Research Highlights and Future Plans

Garving Kevin Luli
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My research covers a wide range of topics in harmonic analysis, with a special focus on problems related to smooth extension of functions.

At the basic level, the extension problems ask for efficient methods to find a function to fit through some data points while minimizing some quantities (e.g. norms, energy functionals). The data points are not assumed to have any geometric structure (e.g. a grid). For $C^{m,\omega}(\mathbb{R}^n)$ (the space of m-times continuously differentiable functions whose highest derivatives have modulus continuity ω), I established that one can construct an extension (that almost minimizes the $C^{m,\omega}$ norm) in a particularly simple way; namely, the value of the extension (as well as its derivatives) at any point is a linear combination of boundedly many given function values (**Adv. Math.**, 2010); the bound depends only on m and n. Representations of this type facilitate the implementation of efficient algorithms. In the setting of Sobolev spaces, we showed that in general a Sobolev extension may not admit a sparse representation (with Fefferman and Israel, **Rev. Mat. Iberoam.**, 2014). However, by introducing some assisted quantities, we are able to give a sharp description for the structure of Sobolev extensions (with Fefferman and Israel, **J. Amer. Math. Soc.**, 2013). In a series of three papers totaling over 300 pages (with Fefferman and Israel) (**Rev. Mat. Iberoam.**, 2016, I, **Rev. Mat. Iberoam.**, 2016, II, **Part III**, to appear), we made all the steps in the 2013 **J. Amer. Math. Soc.** paper algorithmically effective and obtained the analogous Fefferman-Klartag algorithms for Sobolev spaces. Our results yield the first ever robust algorithms for solving Dirichlet type of problems (i.e. minimizing energy functionals subject to boundary constraints) on arbitrary finite sets.

The work on the extension problems resonates with various topics in pure and applied mathematics. J. Kollár is one of the first people to study the continuous closure of a polynomial ideal, i.e., the set of functions that can be written as a linear combination (with continuous functions as the coefficients) of the polynomials in an ideal. This problem lies at the intersection between analysis and real-algebraic geometry. As such it calls for tools from both disciplines. In a joint work (with Fefferman, **Rev. Mat. Iberoam.**, 2013), we initiated the study of $C^m, C^{m,\omega}$ (i.e., differentiable) closures of polynomial ideals; we gave a necessary and sufficient condition to decide when a given function is in the C^m or $C^{m,\omega}$ closure of a given polynomial ideal by means of a solving a vector-valued extension problem. Recently, we showed how to classify all functions that are in the C^m and $C^{m,\omega}$ closures of a given polynomial ideal. Roughly speaking, we show that given a matrix A of functions we can compute a finite list of linear differential operators such that $AF = f$ admits a $C^m(\mathbb{R}^n)$ solution F if and only if f is annihilated by the linear differential operators. Moreover, we can algorithmically compute the generators for all f for which $AF = f$ admits a $C^m(\mathbb{R}^n)$ solution F. This is akin to the classical Frobenius theorem, which allows one to find the maximal set of independent solutions to a system of first order linear PDEs. Our theorem allows higher-order differential operators. Our results completely resolve the original problem raised by Hochster and Kollár.

In a recent paper (**Geometric and Functional Analysis**, 2016), together with my co-authors, we completely resolved the Brudnyni-Shvartsman Conjecture (1994), which states the following: Suppose $E \subset \mathbb{R}^n$ (finite) and at each point $x \in E$ we are given a convex sets $K(x) \subset \mathbb{R}^d$. If for any $S \subset E$ with $\#(S) \le c(m,n,d)$ for some constant $c(m,n,d)$ there exists $F^S \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that $F^S(x) \in K(x)$ for all $x \in S$ and with $||F^S||_{C^m(\mathbb{R}^n,\mathbb{R}^d)} \le 1$; then there exists $F \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that $||F||_{C^m(\mathbb{R}^n,\mathbb{R}^d)} \le C(m,n,d)$ and $F(x) \in K(x)$ for all $x \in E$. In other words, from the effective local properties, we can deduce a global property. For $d = 1, 2$ and when $m = 0, 1$, the conjecture was settled by Brudnyi-Shvartsman. If we assume further that $K(x)$ are symmetric about a point in $\mathbb{R}^d$ and in the conclusion we allow $F(x) \in CK(x)$ for some C depending only on

m, n, d, Fefferman had also proved the conjecture.

In a related paper (**2016**), we showed how in principle when given the function values (nonnegative) on a closed subset $E \subset \mathbb{R}^n$, one can extend the function to all of $\mathbb{R}^n$ while making the $||\cdot||_{C^m}$ norm within a universal factor of the least possible and preserving the nonnegativity on all of $\mathbb{R}^n$.

My current research supported by an NSF CAREER grant deals with solving variational problems with boundary and obstacle constraints on arbitrary subsets of $\mathbb{R}^n$ using extension theory. Rather than resorting to techniques in partial differential equations to solve the obstacle problems, we will construct the solutions directly using the recent theory developed for extension problems. I would like to develop a complete theory revolving around the belief that any variational problem that can be solved using PDE theory can also be dealt with using extension theory. I would also like to study semi-algebraic extensions that preserve smoothness.