A SMOOTH FOLIATION OF THE 5-SPHERE BY COMPLEX SURFACES

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Dedicated to John Milnor for his 70th birthday.

0. Introduction

Thirty years ago, in a landmark paper in the theory of foliations, Blaine Lawson discovered the first example of a smooth codimension one foliation of the sphere $\mathbb{S}^5$ [La]. Each leaf of this foliation can be made *separately* into a complex surface. Thus a natural question is: *Does there exist a smooth codimension one foliation $\mathcal{F}$ of $\mathbb{S}^5$ such that the tangent bundle of the foliation $T\mathcal{F}$ has an integrable almost complex structure?*

The question above amounts to finding a (maximal) smooth foliated atlas $\mathcal{A} = \{(U_i, \phi_i)_{i\in I} \quad | \quad \phi_i(U_i) \subset \mathbb{C}^2 \times \mathbb{R} \simeq \mathbb{R}^5\}$ such that the changes of charts

$$(z,t) \in \phi_i(U_i \cap U_j) \longmapsto \phi_j \circ \phi_i^{-1}(z,t) := (\xi_{ij}(z,t), \zeta_{ij}(t)) \in \phi_j(U_i \cap U_j)$$

are holomorphic in the tangential direction, i.e. the map ξ_{ij} is holomorphic for fixed t. In other words, this means that we can endow each leaf with a complex structure and that the tangent bundle of the foliation defines a smooth codimension one CR-structure on $\mathbb{S}^5$, which is integrable and Levi flat.

In this paper, we give an affirmative answer to this problem by endowing a slight variation of the Lawson foliation with such a CR-structure.

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1. Preliminaries

1.1. CR-structures.

Let M be a smooth manifold without boundary of odd-dimension $2n+1$. A (codimension one) *CR-structure* on M is a smooth $2n$-plane distribution $T\mathcal{F} = \cup_{x\in M} T_x\mathcal{F}$ together with a smooth almost complex structure:

$$J \ : \ (x,v) \in T\mathcal{F} \longmapsto (x, J_x \cdot v) \in T\mathcal{F}$$

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where J_x is a linear operator on $T_x\mathcal{F}$ satisfying $J_x^2 \equiv -Id$. The natural extension of J to the complex distribution $T_{\mathbb{C}}\mathcal{F} = T\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$ is diagonalizable with eigenvalues $-i$ and i. This induces a decomposition:

$$T_{\mathbb{C}}\mathcal{F} = T^{(1,0)}M \oplus T^{(0,1)}M \ .$$

The CR-structure is said to be *integrable* if the Lie bracket of two (local) tangentially anti-holomorphic vector fields (that is smooth local sections of the bundle $T^{(0,1)}M$) is again tangentially anti-holomorphic. The CR-structure is said to be *Levi flat* if the Lie bracket of two (local) tangentially holomorphic vector fields (that is smooth local sections of the bundle $T^{(1,0)}M$) is again tangentially holomorphic.

A Levi flat CR-structure $(T\mathcal{F}, J)$ defines in particular an involutive distribution $T\mathcal{F}$ and thus a smooth foliation on M. Moreover, if it is integrable, the leaves inherit from the operator J an integrable smooth almost complex structure and so are complex manifolds by Newlander-Nirenberg Theorem [N-N]. In this paper, we will call such a foliation *a foliation by complex manifolds.*

Let M be a smooth manifold *with* boundary. A (codimension one) *CR-structure* on M is the data of a CR-structure (in the previous sense) on the interior of M and of an almost complex structure on the boundary ∂M such that the following gluing condition is verified:

Let N be the open manifold obtained by adding a collar $\partial M \times [0,1[$ to M. Then the CR-structure on M extends to a CR-structure on N by considering on the collar the distribution tangent to the submanifolds $\partial M \times \{t\}$ $(0 \leq t < 1)$ equipped with the almost complex structure inherited from ∂M.

We define in the same way as before integrability and Levi-flatness. Again, an integrable, Levi-flat CR-structure on M induces a smooth foliation with complex leaves on M (cf the definition of [La]). We will call such a foliation a *foliation by complex manifolds.* Note that each connected component of the boundary is a leaf.

The next Lemma follows directly from the gluing condition.

Lemma 1. *Let M and M' be two manifolds with diffeomorphic, non vacuous boundary. Assume there is a CR-structure on both of them such that the boundaries are CR-isomorphic. Let N be the manifold obtained by gluing M and M' on their boundary by a CR-isomorphism. Then N admits a CR-structure too. It is integrable if and only if the structures on M and M' are. It is Levi-flat if and only if the structures on M and M' are.*

As a consequence, let N be a submanifold with boundary of a compact manifold M. If both N and $M \setminus \mathring{N}$ (where $\mathring{N}$ is the interior of N) have a foliation by complex manifolds with boundary leaves biholomorphic, then so has M.

1.2. Lawson foliation.

Let us now recall Lawson's construction of a smooth codimension one foliation of $\mathbb{S}^5$. Let

$$F \ : \ (z_1, z_2, z_3) \in \mathbb{C}^3 \longmapsto z_1^3 + z_2^3 + z_3^3 \in \mathbb{C}$$

and let $V = F^{-1}(0)$. The manifold $W := V \setminus \{(0,0,0)\}$ intersects transversally the euclidean unit sphere in the smooth compact manifold K. Moreover, it projects onto the projective space $\mathbb{CP}^2$ as an elliptic curve $\mathbb{E}_\omega$ of modulus ω. This curve admits an automorphism of order three hence $\omega^3 = 1$. The canonical projection:

$$p : W \to \mathbb{E}_\omega$$

describes W as a holomorphic principal $\mathbb{C}^*$-bundle over the elliptic curve $\mathbb{E}_\omega$, with first Chern class equal to -3. By passing to the unit bundle, one has that K is a circle-bundle over a torus with Euler class equal to -3 (see [Mi4] Lemma 7.1 and Lemma 7.2). More precisely, let

$$A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

then K is the suspension of the unipotent isomorphism induced on the two-dimensional torus by the matrix A. A presentation of the fundamental group of K is given by:

$$\pi_1(K) = \{a, b, c \,|\, ab = ba,\ cac^{-1} = a,\ cbc^{-1} = ba^{-3}\}$$

Therefore, $\pi_1(K)$ is isomorphic to the nilpotent group $\mathfrak{N}$ of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ a,\ b,\ c\ \in 3\mathbb{Z}$$

The above description shows that $H_1(K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. In fact, Milnor [Mi4, §8] showed that K is the nilmanifold $\mathfrak{H}/\mathfrak{N}$, where $\mathfrak{N}$ is the three-dimensional Heisenberg group.

The Milnor fibration Theorem [Mi2, Theorem 4.8] shows that $\mathbb{S}^5 \setminus K$ fibers over $\mathbb{S}^1$ and describes $\mathbb{S}^5$ as an open book in the sense of Winkelnkemper [Wi] with fibers diffeomorphic to $F^{-1}(z)$ for $z \in \mathbb{C}^*$. On the other hand, K is smoothly embedded in $\mathbb{S}^5$ with trivial normal bundle [Mi1, §7] and therefore has a closed tubular neighborhood diffeomorphic to $K \times D^2$, where D^2 is the closed two-disk. Thus it follows that both a closed tubular neighborhood of K in $\mathbb{S}^5$ and the closure of its complement fiber over the circle by fibrations which are also fibrations when restricted to the boundary. Hence they are smoothly foliated by using a standard *tourbillonnement* Lemma:

Lemma [La]. *Let M be a compact manifold with boundary ∂M, and suppose there exists a C^∞-submersion $\psi : M \to \mathbb{S}^1$ such that $\psi|\partial M$ is a submersion of the boundary. Then there exists a codimension-one C^∞-foliation of M.*

Gluing carefully those pieces together by a diffeomorphism, Lawson obtained a smooth codimension one foliation of $\mathbb{S}^5$ with a unique compact leaf diffeomorphic to $K \times \mathbb{S}^1$.

In the next Sections, we will modify the foliations given by Lawson on those two pieces to obtain in fact a foliation of $\mathbb{S}^5$ by complex surfaces. A key idea is to take appropriately chosen infinite cyclic coverings [Mi3] of each piece that fibers over the circle.

First, we will put such a foliation on a tubular neighborhood $K \times D^2$.

2. The CR-structure of $K \times D^2$

In all that follows, $\mathcal{N}$ will be a closed tubular neighborhood of K in $\mathbb{S}^5$ identified, via a trivialization, with $K \times D^2$ and $\overset{\circ}{\mathcal{N}}$ its interior. Let $\mathcal{M} := \mathbb{S}^5 \setminus \overset{\circ}{\mathcal{N}}$ and $\overset{\circ}{\mathcal{M}} = \mathbb{S}^5 \setminus \mathcal{N}$ be its interior.

The following Lemma, which indicates how to construct a foliation by complex manifolds on a solid torus will play the rôle of the *tourbillonnement* Lemma of Section 1.

Lemma 2. *Let $B := D^2 \times \mathbb{S}^1$ be a solid torus. Let $\tau \in \mathbb{H}$ be a complex number in the upper half plane. Let $\mathbb{E}_\tau$ be the elliptic curve of modulus τ. Then there exists a codimension one integrable and Levi flat CR-structure on B such that:*
(i) The structure on the boundary is that of $\mathbb{E}_\tau$.
(ii) The induced smooth foliation is the Reeb component.

Furthermore, given any translation of the elliptic curve, it extends as a CR-isomorphism of B.

Remark 1. The leaves of the foliation are the elliptic curve $\mathbb{E}_\tau$ as the boundary leaf and biholomorphic to $\mathbb{C}$ in the interior.

Proof. Consider $\mathbb{C}\times[0,\infty[$ foliated by the foliation whose leaves are $\mathbb{C}\times\{t\}$ for $t \in [0,\infty[$. Remove the point $\{(0,0)\}$. The elliptic curve $\mathbb{E}_\tau$ is the quotient of $\mathbb{C}^*$ by the action of the group generated by the homothetic transformation $w \mapsto w\exp(2i\pi\tau)$.

Let T be the diffeomorphism given by:

$$(w,t) \in \mathbb{C}\times[0,\infty[\setminus\{(0,0)\} \longmapsto T(w,t) = (w\exp(2i\pi\tau), d(t))$$

where d is a smooth diffeomorphism which is equal to t for $t \leq 0$ and satisfies $d'(t) > 1$ for $t > 0$. For instance $d(t) = t$ for $t \leq 0$ and $d(t) = t + e^{-\frac{1}{t}}$ for $t \geq 0$. Let Γ be the group generated by T. Consider the action of Γ on $\tilde{B} = (\mathbb{C}\times[0,\infty[)\setminus\{(0,0)\}$. On $\mathbb{C}\times]0,\infty[$, the action identifies the leaf $\mathbb{C}\times\{t\}$ with the leaf $\mathbb{C}\times\{d(t)\}$ (where $d(t) > t$) so the quotient is an open solid torus foliated by complex lines, that is an open Reeb component with a CR-structure. Moreover, this action extends to the boundary $\mathbb{C}^* \times \{0\}$ as the homothetic action giving $\mathbb{E}_\tau$ as quotient and the gluing condition for CR-structures is verified. Thus the quotient space of $\tilde{B}$ by this action is the solid torus B endowed with a foliation by Riemann Surfaces satisfying (i) and (ii).

Any translation of $\mathbb{E}_\tau$ lifts to a multiplication by a non zero complex number on the covering space $\mathbb{C}^* \to \mathbb{E}_\tau$. For each $\alpha \in \mathbb{C}^*$, consider the automorphism $\tilde{g}_\alpha$ of $\tilde{B}$ given by:

$$(w,t) \in \tilde{B} \longmapsto (\alpha\cdot w, t) \in \tilde{B} \ .$$

Since $\tilde{g}_\alpha$ commutes with T, it projects onto an automorphism g_α of B which preserves the foliation, induces a biholomorphism between the leaves and gives any translation (by suitably choosing α) on the boundary $\mathbb{E}_\tau$. $\square$

Consider the infinite cyclic covering $\mathbb{C}^* \to \mathbb{E}_\omega$ obtained by taking the quotient of $\mathbb{C}^*$ by the subgroup generated by $\exp(2i\pi\omega)$ and take the pull-back of the bundle $W \to \mathbb{E}_\omega$ by this covering. As $\mathbb{C}^*$ is Stein, this pull-back is trivial and therefore we obtain the following diagram:

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{C}^* \arrow[r, "\tilde{\pi}"] \arrow[d, "p_1"'] & W \arrow[d, "p"'] \\ \mathbb{C}^* \arrow[r, "\pi"] & \mathbb{E}_\omega \end{tikzcd}$$

Moreover W is biholomorphic to the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by an action of the form:

$$(A) \qquad (p, w_1, w_2) \in \mathbb{Z}\times\mathbb{C}^*\times\mathbb{C}^* \longmapsto (\exp(2i\pi\omega p)\cdot w_1, (\psi(z))^{-3p}\cdot w_2) \in \mathbb{C}^*\times\mathbb{C}^*$$

where ψ is an automorphic factor associated to the $\mathbb{C}^*$-bundle which is given as a quotient of θ-functions.

On the other hand, let λ be a real number strictly greater than one. The homothetic transformation H of $\mathbb{C}^3$ defined as $H(z_1, z_2, z_3) = (\lambda\omega \cdot z_1, \lambda\omega \cdot z_2, \lambda\omega \cdot z_3)$ leaves W invariant. The quotient of W by the subgroup generated by H is a complex surface denoted by $\mathcal{S}_\lambda$. It is diffeomorphic to $K \times \mathbb{S}^1$ and is a holomorphic principal bundle over $\mathbb{E}_\omega$ with fiber $\mathbb{E}_\tau$ where $\tau := -1/2i\pi \log(\lambda\omega) = -1/2i\pi \log \lambda - 1/3$. From the presentation of the fundamental group of K given in Section 1 and from the exact sequence in homotopy of the fibration $\mathcal{S}_\lambda \to \mathbb{E}_\omega$, we deduce that the first Betti number of $\mathcal{S}_\lambda$ is 3. Hence $\mathcal{S}_\lambda$ is not Kähler.

As before, we may consider the cyclic covering $\mathbb{C}^* \to \mathbb{E}_\omega$ and take the pull-back of the bundle $\mathcal{S}_\lambda \to \mathbb{E}_\omega$. Using again the fact that $\mathbb{C}^*$ is Stein, and using the short exact sequence of sheaves over $\mathbb{C}^*$:

$$0 \to \Lambda \to \mathcal{O} \to \mathcal{O}_\Lambda \to 0$$

(where $\Lambda \simeq \mathbb{Z}^2$ is a lattice in $\mathbb{C}$ and $\mathcal{O}_\Lambda$ the sheaf of germs of holomorphic functions on $\mathbb{C}^*$ modulo translations by the lattice), one obtains that every holomorphic principal bundle over $\mathbb{C}^*$ with fiber an elliptic curve is holomorphically trivial. Therefore, we have the following diagram:

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{E}_\tau & \xrightarrow{\widehat{\pi}} & \mathcal{S}_\lambda \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle \widehat{p}}\downarrow{\scriptstyle \mathbb{E}_\tau} \\ \mathbb{C}^* & \xrightarrow{\pi} & \mathbb{E}_\omega \end{array}$$

Note that the action (A) projects onto a free action of $\mathbb{Z}$ on $\mathbb{C}^* \times \mathbb{E}_\tau$ whose quotient is biholomorphic to $\mathcal{S}_\lambda$. Let us call this action $(\widetilde{A})$.

Recall that the boundary of $K \times D^2$ is diffeomorphic to $\mathcal{S}_\lambda$. On the other hand, $K \times D^2$ fibers (smoothly) over $\mathbb{E}_\omega$ with fiber $B := \mathbb{S}^1 \times D^2$. Taking the pull-back by π, we obtain the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi^*} & K \times D^2 \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle B} \\ \mathbb{C}^* & \xrightarrow{\pi} & \mathbb{E}_\omega \end{array}$$

where X is diffeomorphic to $\mathbb{C}^* \times B$. Besides, the boundary of X admits a natural complex structure as $\mathbb{C}^* \times \mathbb{E}_\tau$.

Let $\mathcal{F}_1$ be the foliation of B by Riemann surfaces given by Lemma 2 for τ. Let $\mathcal{F}$ be the foliation of X by complex manifolds whose leaves are $\mathbb{C}^* \times$ leaf of $\mathcal{F}_1$ via the identification of X with $\mathbb{C}^* \times B$. The boundary of X is biholomorphic to $\mathbb{C}^* \times \mathbb{E}_\tau$ as a leaf. Again by Lemma 2, the action of $\mathbb{Z}$ on $\mathbb{C}^* \times \mathbb{E}_\tau$ whose quotient is $\mathcal{S}_\lambda$ extends to an action of $\mathbb{Z}$ on X whose quotient is $K \times D^2$ as a translation inside the solid torus.

To complete the construction, it would remain only to prove that the action preserves the foliation by complex surfaces. Hence, by taking the quotient, we would obtain a foliation of $\mathcal{N}$ by complex surfaces.

However, this is definitely not true and we have to proceed as follows. Let:

$$\widetilde{X} = \mathbb{C}^* \times (\mathbb{C} \times [0, \infty[\setminus\{(0,0)\})$$

and let Γ be the group generated by the commuting diffeomorphisms T and S defined as follows:

$$\forall (z,u,t) \in \widetilde{X}, \qquad T(z,u,t) = (z, \lambda\omega \cdot u, d(t))$$
$$\text{and} \quad S(z,u,t) = (\exp(2i\pi\omega)\cdot z, (\psi(z))^{-3}\cdot u, t)$$

where d was defined in the proof of Lemma 2.

The T-action (that is the action generated by T) on $\widetilde{X}$ leaves the first coordinate z invariant and acts on $(w,t) \in \mathbb{C} \times [0,\infty[\setminus\{(0,0)\}$ exactly as the action defined in Lemma 2 so the quotient is $\mathbb{C}^* \times B$. Now the S-action, when considered on $\mathbb{C}^* \times B$, extends precisely $(\widetilde{A})$ to $\mathbb{C}^* \times B$. Therefore $\widetilde{X}$ is a regular covering of $K \times D^2$ with group of deck transformations Γ.

Let $\mathcal{F}_i$ be the foliation whose leaves L_t are the level sets in $\widetilde{X}$ of the projection on the third factor. These leaves are naturally complex manifolds biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ for $t > 0$ and $\mathbb{C}^* \times \mathbb{C}^*$ for $t = 0$. The group Γ preserves the foliation $\mathcal{F}$ and sends one leaf biholomorphically onto its image. Therefore, this foliation descends to a foliation by complex surfaces on a manifold with boundary diffeomorphic to $K \times D^2$. Furthermore, the boundary leaf is biholomorphic to $\mathcal{S}_\lambda$. To determine the nature of the other leaves, note that a leaf $\mathbb{C}^* \times \mathbb{C}$ in $\widetilde{X}$ descends to the product of $\mathbb{C}^*$ by a leaf $\mathbb{C}$ of the Reeb component in $\mathbb{C}^* \times B$, once we have taken the quotient by the T-action. Now remark that the S-action on such a leaf is just the extension to $\mathbb{C}^* \times \mathbb{C}$ of the action (A) on $\mathbb{C}^* \times \mathbb{C}^*$. Thus, the interior leaf is a line bundle over $\mathbb{E}_\omega$ obtained from the principal $\mathbb{C}^*$-bundle $W \to \mathbb{E}_\omega$ by adding a zero-section.

3. The CR-structure of $\mathcal{M} := \mathbb{S}^5 \setminus \overset{\circ}{\mathcal{N}}$

Let $f : \mathbb{S}^5 \setminus K \to \mathbb{S}^1$ be the Milnor fibration associated to the polynomial F. The fibration f is given explicitly by $F/|F|$. The fibers $F_\theta := f^{-1}(\{\exp(2i\pi\theta)\})$ are transverse to the orbits of the Hopf fibration, therefore, the monodromy of f is given by the function [Mi2]:

$$(z_1, z_2, z_3) \in F_\theta \mapsto (\omega \cdot z_1, \omega \cdot z_2, \omega \cdot z_3) \in F_\theta \ .$$

The Milnor number of the isolated singularity determined by F is equal to 8 and therefore F_θ has the homotopy type of a bouquet of 8 two-spheres [Mi2, Theorems 6.5 and 7.2].

Let $Y := F^{-1}(\mathbb{R}^+) \setminus \{(0,0,0)\} \subset \mathbb{C}^3$, where $\mathbb{R}^+ \subset \mathbb{C}$ is the set of nonnegative real numbers. We have that $Y = F^{-1}(\mathbb{R}^\bullet) \cup W$, where $\mathbb{R}^\bullet \subset \mathbb{C}$ is the multiplicative group of positive real numbers. Let $T : Y \to Y$ be the map defined by

$$T(z_1, z_2, z_3) = (\lambda\omega \cdot z_1, \lambda\omega \cdot z_2, \lambda\omega \cdot z_3)$$

Then T acts properly and discontinuously on Y.

If we let $\widehat{Y}$ denote the quotient of Y by the group generated by T we obtain a compact manifold with boundary. The boundary is exactly $\mathcal{S}_\lambda$ (see Section 2). On the other hand, $F^{-1}(\mathbb{R}^\bullet)$ is diffeomorphic to $F_\theta \times \mathbb{R}^\bullet$ for a fixed $\theta \in \mathbb{R}$, and the interior of $\widehat{Y}$ is diffeomorphic to a quotient of $F_\theta \times \mathbb{R}^\bullet$ by the action of the group generated by $h(x,t) = (\omega \cdot x, \lambda t)$. Thus it fibers over $\mathbb{S}^1$ with fiber F_θ and with the same monodromy as the Milnor fibration. As a consequence, it is diffeomorphic to $\mathbb{S}^5 \setminus K$ and thus to $\overset{\circ}{\mathcal{M}}$ (see [Mi3]).

Now, Y has a natural smooth foliation by complex manifolds whose leaves are $F^{-1}(\{t\})$ for $t > 0$, and $F^{-1}(\{0\}) \setminus \{0\}$ for $t = 0$. The action of $\widetilde{\Gamma}$ on Y sends each leaf of this foliation biholomorphically onto its image. Therefore, this foliation descends to Y as a foliation by holomorphic leaves.

This gives a smooth foliation of $\overset{\circ}{\mathcal{M}}$ by complex surfaces with boundary leaf biholomorphic to $\mathcal{S}_\lambda$. The other leaves are biholomorphic to $F^{-1}(\{t\})$ for $t > 0$. However, the gluing condition is *not* satisfied. In order to obtain a foliation in which the gluing condition is satisfied we need a variation of the previous construction and proceed as follows:

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\phi(t) = 0$, for $t \leq 0$, and $\phi(t) = e^{-\frac{1}{t}}$ $0 < t \leq 1/2$, and, finally, ϕ restricted to $\mathbb{R}^\bullet$ is a diffeomorphism of $\mathbb{R}^\bullet$ such that $\phi'(t) > 0$, for $t > 0$. Let $\widehat{h} : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\widehat{h}(t) = \phi^{-1}(\lambda^3 \phi(t))$ for $t > 0$ and $\widehat{h}(0) = 0$.

Then, there exists a such that $0 < a < 1/2$ and such that, for $0 \leq t \leq a$, the function $\widehat{h}$ coincides with the parabolic Möbius transformation α, defined by $\alpha(t) = \frac{t}{1 - 3(\log \lambda)t}$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a C^∞ diffeomorphism such that:

(1) $h(t) = t$ for $t \leq -1$.
(2) h coincides with $\widehat{h}$ for $t \geq 0$, and h coincides with the Möbius transformation α for $-a \leq t \leq a$.
(3) The only fixed points of h are 0 and the real numbers t, such that $t \leq -1$.

That such a diffeomorphism exists is for the following reason:

Let $\alpha : \mathbb{R} \cup \{\infty\} = \mathbb{S}^1 \to \mathbb{R} \cup \{\infty\} = \mathbb{S}^1$ be the extension of the Möbius transformation to the real projective line $\mathbb{S}^1$. Since α is a parabolic projective transformation, it is conjugate to the translation $t \mapsto t+1$, $\infty \mapsto \infty$. Therefore, α embeds in a flow. In other words, there exists a smooth vector field v on the circle $\mathbb{S}^1$ which integrates to a flow $f_t : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f_1 = \alpha$. If we consider now the vector field $k(x)v(x)$ on the real line $\mathbb{R} = \mathbb{S}^1 \setminus \{\infty\}$, where $k : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $k(x) = 0$ for $x \leq -1$, $k(x) > 0$ for $x > -1$ and $k(x) = 1$, for $x \geq -a$ we obtain a complete flow on the real line $\psi_t : \mathbb{R} \to \mathbb{R}$, such that the function equal to $\psi_1(t)$ for $t \leq 0$ and $\widehat{h}(t)$ for $t \geq 0$ is a diffeomorphism satisfying the conditions required for h.

Let $g : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}$ be the function defined by

$$g((z_1, z_2, z_3), t) = {z_1}^3 + {z_2}^3 + {z_3}^3 - \phi(t)$$

Let $\widehat{\Xi} = g^{-1}(\{0\})$ and $\Xi = \widehat{\Xi} \setminus (\{(0,0,0)\} \times \mathbb{R})$. Then, 0 is a regular value of the restriction of g to $\mathbb{C}^3 \setminus (\{(0,0,0)\} \times \mathbb{R})$. Therefore, Ξ is a smooth manifold (not properly embedded in $\mathbb{C}^3 \times \mathbb{R}$) and is the union of $g^{-1}(\{0\}) \cap \{t \leq 0\}$ and $g^{-1}(\{0\}) \cap \{t \geq 0\}$. The intersection of these two sets is diffeomorphic to W, furthermore, $g^{-1}(\{0\}) \cap \{t \leq 0\}$ is diffeomorphic to $W \times]-\infty, 0]$ and $g^{-1}(\{0\}) \cap \{t \geq 0\}$ is diffeomorphic to Y. Therefore, Ξ is a manifold diffeomorphic to the manifold obtained from Y when we attach the collar $W \times]-\infty, 0]$ to its boundary.

The manifold Ξ has a natural smooth foliation $\mathcal{F}_e$ by complex manifolds whose leaves $\{L_t\}_{t\in\mathbb{R}}$ are parametrized by the reals:

$$L_t := \{((z_1, z_2, z_3), t) \quad | \quad {z_1}^3 + {z_2}^3 + {z_3}^3 = \phi(t)\}, \text{ for } t \geq 0$$

and

$$L_t = \{((z_1, z_2, z_3), t) \quad | \quad {z_1}^3 + {z_2}^3 + {z_3}^3 = 0\}, \text{ for } t \leq 0$$

Let $G : \Xi \to \Xi$ be the diffeomorphism given by

$$G((z_1, z_2, z_3), t) = ((\lambda\omega \cdot z_1, \lambda\omega \cdot z_2, \lambda\omega \cdot z_3), h(t))$$

The diffeomorphism G sends leaves of the foliation $\mathcal{F}_e$ holomorphically onto leaves. In fact: $G(L_t) = L_{h(t)}$.

The group Γ, generated by G, acts properly and discontinuously on Ξ. The diffeomorphism G leaves invariant the submanifold with boundary:

$$Z := \{((z_1, z_2, z_3), t) \in \Xi \quad | \quad t \geq -1\}$$

The action of Γ on Z is proper and discontinuous, preserves the leaves of the foliation $\mathcal{F}_e$, and Γ sends each leaf L_t onto itself if $t \leq -1$. Therefore, the quotient manifold Z/Γ has a smooth codimension one foliation with holomorphic leaves. The boundary is a leaf biholomorphic to $\mathcal{S}_\lambda$ and the gluing condition is verified. The manifold Z is diffeomorphic to Y with a collar $W \times [-1, 0]$ attached to its boundary and, therefore, diffeomorphic to Y. The foliation induced in the quotient manifold $\widehat{Y} \simeq \mathcal{M}$ satifies the gluing condition and has only two compact leaves: the boundary and the leaf that corresponds to $t = 0$. They are both biholomorphic to $\mathcal{S}_\lambda$. The other leaves are either biholomorphic to the typical Milnor fibre F_θ or else biholomorphic to W, which is an infinite cyclic covering of $\mathcal{S}_\lambda$.

4. Conclusion

By use of Lemma 1, the foliations of $\mathcal{N}$ and $\mathcal{M}$ obtained respectively in Section 2 and Section 3 glue together to give a foliation of $\mathbb{S}^5$ by complex surfaces. We thus have:

Theorem. *There exists on $\mathbb{S}^5$ a smooth, codimension one, integrable and Levi flat CR-structure. The induced foliation by complex surfaces satisfies:*
(i) There are only two compact leaves both biholomorphic to an elliptic bundle over the elliptic curve $\mathbb{E}_\omega$. Since this surface has odd first Betti number it is not Kähler.
(ii) One compact leaf is the boundary of a compact set in $\mathbb{S}^5$ whose interior is foliated by line bundles over $\mathbb{E}_\omega$ with Chern class -3, obtained from W by adding a zero-section. The two compact leaves are the boundary components of a collar and the leaves in the interior of this collar are infinite cyclic coverings of the compact leaves and biholomorphic to W, thus they are principal $\mathbb{C}^*$-bundles over the elliptic curve $\mathbb{E}_\omega$.
(iii) The other leaves have the homotopy type of a bouquet of eight copies of $\mathbb{S}^2$ and they are all biholomorphic to the affine complex smooth manifold $F^{-1}(\{z\})$, $z \in \mathbb{C}^*$.

Remark 2. Due to the compact leaves, this CR-structure is not embeddable.

Remark 3. The construction above depends on the choice of a real parameter λ. Changing the value of this parameter changes the complex structure on the compact leaves. Thus the foliation presented here has a non trivial space of deformations.

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