\eta : (T^2 \times I)/\sigma \to T^2/\sigma$ is the orbifold I-bundle. This projection is the quotient of the projection if $T^2 \times I$ to the 1st factor.

Definition of W_i's We define W to be the suborbifold of $(T^2 \times I)/\sigma$ obtained by removing $\eta^{-1}(D) \cong D \times I$, where D is a nonsingular 2-disk in $S^2(4 \times 2)$. In particular, $\eta^{-1}(\partial D)$ is an annulus. Then the orbifolds W_i, $i = 1, \dots, 4$ are copies of W above. They will be attached to $F \times I$ by gluing the annuli $\eta^{-1}(\partial D)$ to the annuli in $\partial F \times I$.

Extension of τ We extend τ to $F \times I$ by the identity to the second factor. Then τ sends each A_{α_i} to itself, where $A_{\alpha_i} \subset F \times \partial I$ is an annular thickening of $\alpha_i \times \{\pm 1\}$. The quotient A_{α_i}/τ is the Moebius band (since τ reverses orientation on F). We are identifying A_{α_i} with the annulus $A_i \subset \partial Z_i$, where $Z_i = T^2(\infty) \times I$ and $A_i = \partial T(\infty) \times I$. Note that τ acts on the annulus $A = A_i = S^1 \times I$ by $\tau(z, t) = (\tau(z), -t)$, where $\tau : S^1 \to S^1$ is an involution. (We now drop the index i since the construction is the same for $i = 1$ and $i = 2$.) Thus, we again take the elliptic involution $\sigma : T^2 \to T^2$. Let $x \in T^2$ be one of its four fixed points. Take a small σ-invariant 2-disk $D \subset T^2$ around x. We then regard $T^2(\infty)$ as $T^2 \setminus int(D)$. The involution σ restricts to the involution $S^1 \to S^1$ of the boundary circle of $T^2(\infty)$ which is isotopic to $\tau : S^1 \to S^1$, so we identify them. Set $Z := T^2(\infty) \times I$. Now, the map $\sigma : Z \to Z$ given by

$$\sigma(z, t) = \big(\sigma(z), -t\big)$$

is the required extension of τ to $Z = Z_i$, $i = 1, 2$. Clearly, the orbifold

$$V = Z/\tau$$

has only 3 singular points. This concludes the construction of $\hat{O}$ and the extension $\tau : \hat{O} \to \hat{O}$. Therefore, we obtain the 3-dimensional orbifold with boundary $\mathcal{O} := \hat{O}/\tau$ which is a thickening of the orbihedron Y^+. Furthermore, the singular locus of $\mathcal{O}$ is finite. We also have the orbifold-fibration

$$\zeta : V \to D^2(2, 2, 2)$$

obtained as the quotient (by τ) of the projection $Z \to T^2(\infty)$ to the first factor.

Topological properties of $\mathcal{O}$ Our goal is to show that the orbifold $\mathcal{O}$ is *hyperbolizable*, i.e., there exists a closed convex subset Q of $\mathbb{H}^3$ and a discrete isometry group $\Gamma_{\mathcal{O}} < PO(3, 1)$, so that the quotient-orbifold $Q/\Gamma_{\mathcal{O}}$ is homeomorphic to the orbifold $\mathcal{O}$. In principle, this could be proven by constructing $\Gamma_{\mathcal{O}}$ by hand, via Maskit combination. Instead, we will show that $\mathcal{O}$ is hyperbolizable by verifying that it is irreducible and atoroidal, in which case $\mathcal{O}$ is hyperbolizable by Thurston's hyperbolization theorem, see e.g., [5, 16].

We first analyze the JSJ decomposition of the orbifold $\mathcal{O}$, see e.g. [5]. Recall that $\mathcal{O}$ is constructed from 5 pieces: Orbifold $N := (F \times I)/\tau$, two copies of the orbifold $V := Z/\tau$ (where $Z = T^2(\infty) \times I$) and two copies of the orbifold $W = (T^2(\infty, \infty) \times I)/\tau$. We now convert each of these orbifolds to an orbifold pair by marking some of their boundary annuli/Moebius bands:

1. Define (N, P_N), where $N = F \times I$ and $P_N = \partial F \times I \sqcup A_{\alpha_1} \times \{1\} \sqcup A_{\alpha_2} \times \{-1\}$. Then set $(U, P_U) := (N/\tau, P_N/\tau)$.
2. Define (V, P_V), where $V := Z/\tau$ and $P_V = \zeta^{-1}(\partial D^2(2, 2, 2))$ is a single Moebius band.
3. Define (W, P_W), where $P_W = \eta^{-1}(\partial D)$ is a single annulus, see above for the definition of $\eta : (T^2 \times I)/\sigma \to T^2/\sigma$.

For each of the orbifolds U, V, W we define its *partial boundary* ∂_P by: $\partial_P U := \partial U \setminus P_U$, etc.

By the construction, each of the orbifolds N, W, V is very good: It admits a finite manifold-cover. Also, each of the orbifolds is strongly atoroidal, i.e., it contains no π_1-injective TORI: Its fundamental group is virtually free and, hence, contains no $\mathbb{Z}^2$.

Lemma 9.2 *The orbifold pairs* $(U \setminus \partial_P U, P_U)$, $(V \setminus \partial_V, P_V)$ *and* $(W \setminus \partial_W, P_W)$ *are all irreducible, boundary-irreducible and acylindrical* (*see* [16] *for the terminology*).

Proof We will give a proof for $(V \setminus \partial_V, P_V)$ since the other pairs are similar. We first note that irreducibility and acylindricity are stable under passing to finite covers. Now, V is finitely covered by the product $M := T^2(\infty) \times I$, so that P_V lifts to the annulus $P_M := \partial T^2(\infty) \times I$. Irreducibility of M is clear. Boundary-irreducibility follows from the fact that the annulus P_M is π_1-injective in M. To see that $(M \setminus \partial_P M, P_M)$ is acylindrical, note that the image of $\pi_1(P_M)$ in $\pi_1(M) \cong \mathbb{Z} \star \mathbb{Z}$ is a maximal cyclic subgroup of $\pi_1(M)$. □

Lemma 9.3 *The orbifold-pair* (U, P_U) *is acylindrical.*

Proof It suffices to prove acylindricity for the 2-fold cover (N, P_N) of (U, P_U). Then surface P_N contains the union of the annuli in $\partial F \times I$. Every annulus properly embedded in $F \times I$ and disjoint from $\partial F \times I$, is isotopic to one of the form $c \times I$, where c is a simple loop in F. On the other hand, every essential simple loop c in F (i.e., a loop which does not bound a disk and does not bound an annulus whose other boundary component is in ∂F) has to cross either α_1 or α_2. Therefore, the corresponding annulus $c \times I$ either crosses A_{α_1} or A_{α_2}. Hence, it cannot be isotopic to the annulus disjoint from the union of circles ∂P_N. Thus, (U, P_U) is acylindrical. □

We now conclude that the sub-orbifolds W_i, $i = 1, 2$ and V_i, $i = 1, 2$ are maximal (up to isotopy) I-bundles in $\mathcal{O}$. Therefore, their union is the *characteristic suborbifold* in $\mathcal{O}$, and, hence, its splitting along the annuli and Moebius bands P_{V_i}, P_{W_j} is its JSJ decomposition. Irreducibility of $\mathcal{O}$ follows from the fact that each annulus and Moebius band P_{V_i}, P_{W_j} is incompressible in $\mathcal{O}$ (as it is incompressible in the pieces of the JSJ decomposition). In particular, $\mathcal{O}$ contains no bad suborbifolds. Atoroidality of $\mathcal{O}$ follows since each essential TORUS in $\mathcal{O}$ has to be contained in one of the characteristic suborbifolds and they are all strongly atoroidal. We thus proved:

Proposition 9.4 *The orbifold* $\mathcal{O}$ *is hyperbolizable*: *It can be realized as the orbifold-quotient of a closed convex subset* Q *of* $\mathbb{H}^3$ *by a discrete isometry* *group $\Gamma_{\mathcal{O}} < PO(3,1)$. In particular, the group $\Gamma_{\mathcal{O}}$ is a convex-cocompact subgroup of $PO(3,1)$.*

We then observe that $\Gamma_{\mathcal{O}} \cong \pi_1(\mathcal{O})$ contains a free nonabelian subgroup, say, $\pi_1(F)$. In particular, the group $\Gamma_{\mathcal{O}}$ is nonelementary. We can now finish the proof of Theorem 1.4. Given a finitely-presented group G we find a finite index subgroup $\tilde{\Gamma} < \pi_1(Y^+) = \Gamma_{\mathcal{O}}$ so that

$$G \cong \tilde{\Gamma}/\langle\langle torsion \rangle\rangle.$$

The group $\tilde{\Gamma}$ is the fundamental group of some orbifold $\hat{\mathcal{O}}$ (a finite covering of $\mathcal{O}$). Since $\mathcal{O}$ is hyperbolizable, we obtain a discrete embedding $\tilde{\Gamma} \hookrightarrow PO(3,1)$. Since all singularities of $\mathcal{O}$ are isolated, so are all singularities of its finite cover $\hat{\mathcal{O}}$. Thus, $\tilde{\Gamma}$ belongs to class $\mathcal{K}$. Since $\Gamma_{\mathcal{O}}$ is convex-cocompact and $\tilde{\Gamma}$ has finite index in $\Gamma_{\mathcal{O}}$, it follows that $\tilde{\Gamma}$ is also convex-cocompact. Thus, $\tilde{\Gamma}$ belongs to the class $\mathcal{K}^2$. Theorem 1.4 follows. □

10 Constructing projective varieties

Proof of Theorem 1.2 Let G be a finitely-presented group. By Theorem 1.4, there exists a nonelementary group $\tilde{\Gamma} < PO(3,1)$ of class $\mathcal{K}^2$, so that $G \cong \pi_1(\mathbb{H}^3/\tilde{\Gamma})$. We let $x \in \mathbb{H}^3$ be a generic base-point, so that the associated Dirichlet tiling $\mathcal{D}_x(\tilde{\Gamma})$ of $\mathbb{H}^3$ is weakly simple. Recall that since $\tilde{\Gamma}$ is convex-cocompact, every face of $\mathcal{D}_x(\tilde{\Gamma})$ is a finitely-sided convex hyperbolic polytope. However, since $\mathbb{H}^3/\tilde{\Gamma}$ has infinite volume, so is the fundamental domain D_x. Therefore, unlike in [17], D_x is not a projective polytope but only a hyperbolic polyhedron. We will see, nevertheless, that this is harmless.

We now define the locally finite hyperbolic polyhedral complex $\tilde{\mathcal{C}} = \mathcal{D}_x(\tilde{\Gamma}) - \mathcal{D}_x^{(0)}(\tilde{\Gamma})$, the punctured complex of $\mathcal{D}_x(\tilde{\Gamma})$ (see Sect. 3). Let $\Gamma < \tilde{\Gamma}$ be a torsion–free normal finite index subgroup in $\tilde{\Gamma}$, so that $\mathcal{C} := \tilde{\mathcal{C}}/\Gamma$ is a simple finite hyperbolic polyhedral complex, see Lemma 3.8. Set $\Theta := \tilde{\Gamma}/\Gamma$. This finite group acts naturally on $\mathcal{C}$ and this action is transitive on facets (since the action of $\tilde{\Gamma}$ is transitive on facets $D_{\gamma x}$ of $\mathcal{D}_x(\tilde{\Gamma})$).

Consider the manifold $M := \mathbb{H}^3/\Gamma$, and let $F \subset M$ denote the finite set which is the image of $\mathcal{D}_x^{(0)}(\tilde{\Gamma})$ in M. If $m \in F$ is a vertex of $\mathcal{D}_x(\tilde{\Gamma})/\Gamma$, it is not fixed by any nontrivial element of Θ (since it is so for vertices of the complex $\mathcal{D}_x(\tilde{\Gamma})$). Therefore,

$$\pi_1\big((M \setminus F)/\Theta\big) \cong \pi_1(M/\Theta) = \pi_1\big(\mathbb{H}^3/\tilde{\Gamma}\big),$$

which is the quotient of $\tilde{\Gamma}$ by the normal closure of the Cartan involutions in $\tilde{\Gamma}$.

Next, we associate with the hyperbolic polyhedral complex $\mathcal{C}$ the projective polyhedral complex $\check{\mathcal{C}}$ as in Sect. 3. Note that the pushouts C and $\check{C}$ of $\mathcal{C}$ and $\check{\mathcal{C}}$ are homotopy-equivalent, since posets and, hence, nerves, of $\mathcal{C}$ and $\check{\mathcal{C}}$ are isomorphic. We then *complexify* the complex $\check{\mathcal{C}}$ as in Sect. 4. The result is a complex $\mathcal{P}$ of projective spaces based on $\check{\mathcal{C}}$. The action $\Theta \curvearrowright \mathcal{C}$ lifts to the action $\Theta \curvearrowright \mathcal{P}$.

Lemma 10.1 *The action $\Theta \curvearrowright \mathcal{P}$ satisfies the Assumption 4.9 in Sect. 4.*

Proof As in Sect. 4, we consider the complex of projective spaces $\tilde{\mathcal{P}}$ based on the complex $\tilde{\mathcal{C}}$. Each stratum $\tilde{X}_i$ is the (complex-projective) span of the corresponding face $\tilde{c}_i$ of $\tilde{\mathcal{C}}$ and $\tilde{X}_i$ projects isomorphically to the corresponding stratum X_i of $\mathcal{P}$. Therefore, it suffices to verify that the action $\tilde{\Gamma} \curvearrowright \tilde{\mathcal{P}}$ satisfies the Assumption 4.9. Suppose that $\gamma \in \tilde{\Gamma}$ fixes a point $p \in \tilde{X}$, the push-out of $\tilde{\mathcal{P}}$. Let $\tilde{X}_i$ be the minimal stratum of $\tilde{X}$ containing p. Then γ preserves $\tilde{X}_i$ and, hence, preserves the corresponding face $\tilde{c}_i$ of $\tilde{\mathcal{C}}$. Thus, γ is elliptic and has to be a Cartan involution since $\tilde{\Gamma}$ is in the class $\mathcal{K}$. Hence, the fixed-point set of γ in $\tilde{X}_i = Span(\tilde{c}_i)$ is the disjoint union of the point p and the dual (with respect to the Lorentzian inner product) projective space $p^{\perp} \subset Span(\tilde{c}_i)$. Since $p^{\perp}$ is disjoint from $\mathbb{H}^3$, it is also disjoint from $\tilde{c}_i$. Lastly, the fact that p belongs to exactly 3 faces of $\tilde{\mathcal{C}}$ follows immediately from Part 2 of Theorem 1.6: Interior of every 2-face is contained in exactly two facets of $\tilde{\mathcal{C}}$. $\square$

We next replace $\mathcal{P}$ with its blowup $b\mathcal{P}$ and let X denote the stratified projective variety which is the push-out of $b\mathcal{P}$, see Sect. 4: By construction, all singularities of X are normal crossings. The finite group Θ acts naturally on X, the quotient $Z = X/\Theta$ is again a projective variety, see e.g. [14, p. 126]. The action of Θ is transitive on top-dimensional strata of X (since Θ acts transitively on facets of $\mathcal{C}$). Let X_c be one of these top-dimensional strata; removing from X_c all sub-strata we obtain a (Zariski) open and connected subset X_c°. Projecting X_c° to Z we get an open, connected and dense subset. Thus, Z is irreducible.

Note that since Γ is nonelementary and torsion-free, it has trivial center, see Lemma 2.3. Thus, by Proposition 4.13,

$$\pi_1(Z) \cong \pi_1(\check{C}/\Theta) \cong \pi_1(C/\Theta) \cong \pi_1\big((M \setminus F)/\Theta\big) \cong \pi_1\big(\mathbb{H}^3/\tilde{\Gamma}\big) = G.$$

All singularities of Z are normal crossings and $\mathbb{Z}_2$-quotients of normal crossing singularities, types Y_1 and Y_2 described in Sect. 4. By blowing up centers of type 1 singularities Y_1 and dividing by Θ, we get a new irreducible projective variety V where all singularities are normal crossings and their quotients of the type $Y_2/\mathbb{Z}_2$. As before, $\pi_1(V) \cong G$. Lastly, by the argument in the end of Sect. 4, we replace the 3-dimensional V with irreducible projective surface W so that $\pi_1(W) \cong \pi_1(V) \cong G$ and all singularities of W are normal crossings and Whitney umbrellas (corresponding to singularities of type $Y_2/\mathbb{Z}_2$ in V).

In order to prove the second assertion of Theorem 1.2, we note that if $\tilde{\Gamma}$ is a torsion-free convex-cocompact subgroup of $PO(3, 1)$, then the group $\Theta = \tilde{\Gamma}/\Gamma$ acts freely on X and, hence, all singularities of $Z = X/\Theta$ are normal crossings. Then one takes $V := Z$ and proceeds as above. This concludes the proof of Theorem 1.2. □

Remark 10.2 It was proven by Carlson and Toledo in [8] that if G is a Kähler group (e.g., fundamental group of a smooth projective variety) which is isomorphic to a nonelementary discrete subgroup Γ of $PO(n, 1)$, then G contains a finite index subgroup isomorphic to the fundamental group of a Riemann surface. (Note that [8] assumes that Γ is cocompact, but it is clear from the proof that nonelementary is enough.)

Acknowledgements This paper grew out of our work [17] with János Kollár and I am grateful to him for questions, comments and suggestions. In particular, he explained to me that Whitney umbrella singularities appear as $\mathbb{Z}_2$-quotients of normal crossings and suggested the dimension reduction from 3 to 2. I am grateful to Akira Ushijima for sharing with me [24]. I am also grateful to the referee for useful remarks and suggestions. Partial financial support for this work was provided by the NSF grants DMS-09-05802 and DMS-12-05312.

References

1. Amorós, J., Burger, M., Corlette, K., Kotschick, D., Toledo, D.: Fundamental Groups of Compact Kähler Manifolds. Mathematical Surveys and Monographs, vol. 44. Am. Math. Soc., Providence (1996)
2. Armstrong, M.: The fundamental group of the orbit space of a discontinuous group. Math. Proc. Camb. Philos. Soc. **64**, 299–301 (1968)
3. Atkinson, M.D.: Primitive spaces of matrices of bounded rank. II. J. Aust. Math. Soc. A **34**(3), 306–315 (1983)
4. Beardon, A., Maskit, B.: Limit points of Kleinian groups and finite-sided fundamental polyhedra. Acta Math. **132**, 1–12 (1974)
5. Bonahon, F.: Geometric structures on 3-manifolds. In: Handbook of Geometric Topology, pp. 93–164. North-Holland, Amsterdam (2002)
6. Bowditch, B.: Geometrical finiteness for hyperbolic groups. J. Funct. Anal. **113**(2), 245–317 (1993)
7. Brown, K.: Cohomology of Groups. Springer, Berlin (1982)
8. Carlson, J., Toledo, D.: Harmonic mappings of Kähler manifolds to locally symmetric spaces. Publ. Math. IHÉS **69**, 173–201 (1989)
9. Diaz, R., Ushijima, A.: On the properness of some algebraic equations appearing in Fuchsian groups. Topol. Proc. **33**, 81–106 (2009)
10. Drumm, T., Poritz, J.: Ford and Dirichlet fundamental domains for cyclic subgroups of $PSL(2, \mathbb{C})$ acting on $\mathbb{H}^3$ and $\partial\mathbb{H}^3$. Conform. Geom. Dyn. **3**, 116–150 (1999)
11. Eisenbud, D., Harris, J.: Vector spaces of matrices of low rank. Adv. Math. **70**(2), 135–155 (1988)

12. Goresky, M., MacPherson, R.: Stratified Morse Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 14. Springer, Berlin (1988)
13. Haefliger, A.: Orbi-espaces. In: Sur les Groupes Hyperboliques d'Aprés Mikhael Gromov. Progress in Mathematics, vol. 83, pp. 203–213. Birkhäuser, Basel (1990)
14. Harris, J.: Algebraic Geometry: A First Course. Graduate Texts in Mathematics, vol. 133. Springer, Berlin (1992)
15. Jorgensen, T., Marden, A.: Generic fundamental polyhedra for Kleinian groups. In: Holomorphic Functions and Moduli, vol. II. Math. Sci. Res. Inst. Publ., vol. 11, pp. 69–85. Springer, New York (1988)
16. Kapovich, M.: Hyperbolic Manifolds and Discrete Groups: Lectures on Thurston's Hyperbolization. Birkhäuser's Series "Progress in Mathematics". Birkhäuser, Basel (2000)
17. Kapovich, M., Kollár, J.: Fundamental groups of links of isolated singularities. Preprint, arXiv:1109.4047 (2011)
18. Lovasz, L.: Singular spaces of matrices and their application in combinatorics. Bol. Soc. Bras. Mat. **20**(1), 87–99 (1989)
19. Maskit, B.: Kleinian Groups. Springer, Berlin (1988)
20. Panov, D., Petrunin, A.: Telescopic actions. Geom. Funct. Anal. **22**(6), 1814–1831 (2012)
21. Ratcliffe, J.: Foundations of Hyperbolic Manifolds, 2nd edn. Graduate Texts in Mathematics, vol. 149. Springer, New York (2006)
22. Selberg, A.: On discontinuous groups in higher-dimensional symmetric spaces. In: Contributions to Function Theory, pp. 147–164. Tata Institute of Fundamental Research, Bombay (1960)
23. Simpson, C.: Local systems on proper algebraic V-manifolds. Pure Appl. Math. Q. **7**, 1675–1760 (2011)
24. Ushijima, A.: Density of the centers of generic fundamental polyhedra for purely loxodromic Kleinian groups. Preprint, January 2012