1. Compute the following limits.

(a) \( \lim_{x \to 3} \frac{e^{1/x}}{x} \)

**Answer:** By continuity, \( \lim_{x \to 3} \frac{e^{1/x}}{x} = e^{1/3} \)

(b) \( \lim_{x \to 3^+} \ln(x - 3) \)

**Answer:** \( \lim_{x \to 3^+} \ln(x - 3) = -\infty \)

(c) \( \lim_{x \to 3} \frac{\ln(x - 2)}{x - 3} \)

**Answer:** By L'Hôpital

\[
\lim_{x \to 3} \frac{\ln(x - 2)}{x - 3} = \lim_{x \to 3} \frac{\frac{1}{x-2}}{1} = 1
\]

(d) \( \lim_{x \to \infty} \frac{8x^5 - 7x^3 + 9}{(3x^2 - 1)(2x^3 - 3)} \)

**Answer:**

\[
\lim_{x \to \infty} \frac{8x^5 - 7x^3 + 9}{(3x^2 - 1)(2x^3 - 3)} = \lim_{x \to \infty} \frac{8 - 7x^{-2} + 9x^{-5}}{(3 - x^{-2})(2 - 3x^{-3})} = \frac{8 - 0 + 0}{3 - 0} = \frac{4}{3}.
\]

(e) \( \lim_{x \to \infty} \frac{e^x}{x^3} \)

**Answer:** Since \( e^x \) grows faster than any polynomial, \( \lim_{x \to \infty} \frac{e^x}{x^3} = \infty \).

(f) \( \lim_{x \to 0} \frac{\arctan x - x}{x^3} \)

**Answer:** By L'Hôpital,

\[
\lim_{x \to 0} \frac{\arctan x - x}{x^3} = \lim_{x \to 0} \frac{1 - 1}{3x^2} = \lim_{x \to 0} \frac{-2x}{6x^3} = -\frac{1}{3}.
\]
2. Compute the derivatives of the following functions.

(a) \( y = x \ln x - x \)
Answer: \( y' = \ln x + 1 - 1 = \ln x \)
(b) \( y = e^{3x^2} \)
Answer: \( y' = 6xe^{3x^2} \)
(c) \( y = (x - 1)^5 \cos x \)
Answer: \( y' = 5(x-1)^4 \cos x - (x-1)^5 \sin x \)
(d) \( y = \sin(\ln x) \)
Answer: \( y' = \cos(\ln x) \cdot \frac{1}{x} \)
(e) \( y = \frac{e^{3x+2}}{\cos x + 2} \)
Answer: \( y' = \frac{3e^{3x+2}(\cos x + 2) + \sin xe^{3x+2}}{(\cos x + 2)^2} \)
(f) \( y = \sin(e^x - x^2 - 1) \)
Answer: \( y' = \cos(e^x - x^2 - 1) \cdot (e^x - 2x) \)
(g) \( y = \frac{\arctan(4x)}{x^2 + 1} \)
Answer: \( y' = \frac{4(x^5+1) - 5x^4 \arctan(4x)}{(x^2+1)^2} \)

3. Find the absolute minima and maxima of the functions on the given domain.

(a) \( f(x) = x^2 e^{-x} \) on \([0, 1]\)
Answer: \( f'(x) = 2xe^{-x} - x^2 e^{-x} = 0 \) iff \( x = 0 \) or \( x = 2 \). The only critical point on the given domain is at \( x = 0 \) and this is also one of the endpoints. Since \( f(0) = 0, f(1) = e^{-1} \) the absolute min is \( f(0) = 0 \) and the absolute max is \( f(1) = e^{-1} \).
(b) \( f(x) = 2x^3 - 3x^2 \) on \([-1, 2]\)
Answer: The critical points are as follows: \( f'(x) = 6x^2 - 6x = 0 \) iff \( x = 0 \) or \( x = 1 \). Since \( f(-1) = -2 - 3 = -5, f(0) = 0, f(1) = -1, f(2) = 16 - 12 = 4, \) the absolute min is \( f(-1) = -5 \) and the absolute max is \( f(2) = 4 \).
(c) \( f(x) = \sin^2 x \) on \([0, \pi]\)
Answer: \( f'(x) = 2\sin x \cos x = 0 \) iff \( \cos x = 0 \) or \( \sin x = 0 \). In the given domain this happens at \( x = 0, \pi/2, \pi \). Since \( f(0) = 0, f(\pi/2) = 1, f(\pi) = 0 \) the absolute max is \( f(\pi/2) = 1 \) and the absolute min is \( f(0) = f(\pi) = 0 \).
(d) \( f(x) = \frac{x}{1+x^2} \) on \([-2, 2]\).
Answer: \( f'(x) = \frac{1-x^2}{(1+x^2)^2} = 0 \) iff \( x = \pm 1 \). Since \( f(-2) = -2/5, f(-1) = -1/2, f(1) = 1/2, f(2) = 2/5, \) the absolute max is \( f(1) = 1/2 \) and the absolute min is \( f(-1) = -1/2 \).
4. Find the equation of the tangent line to the curve at the given point:

(a) \( y = \ln x \) at \( x = 5 \).

**Answer:** \( y' = 1/x \) so the tangent line is of the form \( y = \frac{1}{5}y + b \). Since the point \((5, \ln 5)\) has to lie on this line, \( \ln 5 = \frac{5}{5} + b \) gives the equation \( y = \frac{1}{5}y + (\ln(5) - 1) \).

(b) \( \ln(x^2 + y - 1) + xy + y^4 = 2 \) at \((x, y) = (1, 1)\).

**Answer:** Implicit differentiation gives

\[
\frac{2x + y'}{x^2 + y - 1} + y + xy' + 4y^3y' = 0.
\]

At \((x, y) = (1, 1)\) this simplifies to

\[
\frac{2 + y'}{1} + 1 + y' + 4y' = 0 \iff y' = -\frac{1}{2}.
\]

In particular the tangent line is of the form \( y = -\frac{1}{2}x + b \) and since \( 1 = -\frac{1}{2} + b \), \( b = \frac{3}{2} \) so the tangent line is given by \( y = -\frac{1}{2}x + \frac{3}{2} \).

5. For the given function \( f(x) \):

- Find the domain of \( f(x) \).
- Find the derivative of \( f(x) \) and determine the open intervals where \( f(x) \) is increasing/decreasing using a sign chart. Determine also the critical points.
- Find the second derivative of \( f(x) \) and determine the open intervals where \( f(x) \) is concave up/down. Determine the inflection points.
- *Good practice but not on the final* Graph the function using the information above.

(a) \( f(x) = 2x^3 - 3x^2 + 1 \)

**Answer:** The domain is \( \mathbb{R} = (-\infty, \infty) \). The derivative is \( f'(x) = 6x^2 - 6x \), which is 0 iff \( x = 0, 1 \). So \((0, f(0)), (1, f(1))\) are the critical points. As \( f'(-1) = 12, f(0.5) = 6/4 - 3 = -3/2, f(2) = 12 \), the sign chart looks like

\[
\begin{array}{ccc}
+ & 0 & -1 & + \\
\end{array}
\]

so the function is increasing on \((0, 0) \cup (1, \infty)\) and decreasing on \((0, 1)\). The second derivative is \( f''(x) = 12x - 6 \) which is zero iff \( x = 1/2 \). As \( f(0) = -6, f(1) = 6 \), the function is concave up when \( x > 1/2 \) and concave down when \( x < 1/2 \). The graph is below.
(b) \( f(x) = xe^{-x} \)

**Answer:** The domain is \( \mathbb{R} \), and the derivative is \( f'(x) = e^{-x} - xe^{-x} \). This is zero when \( x = 1 \), so this is the only critical point on the domain. Since \( f'(0) = e^{-1} \), \( f'(2) = -e^{-2} \) the sign chart looks like

\[
\begin{array}{c|c|c}
\text{Interval} & f'(x) & \text{Sign} \\
\hline
(-\infty, 1) & + & + \\
(1, \infty) & - & - \\
\end{array}
\]

and therefore the function is increasing for \( x < 1 \) and decreasing for \( x > 1 \). The second derivative is \( f''(x) = -2e^{-x} + xe^{-x} \), so there is an inflection point at \( x = 2 \). As \( f''(0) = -2, f''(3) = e^{-3} \), the function is concave up when \( x > 2 \) and concave down when \( x < 2 \). See the graph below.
(c) \( f(x) = \ln(x^2 + 1) \)

**Answer:** The domain is wherever \( x^2 + 1 > 0 \), which is everywhere i.e. \( \mathbb{R} \). The derivative is \( f'(x) = \frac{2x}{x^2+1} \), and the only critical point is at \( x = 0 \). The sign chart is

\[
\begin{array}{c|cc}
\text{x} & - & + \\
\hline
\text{f'(x)} & - & + \\
\end{array}
\]

and thus the function is increasing on \( x > 0 \) and decreasing on \( x < 0 \). The second derivative is \( f''(x) = \frac{2(x^2+1)-4x^2}{(x^2+1)^2} \) which is zero iff \( 2 - 2x^2 = 0 \) so \( x = \pm 1 \). Therefore the function has inflection points at \( x = \pm 1 \), and as \( f''(-2) = -6/25, f''(0) = 2, f''(2) = -6/25 \) the function is concave down on \((-\infty, -1) \cup (1, \infty)\) and concave up on \((-1, 1)\).

(d) \( f(x) = \frac{x-1}{x+1} \)

**Answer:** The domain is where \( x \neq -1 \) and the derivative is \( f'(x) = \frac{2}{(x+1)^2} \). This is never zero so there are no critical points in the domain. However, at \( x = -1 \) the derivative is also undefined so this should be considered a critical point. The derivative is always positive, so the sign chart is

\[
\begin{array}{c|cc}
\text{x} & + & + \\
\hline
\text{f'(x)} & + & + \\
\end{array}
\]

The second derivative is \( f''(x) = \frac{2(x-1)}{(x+1)^3} - \frac{2}{(x+1)^2} = -\frac{4}{(x+1)^3} \). This is never zero but undefined at \( x = -1 \). The function is concave up on \((-\infty, -1)\) and concave down on \((-1, \infty)\).
6. Consider the function

\[ f(x) = \begin{cases} 
  x + 1, & \text{if } x < -1, \\
  x^2 + ax + b, & \text{if } x \geq -1.
\end{cases} \]

(a) For which values of the parameters \(a, b\) is this function continuous?

\textbf{Answer:} The function is continuous everywhere iff \(\lim_{x \to c} f(x) = f(c)\) for all \(c\). Since the branches are given by (restrictions of) continuous functions, the continuity condition only needs to be checked at \(x = -1\). Since \(\lim_{x \to -1^-} x + 1 = 0\) and \(\lim_{x \to -1^+} x^2 + ax + b = 1 - a + b\), we need to find \(a, b\) such that these two limits equal. So the function is continuous for all \((a, b)\) such that \(0 = 1 - a + b\).

(b) For which parameter values is the function differentiable everywhere?

\textbf{Answer:} The only problematic point is again at \(x = -1\). As the branches have derivatives \(1\) and \(2x + a\), the function is differentiable everywhere iff \(-2 + a = 1\) so \(a = 3\). This implies by part (a) that \(b = 2\).
7. Consider the curve given by the equation \( x^{2/3} + y^{2/3} = 1 \). Find \( y' \) using implicit differentiation and find the equation of the tangent line to the curve at \((x, y) = (0, 1)\). Sketch the curve.

**Answer:** Implicit differentiation gives
\[
\frac{2}{3} (x^{-1/3} + y^{-1/3} y') = 0 \iff y' = -\left(\frac{y}{x}\right)^{1/3}.
\]
At \((x, y) = (0, 1)\) the right-hand side makes no sense a priori, but at least informally it is easy to see \( y' = -\frac{1}{0} = -\infty \). Therefore the tangent line is vertical, and is given by the equation \( x = 0 \). To sketch the curve, one can note that the points \((0, \pm 1), (\pm 1, 0)\) are on the curve and analyze the concavity properties given by \( y'' = -\frac{1}{3} \left(\frac{x}{y}\right)^{3/2} \cdot \frac{y - y x}{y^3} \).

8. You are designing a rectangular poster to contain \( 50 \text{ in}^2 \) of printing with 4in margins at the top and bottom and 2in margins at each side. What overall dimensions will minimize the amount of paper used?

**Answer:** Let the width of the print be \( x \) and the height of the print be \( y \). Then the poster has dimensions \( x + 2 \cdot 2, y + 2 \cdot 4 \) and area \( A = (x+4)(y+8) \). Since \( xy = 50 \) and \( x \neq 0 \), we know \( y = \frac{50}{x} \). Therefore the total area of the poster, which is directly proportional to the amount of paper used, is given by
\[
A = (x+4)\left(\frac{50}{x} + 8\right) = 50 + 8x + \frac{200}{x} + 32.
\]
The relevant (\( x \) has to be positive) critical points of this function are \( A' = 8 - \frac{200}{x^2} = 0 \implies x = 5 \). In this case \( A = 9 \cdot 18 = 162 \). There are no endpoints to check, in particular it is easy to see that \( A \to \infty \) if \( x \to \infty \), so this has to be a local minimum. Therefore the dimensions are \( x = 9, y = 18 \).
9. An open rectangular box with square base is to be made from wood. There is a total of 1 m² wood available. What dimensions of the box will result in a box with the largest possible volume?

**Answer:** If the top is is open and the base is a square, the area of the box with height $z$ and width $x$ is $A = 1 = x^2 + 4xz$. As the volume is given by $V = x^2z$, we can solve the area constraint in terms of $z$ to get $z = \frac{1-x^2}{4x} = \frac{1}{4x} - \frac{x}{4}$ and therefore

$$V = x^2 \left( \frac{1}{4x} - \frac{x}{4} \right) = \frac{1}{4} x - \frac{x^3}{4}.$$ 

This has critical points

$$V' = \frac{1}{4} - \frac{3}{4} x^2 = 0 \iff x = \pm 1/\sqrt{3}.$$ 

The only relevant one is $x = \frac{1}{\sqrt{3}}$, so $z = \frac{1-\frac{1}{\sqrt{3}}^2}{4 \cdot \frac{1}{\sqrt{3}}} = \frac{1}{2\sqrt{3}}$. As $V \to 0$ with $x \to 0$ and $V \to -\infty$ as $x \to \infty$, the maximum volume is given by these dimensions.

10. A TV set costs 100 dollars. If its price is lowered by $a\%$, the sales will increase by $2a\%$. Find the discount percentage $a$ resulting in maximal profit.

**Answer:** The new price after a discount of $a\%$ is exactly $P = 100 - a$ dollars. The profit is given by the volume of the sales times the price, $Q = \frac{100+2a}{100} \cdot (100 - a) \cdot S$, where $S$ is the original amount of sales (which is a constant). Therefore

$$Q'(a) = S \left( 1 - \frac{a}{25} \right) = 0 \iff a = 25.$$ 

At the left endpoint where $a = 0$ we get $Q = 100 \cdot S$ and at the critical point we get $Q = \frac{25}{2} \cdot S$ which is larger. So the discount percentage maximizing profit is $a = 25\%$.

11. The surface area of a cube is increasing at the rate of $6 ft^2/min$. Determine the rate at which the volume of the cube is changing, when the edge of the cube is 2 ft long.

**Answer:** If the sidelenes is $x$, the area is given by $A = 6x^2$ and the volume is $V = x^3$. As $A'(t) = 12xx' = 6$ is $A'(t_0) = 12 \cdot 2 \cdot x'(t_0)$ when the edge is 2 feet long, $x'(t_0) = \frac{1}{4} ft/min$. In particular, $V'(t_0) = 3x^2(t_0)x'(t_0) = 12/4 = 3 ft^3/min$. 

12. Among all tangent lines to the graph of \( y = \frac{1}{1+x^2} \), find the equation of the tangent line with minimal slope.

**Answer:** The tangent lines have slope \( y' = \frac{-2x}{(1+x^2)^2} \) at \( x \). As a function of \( x \), this has derivative

\[
y'' = \frac{6x^2 - 2}{(1 + x^2)^3} = 0 \iff x = \pm \sqrt{\frac{1}{3}}.
\]

In particular, the slope has local extrema at these points. Since \( y'(1/\sqrt{3}) = \frac{-2/\sqrt{3}}{16/9} < 0 \) and \( y'(-1/\sqrt{3}) = \frac{2/\sqrt{3}}{16/9} > 0 \), and it is clear that \( y' \to 0 \) as \( x \to \pm \infty \), the minimal slope is attained at \( x = 1/\sqrt{3} \). Since \( y(1/\sqrt{3}) = \frac{3}{4} \), and \( y'(1/\sqrt{3}) = -\frac{3\sqrt{3}}{8} \), we get that the equation of the tangent line is

\[
y = -\frac{3\sqrt{3}}{8} x + b,
\]

where \( \frac{3}{4} = -\frac{3\sqrt{3}}{8} \cdot \frac{1}{\sqrt{3}} + b \iff b = \frac{9}{8}. \)