Van Kampen's embedding obstruction for discrete groups

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Abstract

We give a lower bound to the dimension of a contractible manifold on which a given group can act properly discontinuously. In particular, we show that the n-fold product of nonabelian free groups cannot act properly discontinuously on $\mathbb{R}^{2n-1}$.

1 Introduction

In [vK33] van Kampen developed an obstruction theory for embeddings of finite n-complexes into $\mathbb{R}^{2n}$. We will briefly review van Kampen's theory in Section 2. It is natural (and straightforward) to remove the dimension restrictions and talk about a cohomological obstruction to embedding a complex into $\mathbb{R}^m$. Complexes where this obstruction does not vanish will be called *m-obstructor complexes*. The precise definition will be given below (see Definition 4). For example, the utilities graph (the join of two 3-point sets) is a 2-obstructor complex, and van Kampen proved that the n-fold join of 3-point sets is a $(2n-2)$-obstructor complex.

We introduce the notion of the *obstructor dimension* $\operatorname{obdim}\Gamma$ of a discrete group Γ (see Definition 10). For example, when the group is hyperbolic or CAT(0) and the boundary contains an m-obstructor complex, then $\operatorname{obdim}\Gamma \geq m+2$. In particular, $\operatorname{obdim} F_2^n = 2n$ (F_2 is the free group of rank 2) since the boundary of F_2^n is the n-fold join of Cantor sets and thus contains the complex considered by van Kampen.

The main theorem in this paper is the following.

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Theorem 1. *If* $\operatorname{obdim}\Gamma \geq m$ *then* Γ *cannot act properly discontinuously on a contractible manifold of dimension* $< m$.

For example, F_2^n cannot act properly discontinuously on $\mathbb{R}^{2n-1}$.
In [BF] the methods of this paper are used to prove:

Theorem 2. *[BF] Let* G *be a connected semisimple Lie group,* $K \subset G$ *a maximal compact subgroup,* G/K *the associated contractible manifold (i.e., the symmetric space when the center of* G *is finite) and* Γ *a lattice in* G. *If* Γ *acts properly discontinuously on a contractible manifold* W, *then* $\dim W \geq \dim G/K$.

There is an application of our results to Geometric Topology. The celebrated theorems of Whitney [Whi44a], [Whi44b] state that every n-manifold can be embedded in $\mathbb{R}^{2n}$ and immersed (for $n > 1$) into $\mathbb{R}^{2n-1}$. A less known theorem of Stallings [St], [DR93] asserts that every n-complex is homotopy equivalent to a complex that embeds in $\mathbb{R}^{2n}$. It is therefore natural to ask whether every n-complex can be immersed up to homotopy into $\mathbb{R}^{2n-1}$.

Corollary 3. *Let* $X = X_1 \times X_2 \times \cdots \times X_n$ *be the* n*-fold product of connected graphs* X_i *with the first betti number 2. Then* X *does not immerse up to homotopy into* $\mathbb{R}^{2n-1}$.

Proof. Suppose $X \simeq Y$ and Y immerses in $\mathbb{R}^{2n-1}$. Then Y has a thickening (an immersed in $\mathbb{R}^{2n-1}$ regular neighborhood) which is an aspherical $(2n-1)$-manifold with fundamental group F_2^n. But then the universal covering action violates Theorem 1. □

It appears that the above comlplex is the first example of aspherical simplicial complex of dimension $n \geq 3$ which does not immerse up to homotopy into $\mathbb{R}^{2n-1}$. A detailed study of thickenings in the case when $K = S^m \cup_\alpha e^n$ has 3 cells was carried out by Cooke [Coo79]. In particular, he constructs such complexes where the minimal thickenings have arbitrarily large codimension.

2 Obstructor complexes

In this section we briefly recall the work of van Kampen [vK33]. At the time his paper was written, cohomology theory was still not fully developed and many of the details were elaborated later in [Sha57] (see also [Wt65]). Van Kampen constructed an n-complex that does not embed into $\mathbb{R}^{2n}$. Van Kampen's complex is the $(n+1)$-fold join of the 3-point set, generalizing the well-known non-planar "utilities" graph. Flores [Flo35] showed that the n-skeleton of the $(2n+2)$-simplex works just as well, thus generalizing the other standard example of a non-planar graph, namely the complete graph on 5 vertices. Flores reduced the claim to the Borsuk-Ulam theorem and no additional cohomological arguments were needed.

In what follows, we shall also need examples of complexes that embed in an even dimensional Euclidean space, but not in one of lower dimension. We will follow the standard practice and blur the distinction between a simplicial complex and its geometric realization. All (co)homology groups are taken with coefficients in $\mathbb{Z}_2$.

Definition 4. Fix a non-negative integer m. A finite simplicial complex K of dimension $\leq m$ is an *m-obstructor complex* if the following holds:

1. There is a collection
$$\Sigma = \{\{\sigma_i, \tau_i\}_{i=1}^k\}$$
of unordered pairs of disjoint simplices of K with $\dim \sigma_i + \dim \tau_i = m$ that determine an m-cycle (over $\mathbb{Z}_2$) in
$$\bigcup \{\sigma \times \tau \subset K \times K | \sigma \cap \tau = \emptyset\}/\mathbb{Z}_2$$
where $\mathbb{Z}_2$ acts by $(x, y) \mapsto (y, x)$.

2. For some (any) general position map $f : K \to \mathbb{R}^m$ the (finite) number
$$\sum_{i=1}^{k} |f(\sigma_i) \cap f(\tau_i)|$$
is odd.

3. For every m-simplex $\sigma \in K$ the number of vertices v such that the unordered pair $\{\sigma, v\}$ is in Σ is even.

It turns out (see [Sha57], [Wt65], [FKT]) that Van Kampen's obstruction (conditions (1) and (2) above) is the only obstruction to existence of embedding of complexes of dimension ≥ 3 into $\mathbb{R}^{2n}$. For 2-dimensional complexes there are other obstructions as well, see [FKT], [Kr00].

2.1 Discussion and basic properties

Let K be an m-obstructor complex. If f and f' are two general position maps $K \to \mathbb{R}^m$ choose a general position homotopy H between them. A standard argument of "watching H" shows that in the presence of item 1 (in Definition 4) the two integers from item 2 for f and f' differ by an even integer. In particular, item 2 implies that K does not embed in $\mathbb{R}^m$; indeed for every map $K \to \mathbb{R}^m$ there exist two disjoint simplices of K whose images intersect.

We will view Σ as a subcomplex of $\bigcup\{\sigma \times \tau \subset K \times K | \sigma \cap \tau = \emptyset\}/\mathbb{Z}_2$. Then item 1 states that Σ is a m-pseudomanifold over $\mathbb{Z}_2$, meaning that every $(m-1)$-cell is the face of an even number of m-cells, and in particular we have the fundamental class $[\Sigma] \in H_m(\Sigma)$. Similarly, the collection $\tilde{\Sigma}$ of *ordered* pairs corresponding to the pairs in Σ can be viewed as a subcomplex of $\bigcup\{\sigma \times \tau \subset K \times K | \sigma \cap \tau = \emptyset\}$ and is an m-pseudomanifold. Further, $(x, y) \mapsto (y, x)$ is the deck transformation of the natural double cover $\tilde{\Sigma} \to \Sigma$. Let $\phi : \Sigma \to \mathbb{R}P^\infty$ be a classifying map for this double cover. We note that item 2 is equivalent to the requirement that

$$\langle \phi^*(w^m), [\Sigma] \rangle \neq 0 \in \mathbb{Z}_2$$

where $w^m \in H^m(\mathbb{R}P^\infty)$ is the nonzero class. Indeed, we can perturb $f : K \to \mathbb{R}^m$ to a map $F = (f, g) : K \to \mathbb{R}^m \times \mathbb{R} = \mathbb{R}^{m+1}$ so that $F(\sigma_i) \cap F(\tau_i) = \emptyset$ for all i. Then we have a classifying map $\phi : \Sigma \to \mathbb{R}P^m \subset \mathbb{R}P^\infty$ defined by

$$\phi(\{x, y\}) = \text{line through } F(x) \text{ and } F(y)$$

where a point of $\mathbb{R}P^m$ is viewed as the set of parallel lines in $\mathbb{R}^m$. Then $\langle \phi^*(w^m), [\Sigma] \rangle$ can be computed as the "degree" of ϕ, which in turn is the number of points of Σ mapped to the "vertical lines" $pt \times \mathbb{R}$, i.e., the number from item 2.

We could have defined the notion of an m-obstructor complex by requiring only items 1 and 2. This definition would then be equivalent to the requirement that $\Phi^*(w^m) \neq 0$, where $\Phi : \big(K \times K \setminus \Delta\big)/\mathbb{Z}_2 \to \mathbb{R}P^\infty$ is the classifying map ($\Delta \subset K \times K$ is the diagonal), and this would be closer in spirit to van Kampen's work. We impose item 3 to ensure that the Join Lemma and the Linking Lemma below hold. A restatement of item 3 is that the projection map $\pi : \tilde{\Sigma} \to K$ (say to the second coordinate) has the property that the pullback of every m-cocycle evaluates trivially on the fundamental class $[\tilde{\Sigma}]$.

Van Kampen's obstruction theory can be summarized in the following proposition.

Proposition 5. *Suppose that K is an m-obstructor complex and that W is a contractible m-manifold. Then for every map $F : K \to W$ there exist* *disjoint simplices σ and τ in K such that $F(\sigma) \cap F(\tau) \neq \emptyset$. In particular, K does not embed into W.*

Proof. The case of $W = \mathbb{R}^m$ was discussed above. For the general case, assume on the contrary that $F : K \to W$ violates the proposition. Define $\phi : \Sigma \to W \times W \setminus \Delta/\mathbb{Z}_2$ by $\phi(\{x, y\}) = \{F(x), F(y)\}$. The following lemma then implies that Σ classifies into $\mathbb{R}P^{m-1}$, a contradiction. $\square$

Lemma 6. *Suppose W is a contractible manifold of dimension m. Then the space $W \times W \setminus \Delta/\mathbb{Z}_2$ of unordered pairs of points in W is homotopy equivalent to $\mathbb{R}P^{m-1}$.*

Proof. We may assume that $n > 2$ since otherwise W is homeomorphic to $\mathbb{R}^n$. Let $U \subset W$ be a (small) open set homeomorphic to $\mathbb{R}^n$. Consider the diagram

$$\begin{array}{ccc} U \times U \setminus \Delta & \hookrightarrow & W \times W \setminus \Delta \\ \downarrow & & \downarrow \\ (U \times U \setminus \Delta)/\mathbb{Z}_2 & \hookrightarrow & (W \times W \setminus \Delta)/\mathbb{Z}_2 \end{array}$$

Note that $U \times U \setminus \Delta$ fibers over U with fiber $U \setminus pt \simeq S^{n-1}$; thus $U \times U \setminus \Delta \simeq S^{n-1}$ and similarly $W \times W \setminus \Delta \simeq S^{n-1}$; moreover, inclusion

$$U \times U \setminus \Delta \hookrightarrow W \times W \setminus \Delta$$

is a homotopy equivalence. Since for $n > 2$ the two spaces in the first row of the above diagram are simply-connected, it follows that

$$(U \times U \setminus \Delta)/\mathbb{Z}_2 \hookrightarrow (W \times W \setminus \Delta)/\mathbb{Z}_2$$

induces an isomorphism in homotopy groups, and is therefore a homotopy equivalence. $\square$

In the simplest instance, the lemma below states that the utilities graph embedded in $\mathbb{R}^3$ links every push-off of itself.

Lemma 7 (The Linking Lemma). *Suppose W is a contractible $(m+1)$-manifold, K is an m-obstructor complex and $G : K \times [0, \infty) \to W$ is a (continuous) proper map. Then there exist two disjoint simplices σ, τ in K such that $G(\sigma \times \{0\}) \cap G(\tau \times [0, \infty)) \neq \emptyset$.*

Proof. Again we first consider the case $W = \mathbb{R}^{m+1}$. Assuming the contrary, consider the homotopy $H_t : \tilde{\Sigma} \to S^m$ defined by declaring that $H_t(x, y)$ is the class of parallel rays containing the ray from $G(x, 0)$ through $G(y, t)$. Then

$H_0 : \tilde{\Sigma} \to S^m$ covers a classifying map and therefore has degree 1. Let B be a Euclidean ball centered at the origin containing $G(K \times \{0\})$ and assume that t is chosen so that $G(K \times \{t\}) \cap B = \emptyset$. There is a homotopy L_s of H_t defined by setting $L_s(x, y)$ to be the equivalence class of rays containing the ray from $(1-s)G(x, 0)$ through $G(y, t)$. Now L_1 visibly factors through the projection $\pi : \tilde{\Sigma} \to K$ and therefore by item 3 in the definition of obstructor complexes the degree of L_1, and hence of H_t, is 0. Contradiction.

For the case of a general W, replace the definition of H_t by $H_t(x, y) = (G(x, 0), G(y, t)) \in W \times W \setminus \Delta$, and replace the ball B by a compact set in which $G(K \times \{0\})$ can be homotoped to a point. $\square$

2.2 Examples

The n-complexes of van Kampen and of Flores are $(2n)$-obstructor complexes in our terminology[1]. The collection Σ consists of all pairs of disjoint n-simplices. The case of the iterated join of three points can be verified inductively noting that the three-point set is a 0-obstructor complex and using the Join Lemma. Note that item 3 is vacuous in both examples (for $n > 0$).

Lemma 8 (The Cone Lemma). *If K is an m-obstructor complex, then the cone cK is an $(m+1)$-obstructor complex.*

Proof. Let $\Sigma = \Sigma(K) = \{\{\sigma_i, \tau_i\}\}$ be the cycle for K. We define $\Sigma(cK)$ to have twice as many elements: for every $\{\sigma_i, \tau_i\} \in \Sigma$ put $\{c\sigma_i, \tau_i\}$ and $\{\sigma_i, c\tau_i\}$ into $\Sigma(cK)$. It is straightforward to check items 1 and 3.

To verify item 2, choose a general position map $f : K \to \mathbb{R}^m$, and let $(f, g) : K \to \mathbb{R}^m \times \mathbb{R}$ be a perturbation to a general position map. Put the cone point high above the hyperplane $\mathbb{R}^m \times \{0\}$ and let $\tilde{G} : cK \to \mathbb{R}^{m+1}$ be the natural extension of (f, g). Then

$$|\tilde{G}(c\sigma_i) \cap \tilde{G}(\tau_i)| + |\tilde{G}(\sigma_i) \cap \tilde{G}(c\tau_i)| = |f(\sigma_i) \cap f(\tau_i)|$$

and the claim follows. $\square$

Lemma 9 (The Join Lemma). *If K_j is an m_j-obstructor complex for $j = 1, 2$ then the join $K_1 * K_2$ is an $(m_1 + m_2 + 2)$-obstructor complex.*

Proof. Let $\Sigma^j = \Sigma(K_j) = \{\{\sigma_i^j, \tau_i^j\}\}$ be the cycle for K_j, $j = 1, 2$. We define $\Sigma(K_1 * K_2)$ to have $2|\Sigma^1||\Sigma^2|$ elements: for each $\{\sigma_i^1, \tau_i^1\} \in \Sigma^1$ and $\{\sigma_l^2, \tau_l^2\} \in \Sigma^2$ we put the following two pairs in $\Sigma(K_1 * K_2)$: $\{\sigma_i^1 * \sigma_l^2, \tau_i^1 * \tau_l^2\}$

[1] Although we will not need the Flores' complexes in what follows.

and $\{\sigma_i^1 * \tau_l^2, \tau_i^1 * \sigma_l^2\}$. Item 3 is vacuous for $\Sigma(K_1 * K_2)$ as there are no (m_1+m_2+2)-simplices in $\Sigma(K_1 * K_2)$.

To verify that $\Sigma(K_1 * K_2)$ is a cycle, suppose first that $\sigma * \tau$ and $\sigma' * \tau'$ are disjoint simplices of $K_1 * K_2$ (with σ and σ' simplices of K_1 and τ, τ' simplices of K_2) and that the sum of their dimensions is m_1+m_2+1. If $\dim(\sigma)+\dim(\sigma') > m_1$ or if $\dim(\tau)+\dim(\tau') > m_2$ then the corresponding (m_1+m_2+1)-cell is not a face of any (m_1+m_2+2)-cells in $\Sigma(K_1 * K_2)$. So without loss of generality we may assume that $\dim(\sigma)+\dim(\sigma') = m_1$ and $\dim(\tau)+\dim(\tau') = m_2 - 1$. Since $\{\tau, \tau'\}$ represents an (m_2-1)-cell, item 1 for K_2 implies that there is an even number m_2-cells $\{\tilde{\tau}_p, \tau'\}$ and $\{\tau, \tilde{\tau}'_q\}$ in Σ^2 that contain $\{\tau, \tau'\}$. Since $\{\sigma * \tilde{\tau}_p, \sigma' * \tau'\}$ and $\{\sigma * \tau, \sigma' * \tilde{\tau}'_q\}$ are precisely the (m_1+m_2+2)-cells in $\Sigma(K_1 * K_2)$ that contain $\{\sigma * \tau, \sigma' * \tau'\}$ the verification of item 1 in this case is finished.

Now suppose that $\sigma * \tau$ and $\sigma' * \tau'$ are disjoint simplices of $K_1 * K_2$ and that the sum of their dimensions is m_1+m_2+1. The number of ways of enlarging this cell to an (m_1+m_2+2)-cell in $\Sigma(K_1 * K_2)$ is either 0 (if $\{\sigma, \sigma'\} \notin \Sigma^1$) or it equals the number of vertices $v \in K_2$ such that $\{\tau, v\} \in \Sigma^2$, which is even by item 3 for K_2. Thus item 1 is verified for $\Sigma(K_1 * K_2)$.

It remains to verify item 2. Let $f_j : K_j \to \mathbb{R}^{m_j}$ be general position maps and I_j the total number of intersection points of f_j-images of unordered pairs of simplices in Σ^j. View $\mathbb{R}^{m_i}$ as $\mathbb{R}^{m_i} \times \{0\} \subset \mathbb{R}^{m_i+1}$ and $\mathbb{R}^{m_1+m_2+2}$ as $\mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1}$. Perturb f_j to a general position map $\tilde{G}_j = (f_j, g_j) : K_j \to \mathbb{R}^{m_j} \times \mathbb{R} = \mathbb{R}^{m_j+1}$ and let $G : K_1 * K_2 \to \mathbb{R}^{m_1+m_2+2}$ be the linear join of $\tilde{G}_1$ and $\tilde{G}_2$. The number of intersection points of G-images of unordered pairs of simplices in $\Sigma(K_1 * K_2)$ is $I_1 I_2$. The details are left to the reader. $\square$

3 The main theorem

Recall that a (continuous) map $h : A \to B$ is *proper* if the preimages of compact sets are compact. We say that maps $h_1 : A_1 \to B$ and $h_2 : A_2 \to B$ into a metric space B *diverge* (from each other) if for every $D > 0$ there are compact sets $C_i \subset A_i$ such that $h_1(A_1 \setminus C_1)$ and $h_2(A_2 \setminus C_2)$ are $> D$ apart. If K is a finite complex, we define the open cone $cone(K) = K \times [0, \infty)/K \times \{0\}$. If K is also an obstructor complex, we say that a proper map $h : cone(K) \to B$ is *expanding* if for disjoint simplices σ, τ in K the maps $h|cone(\sigma)$ and $h|cone(\tau)$ diverge. It will also be convenient to make the analogous definition on the level of 0-skeleta. Triangulate $cone(K)$ so that $cone(\sigma)$ is a subcomplex whenever σ is a simplex of K. We say that a proper map $h : cone(K)^{(0)} \to B$ is *expanding* if for all pairs σ, τ of disjoint simplices in K the restrictions $h|cone(\sigma)^{(0)}$ and $h|cone(\tau)^{(0)}$ diverge. We also equip $cone(K)^{(0)}$ with the edge-path metric, so that a map $h : cone(K)^{(0)} \to B$ is Lipschitz if there is a uniform upper bound on the distance between the images of adjacent vertices in $cone(K)^{(0)}$.

Note that if $h : cone(K) \to B$ is a proper expanding map, then there is $t_0 \geq 0$ such that the map $G : K \times [0, \infty) \to B$ defined by $G(x, t) = h([x, t + t_0])$ satisfies $G(\sigma \times \{0\}) \cap G(\tau \times [0, \infty)) = \emptyset$ for any two disjoint simplices σ, τ of K.

A proper map $h : A \to B$ between proper metric spaces is *uniformly proper* if there is a proper function $\phi : [0, \infty) \to [0, \infty)$ such that

$$d_B(h(x), h(y)) \geq \phi(d_A(x, y))$$

for all $x, y \in A$. This notion is weaker than the notion of a quasi-isometric embedding, which would require ϕ to be a linear function.

Let Γ be a finitely generated group equipped with the word-metric with respect to some finite generating set. We make the following definitions.

Definition 10. The *obstructor dimension* $\operatorname{obdim}(\Gamma)$ is defined to be 0 for finite groups, 1 for 2-ended groups, and otherwise $m+2$ where m is the largest integer such that for some m-obstructor complex K and some triangulation of the open cone $cone(K)$ as above there exists a proper, Lipschitz, expanding map $f : cone(K)^{(0)} \to \Gamma$. If no maximal m exists we set $\operatorname{obdim}(\Gamma) = \infty$.

Remark 11. Clearly, one can replace Γ in the above definition by any quasi-isometric proper metric space. In particular, if Γ acts cocompactly, properly discontinuously, and isometrically on a proper geodesic metric space X, we can substitute X for Γ. Moreover, if Γ is of type F_{m+1} (see e.g., [Br82]) so that X can be chosen to be m-connected, then f can be extended to a proper, expanding map $\tilde{f} : cone(K) \to X$ with a uniform bound on the diameter of the image of any simplex. One advantage of having the (continuous) map defined on the whole cone is that the requirement that the map be Lipschitz can be dropped: one can always triangulate $cone(K)$ to make the same map Lipschitz.

Note that if Γ is infinite and not 2-ended, then we can take K to consist of 3 points, so $\operatorname{obdim}(\Gamma) \geq 2$.

Definition 12. The *uniformly proper dimension* $\operatorname{updim}(\Gamma)$ is the smallest integer n such that there is a contractible n-manifold W equipped with a proper metric d_W so that there is a Lipschitz, uniformly proper map $g : \Gamma \to W$ and so that in addition there is a *contractibility function* $\rho : (0, \infty) \to (0, \infty)$ such that any ball of radius r centered at a point of the image of g is contractible in the ball of radius $\rho(r)$ centered at the same point. If no such n exists we set $\mathrm{updim}(\Gamma) = \infty$.

Remark 13. One usually requires of the contractibility function that the statement about balls be true regardless of where the center is. If we omit the requirement altogether, the invariant would be trivial: every finitely generated group admits a uniformly proper map into $[0, \infty)$. Just choose an injective map $g : \Gamma \to \mathbb{N} \subset [0, \infty)$. The largest metric on $[0, \infty)$ that makes g 1-Lipschitz and makes all $[n, n+1]$ isometric to a standard closed interval (of length dependent on n) is proper. Of course, this metric is not a path-metric, but insisting on path-metrics would only raise the dimension by 1: For every Γ there is a proper path-metric on $\mathbb{R}^2$ and a uniformly proper map $\Gamma \to \mathbb{R}^2$.

Definition 14. The *action dimension* $\mathrm{actdim}(\Gamma)$ is the smallest integer n such that Γ admits a properly discontinuous action on a contractible n-manifold. If no such n exists, then $\mathrm{actdim}(\Gamma) = \infty$.

Denote also by $\mathrm{gdim}(\Gamma)$ the *geometric dimension* of Γ, i.e., the minimal n such that Γ admits a properly discontinuous action on a contractible n-complex. Recall that for virtually torsion-free groups Γ, $\mathrm{gdim}(\Gamma)$ is conjectured to be equal to the virtual cohomological dimension vcdim of Γ and that the only potential counterexamples would have $\mathrm{gdim} = 3$ and $\mathrm{vcdim} = 2$ (see [Br82]).

We note that $\mathrm{updim}(\Gamma) \leq \mathrm{actdim}(\Gamma)$ by choosing a proper invariant metric on W and taking an orbit of the action, and that for torsion-free groups Γ we have $\mathrm{actdim}(\Gamma) \leq 2 \cdot \mathrm{gdim}(\Gamma)$ by the Stallings theorem cited in the introduction. Alternatively, we could find a $(2n)$-dimensional thickening of an n-complex by immersing it in $\mathbb{R}^{2n}$ and taking a regular neighborhood. The inequality $\mathrm{actdim}(\Gamma) \leq 2 \cdot \mathrm{gdim}(\Gamma)$ is false for groups with torsion; indeed, the free product $A_5 * A_5$ acts properly discontinuously on a tree but not on the plane. On the other hand, $\Gamma = A_5 * A_5$ contains a free subgroup Γ' of finite index, hence $2 = \mathrm{actdim}(\Gamma') < \mathrm{actdim}(\Gamma)$.

The main theorem in this note is:

Theorem 15. $\mathrm{obdim}(\Gamma) \leq \mathrm{updim}(\Gamma)$.

Proof. The special cases when $\mathrm{obdim}(\Gamma) \leq 1$ are clear. Let K be an m-obstructor complex and $f : cone(K)^{(0)} \to \Gamma$ a proper, Lipschitz, expanding map. Let W be a contractible n-manifold with a proper metric and $g : \Gamma \to W$ a uniformly proper Lipschitz map satisfying the contractibility function requirement. Consider the composition $gf : cone(K)^{(0)} \to W$. Now extend gf inductively over the skeleta of $cone(K)$ to get a map $G : cone(K) \to W$.

Using the contractibility function, we can arrange that the diameter of the image of each simplex of $cone(K)$ is uniformly bounded. It follows that G is a proper expanding map, and therefore $n \geq m+2$ by the Linking Lemma 7. $\square$

This theorem immediatately implies the following chain of inequalities (with the last inequality only for torsion-free groups):

$$\operatorname{obdim}(\Gamma) \leq \operatorname{updim}(\Gamma) \leq \operatorname{actdim}(\Gamma) \leq 2 \operatorname{gdim}(\Gamma) \tag{1}$$

The second inequality can be strict. The Baumslag-Solitar group

$$B = \langle x, t | xt = t^2 x \rangle$$

is not a 3-manifold group and so $\operatorname{actdim}(B) = 2 \cdot \operatorname{gdim}(B) = 4$. On the other hand, $\operatorname{obdim}(B) = \operatorname{updim}(B) = 3$. The group B admits a uniformly proper map into $\mathbb{H}^3$ and the universal cover of the presentation 2-complex admits a proper expanding map defined on the open cone on the tripod, which is a 1-obstructor complex. All three invariants in (1) are monotone, in the sense that if Γ' is a finitely generated subgroup of Γ, then $anydim(\Gamma') \leq anydim(\Gamma)$. We also note that both obdim and updim are invariant under quasi-isometries. This is not the case for actdim (even for torsion-free groups) as there are examples of torsion-free groups that are not 3-manifold groups but contain 3-manifold groups as finite index subgroups [KK].

Lemma 16.

$$\operatorname{obdim}(\Gamma_1 \times \Gamma_2) \geq \operatorname{obdim}(\Gamma_1) + \operatorname{obdim}(\Gamma_2)$$

while

$$\operatorname{updim}(\Gamma_1 \times \Gamma_2) \leq \operatorname{updim}(\Gamma_1) + \operatorname{updim}(\Gamma_2)$$

and

$$\operatorname{actdim}(\Gamma_1 \times \Gamma_2) \leq \operatorname{actdim}(\Gamma_1) + \operatorname{actdim}(\Gamma_2).$$

Proof. The latter two statements are obvious, while the first one follows from the Join Lemma 9. The product $cone(K_1) \times cone(K_2)$ can naturally be viewed as $cone(K_1 * K_2)$ and the product map into $\Gamma_1 \times \Gamma_2$ satisfies the requirements. (If one of the two groups is 2-ended, use the Cone Lemma instead.) $\square$

Corollary 17. *In particular, we see that for* $\Gamma = F_2^n$ *all three invariants* obdim, *updim and* actdim *are* $2n$ *and the inequalities in the chain (1) are equalities.*

If Γ has a reasonable boundary, it may be easier to compute obdim(Γ). The following definition is taken from [Bes96].

Definition 18. Let Γ be a group. A *$\mathcal{Z}$-structure* on Γ is a pair $(\tilde{X}, Z)$ of spaces satisfying the following four axioms.

- $\tilde{X}$ is a Euclidean retract.
- Z is a Z-set in $\tilde{X}$.
- $X = \tilde{X} \setminus Z$ admits a covering space action of Γ with compact quotient.
- The collection of translates of a compact set in X forms a null-sequence in $\tilde{X}$, i.e., for every open cover $\mathcal{U}$ of $\tilde{X}$ all but finitely many translates are $\mathcal{U}$-small.

A space Z is a *boundary* of Γ if there is a $\mathcal{Z}$-structure $(\tilde{X}, Z)$ on Γ.

For example, torsion-free hyperbolic groups and $CAT(0)$ groups[2] admit a boundary. However, unlike for hyperbolic groups, boundary of a $CAT(0)$-group G is not uniquely determined by G (up to a homeomorphism) [CK00].

Corollary 19. *Suppose Z is a boundary of Γ and $f : K \to Z$ is a map from an m-obstructor complex that sends disjoint simplices disjointly (e.g. f could be an embedding). Then* obdim$(\Gamma) \geq m + 2$. ☐

Proof. Let $X, \tilde{X}$ be as in the definition. Since Z is a Z-set in $\tilde{X}$, there is a homotopy $H : K \times [0, 1] \to \tilde{X}$ with $H(x, 0) = *$, $H(x, 1) = f(x)$ and $H(K \times (0, 1]) \cap Z = \emptyset$. Restricting to $K \times [0, 1)$ and reparametrizing yields an expanding map $cone(K) \to X$. ☐

It is convenient to introduce the notation

$$\text{“}K \subset \partial\Gamma\text{”}$$

to mean that there is a proper expanding Lipschitz map $cone(K)^{(0)} \to \Gamma$ as in the definition of obdim. The above corollary implies

$$K \subset \partial\Gamma \Rightarrow \text{“}K \subset \partial\Gamma\text{”}$$

[2] I.e. groups which admit discrete cocompact isometric action on a $CAT(0)$-space.

Example 20. The n-fold join of Cantor sets is a boundary of F_2^n and it contains van Kampen's $(2n-2)$-obstructor complex. Thus $\mathrm{obdim}(F_2^n) = 2n$ and all inequalities in the chain (1) are equalities.

Remark 21. It seems to be believed by the experts that there are n-dimensional torsion-free hyperbolic groups Γ with boundary the Menger universal $(n-1)$-dimensional compactum. For such a group all inequalities would be equalities as well, but no such examples of hyperbolic groups are known except for small n.

Somewhat more generally, consider a group G acting discretely isometrically on a CAT(0)-space X with the ideal boundary $D = \partial_\infty X$. (We do not assume that this action is cocompact.) Pick a base point $x \in X$ and $C \in \mathbb{R}_+$. The C-cone limit set $\Lambda_C(G)$ of G consists of points $\xi \in D$ such that for the geodesic ray ρ in X emanating from x and representing ξ, there exists an infinite sequence $g_n \in G$ such that $d(g_n x, \rho) \leq C$. The arguments from Corollary 19 imply

Corollary 22. *If for some C, $\Lambda_C(G)$ contains an m-obstructor complex, then* $\mathrm{obdim}(G) \geq m$.

4 Short exact sequences

We now investigate the obstructor dimension of a group G that fits in a short exact sequence

$$1 \to H \to G \xrightarrow{\pi} Q \to 1$$

where all groups are finitely generated. The natural guess is that

$$\mathrm{obdim}\, G \geq \mathrm{obdim}\, H + \mathrm{obdim}\, Q \tag{2}$$

and this is what we prove under certain technical assumptions on π (admits a Lipschitz section) and H (weakly convex). All groups are equipped with word metrics. We note that *some* restrictions are clearly neccessary for (2) to hold. For instance, Rips in [Rip82] constructs examples of 2-dimensional hyperbolic groups G which admit epimomorphisms $G \to Q$ (where Q is the prescribed finitely presented group) so that the kernel H is finitely generated (and is neither finite nor 2-ended). Note that G can be assumed to have Menger curve boundary [KK00]. Then $obdim(G) = 4$, $\mathrm{obdim}\, H \geq 2$ and Q can be chosen to have $\mathrm{obdim}(Q)$ as large as one likes. See also Example 28.

Definition 23. We say that a finitely generated group Γ is *weakly convex* if there is a collection of (discontinuous, of course) paths $\{\phi_{z,w} : [0,1] \to \Gamma\}_{z,w \in \Gamma}$ and a constant $M > 0$ satisfying the following properties:

1. $\phi_{z,w}(0) = z$ and $\phi_{z,w}(1) = w$.
2. There is a function $\gamma : [0, \infty) \to [0, \infty)$ such that
$$d(z, w) \leq R \Longrightarrow \operatorname{diam}(Im(\phi_{z,w})) \leq \gamma(R).$$
3. For all $z, w \in \Gamma$ there is $\epsilon > 0$ such that $\phi_{z,w}$ sends subintervals of length $< \epsilon$ to sets of diameter $< M$.
4. If $d(z, z') \leq 1$ and $d(w, w') \leq 1$ then for all $t \in [0, 1]$
$$d(\phi_{z,w}(t), \phi_{z',w'}(t)) \leq M.$$

Remark 24. The paths are to be thought of as being piecewise constant. We could avoid talking about discontinuous functions by requiring that they be defined only on the rationals in $[0, 1]$. It is more standard to think of paths in Γ as eventually constant 1-Lipschitz functions defined on non-negative integers; however, for what follows it is important that all paths be defined on the same bounded set. It is possible to reparametrize such paths by "constant speed" paths defined on $[0, 1]$. The collection of paths as above is usually called a "combing" (except for the domain being $[0, 1]$). Condition 2 is then a weak version of the requirement that the combing be quasi-geodesic and it follows automatically if the combing is equivariant (i.e., $\phi_{gz,gw} = L_g \circ \phi_{z,w}$, where $L_g : \Gamma \to \Gamma$ denotes left translation by g). Condition 3 is the replacement of the 1-Lipschitz requirement. Condition 4 is the "Fellow Traveller" property.

If Γ and Γ' are quasi-isometric and one is weakly convex, so is the other. Hyperbolic groups, $CAT(0)$ groups, and semi-hyperbolic groups [AB95] are weakly convex.

We can regard the given paths in the definition of weak convexity as a recipe for extending maps into Γ defined on the (ordered) vertices of a 1-simplex to the whole 1-simplex. It is easy to see that one can similarly extend maps defined on the vertices of an n-simplex for any $n > 0$, with the constant $M = M(n)$ above depending on n. By
$$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} | t_i \geq 0, t_0 + t_1 + \cdots + t_n = 1\}$$
we denote the standard n-simplex, and by $I_{n,k}$ the standard face inclusion $\Delta^{n-1} \hookrightarrow \Delta^n$ onto the face $t_k = 0$ given by
$$I_{n,k}(t_0, t_1, \cdots, t_{n-1}) = (t_0, t_1, \cdots, 0, \cdots, t_{n-1}).$$

Proposition 25. *Let Γ be a weakly convex group. Then for every $n > 0$ there is a constant $M(n)$ and for every $(n+1)$-tuple $(z_0, z_1, \cdots, z_n) \in \Gamma^{n+1}$ there is a function $\phi_{z_0,z_1,\cdots,z_n} : \Delta^n \to \Gamma$ such that*

- $\phi_{z_0,z_1,\cdots,z_n}(v_k) = z_k$ *where $v_k \in \Delta^n$ is the vertex with $t_k = 1$.*
- *There is a function $\gamma_n : [0, \infty) \to [0, \infty)$ such that*

$$d(z_i, z_j) \leq T \text{ for all } i, j \Longrightarrow \operatorname{diam}(Im(\phi_{z_0,z_1,\cdots,z_n})) \leq \gamma_n(T)$$

- *For all $z_0, z_1, \cdots, z_n$ there is $\epsilon > 0$ such that the $\phi_{z_0,z_1,\cdots,z_n}$-images of sets of diameter $< \epsilon$ have diameter $< M(n)$.*
- *If $d(z_i, w_i) \leq 1$ then*

$$d(\phi_{z_0,z_1,\cdots,z_n}(t), \phi_{w_0,w_1,\cdots,w_n}(t)) \leq M(n)$$

- $\phi_{z_0,z_1,\cdots,z_n} \circ I_{n,k} = \phi_{z_0,z_1,\cdots,\hat{z}_k,\cdots,z_n}$

Proof (sketch). Functions $\phi_{z_0,z_1,\cdots,z_n}$ are constructed by induction on n, with the case $n = 1$ being the definition. The inductive step consists of defining $\phi_{z_0,z_1,\cdots,z_n}$ on the boundary of Δ^n so that the last item above holds and then extending to the interior by coning off from the first vertex. More precisely, if $\psi : [0, 1] \to \Delta^n$ is a linear map with $\psi(0) = v_0$ and $\psi(1)$ belongs to the face with $t_0 = 0$ then

$$\phi_{z_0,z_1,\cdots,z_n}(\psi(t)) = \phi_{z_0,w}(t)$$

where $w \in \Gamma$ is the image of $\psi(1)$ under the (partially defined) $\phi_{z_0,z_1,\cdots,z_n}$. Checking the properties listed above is straightforward (by construction, the restriction of the function to a face containing v_0 is already the cone on the opposite face). □

Theorem 26. *Let*

$$1 \to H \to G \xrightarrow{\pi} Q \to 1$$

be a short exact sequence of finitely generated groups. Suppose that H is weakly convex and that π admits a Lipschitz section $s : Q \to G$. Then

$$\operatorname{obdim} G \geq \operatorname{obdim} H + \operatorname{obdim} Q$$

Proof. If H (or Q) is finite, then G is quasi-isometric to Q (or H) and equality holds. If H (or Q) is 2-ended, we can use $K_H = point$ (or $K_Q = point$) in the proof below and appeal to the Cone Lemma. If both H and Q are 2-ended, then G is virtually $\mathbb{Z} \times \mathbb{Z}$ and thus $\mathrm{obdim}(G) = 2$, $\mathrm{obdim}(H) = \mathrm{obdim}(Q) = 1$, so equality again holds.

Let $\alpha : cone(K_H)^{(0)} \to H$ and $\beta : cone(K_Q)^{(0)} \to Q$ be proper Lipschitz expanding maps defined on the vertices of a fine triangulation of the cones on obstructor complexes K_H and K_Q. Define

$$f : cone(K_H * K_Q)^{(0)} = cone(K_H)^{(0)} \times cone(K_Q)^{(0)} \to G$$

by

$$f(x, y) = \alpha(x) \cdot s\beta(y).$$

Claim 1. f is a proper map.

Indeed, let (x_i, y_i) be a sequence in $cone(K_H)^{(0)} \times cone(K_Q)^{(0)}$ leaving every finite set. If the sequence $\pi f(x_i, y_i) = \beta(y_i) \in Q$ leaves every finite set, the same is true for $f(x_i, y_i) \in G$. Otherwise, after passing to a subsequence, we may assume that the sequence $\pi f(x_i, y_i) = \beta(y_i) \in Q$ stays in a finite set $D \subset Q$. Then $s\beta(y_i)$ stays in the finite set $s(D)$. Since β is a proper map, the sequence $y_i \in cone(K_Q)^{(0)}$ stays in a finite set, and thus the sequence $x_i \in cone(K_H)^{(0)}$ leaves every finite set. Since α is a proper map, we see that the sequence $f(x_i, y_i) = \alpha(x_i) \cdot s\beta(y_i)$ leaves every finite set.

Claim 2. If $\sigma = \sigma_H * \sigma_Q$ and $\tau = \tau_H * \tau_Q$ are disjoint simplices of $K_H * K_Q$, then $f|cone(\sigma)^{(0)}$ and $f|cone(\tau)^{(0)}$ diverge.

Indeed, let (x_i, y_i) and (x'_i, y'_i) be sequences in $cone(\sigma_H)^{(0)} \times cone(\sigma_Q)^{(0)}$ and $cone(\tau_H)^{(0)} \times cone(\tau_Q)^{(0)}$ respectively, leaving every finite set. Note that π is a Lipschitz map, so if one of two sequences $\pi f(x_i, y_i) = \beta(y_i)$ and $\pi f(x'_i, y'_i) = \beta(y'_i)$ leaves every finite set in Q, then $d_Q(\beta(y_i), \beta(y'_i)) \to \infty$ (since $\beta|cone(\sigma_Q)^{(0)}$ and $\beta|cone(\tau_Q)^{(0)}$ diverge) and consequently

$$d_G(f(x_i, y_i), f(x'_i, y'_i)) \to \infty.$$

Now assume that both sequences $\beta(y_i)$ and $\beta(y'_i)$ are contained in a fixed finite set $D \subset Q$. Then we have

$$\begin{aligned} d_G(f(x_i, y_i), f(x'_i, y'_i)) = d_G(\alpha(x_i) \cdot s\beta(y_i), \alpha(x'_i) \cdot s\beta(y'_i)) = \\ d_G(1, s\beta(y_i)^{-1}\alpha(x_i)^{-1}\alpha(x'_i)s\beta(y'_i)) \end{aligned}$$

Since $s\beta(y_i)$ and $s\beta(y'_i)$ stay in a finite set and

$$d_H(1, \alpha(x_i)^{-1}\alpha(x'_i)) = d_H(\alpha(x_i), \alpha(x'_i)) \to \infty$$

it follows that $d_G(1, \alpha(x_i)^{-1}\alpha(x_i')) \to \infty$ and

$$d_G(1, s\beta(y_i)^{-1}\alpha(x_i)^{-1}\alpha(x_i')s\beta(y_i')) \to \infty,$$

and the claim is proved.

The remaining problem is that f is not Lipschitz.

Claim 3. The restriction of f to $\{q\} \times cone(K_Q)^{(0)}$ is Lipschitz with the Lipschitz constant independent of q.

Indeed, let x, y be two adjacent vertices in $cone(K_Q)^{(0)}$.

$$d_G(f(q,x), f(q,y)) = d_G(\alpha(q) \cdot s\beta(x), \alpha(q) \cdot s\beta(y)) = d_G(s\beta(x), s\beta(y))$$

and the claim follows from the assumption that s is Lipschitz.

For every $p \in cone(K_Q)^{(0)}$ let f_p denote the restriction of f to the slice $cone(K_H)^{(0)} \times \{p\}$. Recall that $L_{s\beta(p)}$ is the left translation by $s\beta(p)$ and it induces an isometry between $H = \pi^{-1}(1)$ (with the G-metric) and $\pi^{-1}(\beta(p))$.

Claim 4. $L^{-1}_{s\beta(p)} f_p : cone(K_H)^{(0)} \times \{p\} \to H$ is Lipschitz with respect to the word-metric on H (but the Lipschitz constant depends on p). In particular, $f_p : cone(K_H)^{(0)} \times \{p\} \to G$ is Lipschitz.

Indeed, $L^{-1}_{s\beta(p)} f_p(x,p) = s\beta(p)^{-1} \cdot \alpha(x) \cdot s\beta(p)$ which is Lipschitz.

We next order all vertices of $cone(K_H)$ and then extend (simplex-by-simplex) $L^{-1}_{s\beta(p)} f_p$ for each p to the map $\tilde{F}_p : cone(K_H) \times \{p\} \to H$ using the weak convexity of H and Proposition 25. Then define

$$\tilde{f}_p = L_{s\beta(p)}\tilde{F}_p : cone(K_H) \times \{p\} \to \pi^{-1}(p).$$

Let

$$\tilde{f} : cone(K_H) \times cone(K_Q)^{(0)} \to G$$

be defined as $\tilde{f}_p$ on each $cone(K_H) \times \{p\}$.

We now note that for $n = \dim cone(K_H)$ and for $M = M(n)$ from Proposition 25 we have that:

- For each $p \in cone(K_Q)^{(0)}$ there is $\epsilon(p) > 0$ so that sets of diameter $< \epsilon(p)$ in a simplex of $cone(K_H) \times \{p\}$ are sent by $\tilde{f}$ to sets of diameter $< M$.

- If p, p' are adjacent vertices in $cone(K_Q)$ then $\tilde{f}(x,p)$ and $\tilde{f}(x,p')$ are kM-close for any $x \in cone(K_H)$, where k is a Lipschitz constant for $s\beta$.

For each $p \in cone(K_Q)^{(0)}$ choose a positive integer $m(p)$ so that the simplices of the $m(p)^{th}$ barycentric subdivision of $\sigma \times \{p'\}$ have diameter $< \epsilon(p')$ for all vertices p' at distance ≤ 1 from p.

We now define a triangulation of $cone(K_H) \times cone(K_Q)$. Start with a decomposition into cells of the form $\sigma \times \tau$ where τ is a simplex of $cone(K_Q)$ and σ is a simplex of the $k(\tau)^{th}$ barycentric subdivision of a simplex of $cone(K_H)$ with $k(\tau) = \min\{m(p), p \in \tau^{(0)}\}$. Now triangulate each such cell inductively on the dimension so that the vertex set of the triangulation is precisely

$$\{(v,p) | p \in cone(K_Q)^{(0)}, v \text{ is a vertex of the } m(p)^{th} \\ \text{barycentric subdivision of } cone(K_H)\}$$

The restriction of $\tilde{f}$ to the vertex set is now a Lipschitz function.

Claim 5. This restriction is still proper and expanding.

The proof closely follows proofs of Claims 1 and 2. If the sequence $\beta(y_i)$ (resp. one of the sequences $\beta(y_i)$ or $\beta(y_i'))$ leaves every finite set, the proof is exactly the same as in Claims 1 and 2. Otherwise, without loss of generality, the sequence $\tilde{f}(x_i, y_i)$ (resp. sequences $\tilde{f}(x_i, y_i)$ and $\tilde{f}(x_i', y_i'))$ belong to finitely many slices of the form $cone(K_H) \times \{p\}$ and therefore lie a bounded distance away from sequences considered in Claims 1 and 2 coming from the vertices of the original triangulation. The proof follows. $\square$

Corollary 27. *If $G = H \rtimes Q$ with H and Q finitely generated and H weakly convex, then* $\operatorname{obdim} G \geq \operatorname{obdim} H + \operatorname{obdim} Q$.

Example 28. Let $G = F_2^n$. Define $\phi : G \to \mathbb{Z}$ by sending the basis elements of each factor to $1 \in \mathbb{Z}$. Let $H = Ker(\phi)$. It is easy to see that H contains a copy of F_2^n and thus $\operatorname{obdim}(G) = \operatorname{obdim}(H) = 2n$. Therefore $G = H \rtimes \mathbb{Z}$, but $\operatorname{obdim}(H) + \operatorname{obdim}(\mathbb{Z}) = 2n + 1 > \operatorname{obdim}(G)$. It follows that H is not weakly convex, and one knows [B76] that H is of type F_{n-1} (in particular, it is finitely generated for $n \geq 2$, finitely presented for $n \geq 3$, etc.).

Remark 29. Example 28 shows that in the above corollary weak convexity of H is a necessary assumption.

We now apply the above theorem to the group $Out(F_n)$ of outer automorphisms of $F_n = \langle x_1, \cdots, x_n \rangle$, the free group of rank n. Recall [CV86] that the virtual cohomological dimension of $Out(F_n)$ is $2n - 3$ $(n > 1)$ and that $Out(F_n)$ is virtually torsion-free. It follows from the Stallings theorem that $\operatorname{obdim}(Out(F_n)) \leq 4n - 6$. We note that equality holds, since $Out(F_n)$ contains as a subgroup a group of the form $F_2^{2n-4} \rtimes F_2$. Indeed, choose $F_2 < Aut(F_2)$ that injects into $Out(F_2)$ and let it act diagonally on F_2^{2n-4}. The corresponding semi-direct product is realized by the subgroup of $Aut(F_n)$ that injects into $Out(F_n)$ as follows. Send an element

$$u = (w_3, v_3, ..., w_n, v_n, \alpha) \in F_2^{2n-4} \rtimes F_2, w_i, v_i \in F_2 = \langle x_1, x_2 \rangle,$$
$$\alpha \in F_2 \subset Aut(F_2)$$

to the automorphism ϕ_u of F_n which acts on $\langle x_1, x_2 \rangle$ as the automorphism α and maps x_i $(i \geq 3)$ to $w_i x_i v_i^{-1}$. The reader will verify that $u \mapsto \phi_u$ is indeed a monomorphism $F_2^{2n-4} \rtimes F_2$ to $Aut(F_n)$ whose image projects injectively to $Out(F_n)$.

We conjecture that obdim for the mapping class group of a closed oriented surface of genus g is $6g-6$, the dimension of the associated Teichmüller space. This is analogous to Theorem 2 stated in the Introduction.

Let B_n denote the braid group on n strands. As evidence for the conjecture in the previous paragraph, we note that $\operatorname{obdim} B_n = 2n-3$ $(n \geq 2)$. This can be seen as follows. Denote by $P_n < B_n$ the pure braid group on n strands. Then $P_n = F_{n-1} \rtimes P_{n-1}$ so the statement that $\operatorname{obdim} P_n = 2n-3$ follows by induction from Corollary 27. But $\operatorname{obdim} B_n = \operatorname{obdim} P_n$ since P_n has finite index in B_n.

5 Questions

We conclude this note with the following questions about the invariants introduced in this paper.

1. Is $\operatorname{obdim}\Gamma = \operatorname{updim}\Gamma$ for all Γ? The answer is probably negative as stated, but the question should be interpreted liberally: Is $\operatorname{updim}\Gamma$ detected homologically? (Compare this with a theorem of Kuratowsky and Claytor [Cl24] that a 1-dimensional continuum without global cut-points is planar provided that it contains neither the complete graph on five vertices, nor the "utility" graph.)
2. Suppose M_i is a compact aspherical n_i-manifold with all boundary components aspherical and incompressible, $i = 1, ..., k$. If M_i is not homotopy equivalent to an $(n_i - 1)$-manifold, and if $G = \pi_1(M_1) \times \cdots \times \pi_1(M_k)$, is $\operatorname{actdim} G = n_1 + \cdots + n_k$? Update (November 2001): Sung Yil Yoon has answered this question in the affirmative in his PhD thesis [Yoon].
3. Is the assumption of the existence of a Lipschitz section in Theorem 26 necessary if in addition H, G, Q have finite type (i.e., have finite Eilenberg-MacLane spaces)?

4. What is actdim(Γ) for uniform/nonuniform S-arithmetic groups? Every such group acts on the product of symmetric spaces and buildings. A natural guess is that the answer equals the sum of dimensions of the symmetric spaces plus twice the sum of dimensions of the buildings.
5. Are there groups of finite type (i.e., groups of type FP) which are not weakly convex? For instance, are the fundamental groups of 3-dimensional Nil-manifolds weakly convex?

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