V*-algebras

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“Set theory” means \textbf{ZF}, the Zermelo-Fraenkel axioms.

\textbf{Theorem (Vitali, 1905)}
Assume \textbf{AC}. There exists a set $X \subseteq \mathbb{R}$ that is not Lebesgue measurable.

\textbf{Definition (Chang, 1971)}
The Chang Model is the smallest inner model of set theory closed under countable unions.

\textbf{Theorem (Woodin, 1994)}
Assume \textbf{AC} and the existence of a proper class of Woodin cardinals that are limits of Woodin cardinals. Then the Chang Model satisfies \textbf{DC}, \textbf{AC}_{a.e.}, and \textbf{AD}^+.

The axiom \textbf{AD}^+ implies that every subset of $\mathbb{R}$ is Lebesgue measurable.
We can verify theorems in the Chang Model by analyzing their statements, rather than their proofs.

Some theorems hold in full generality:

- Spectral theorem
- Double commutant theorem

Some theorems hold only for separable C*-algebras:

- Gelfand duality
- Gelfand-Naimark theorem

Theorems about separable operator algebras generally hold:

- Kirchberg-Phillips classification
- Classification of hyperfinite factors with separable predual
Everything that follows is formulated in the Chang Model.

Let $\mathcal{H}$ be a Hilbert space.

**Definition**

The continuum weak topology on $\mathcal{B}(\mathcal{H})$ is the initial topology for the functionals

$$x \mapsto \int_{\mathbb{R}} \langle \xi_t | x \eta_t \rangle \, dt$$

for $(\xi_t), (\eta_t) \in L^2(\mathbb{R}, \mathcal{H})$.

The continuum weak topology is finer than the ultraweak topology. However, if $x_n \to x$ ultraweakly, then $x_n \to x$ continuum weakly.
**Definition**

A concrete C*-algebra $A \subseteq B(\mathcal{H})$ is a V*-algebra iff it is closed in the continuum weak topology.

- Every von Neumann algebra is a V*-algebra.
- Every nondegenerate V*-algebra on a *separable* Hilbert space is a von Neumann algebra.

**Example**

The C*-algebra of bounded functions $\mathbb{R} \to \mathbb{C}$ with countable support is a nonunital V*-algebra.

The projections of a V*-algebra form an approximate unit.
Let $E$ be a V*-algebra.

**Lemma**

Let $(\varphi_t : t \in \mathbb{R})$ be an indexed family of continuum weakly continuous functionals on $E$ such that $(\|\varphi_t\| : t \in \mathbb{R}) \in L^1(\mathbb{R})$. Then

$$\varphi : x \mapsto \int \varphi_t(x) \, dt$$

is continuum weakly continuous.

**Lemma**

If $\varphi : E \rightarrow \mathbb{C}$ is a continuum weakly continuous state, then the GNS representation $\gamma_\varphi : E \rightarrow \mathcal{B}(\mathcal{G}_\varphi)$ is continuum weakly continuous.
The enveloping $V^*$-algebra

Let $A$ be an abstract C*-algebra, and let $\gamma_S$ be its universal representation:

$$
\gamma_S : a \mapsto \bigoplus_{\varphi \in S} \gamma_{\varphi}(a)
$$

**Definition**

The enveloping $V^*$-algebra of $A$ is $V^*(A) = \overline{\gamma_S(A)}^{cw}$.

**Theorem**

For all $V^*$-algebras $E$, and all $\ast$-homomorphisms $\pi : A \to E$, there is a unique continuum weakly continuous $\ast$-homomorphism $\rho : V^*(A) \to E$ such that $\rho \circ \gamma_S = \pi$.

$$
\begin{array}{ccc}
A & \xrightarrow{\gamma_S} & V^*(A) \\
\downarrow{\pi} & & \downarrow{\rho} \\
E & \downarrow{!} & \\
\end{array}
$$
If $A$ is a separable $C^*$-algebra, then $A \cong C_0(X)$ for some locally compact Hausdorff $X$, and $V^*(A) \cong \ell_\infty(X)$.
Strongly affine functions

**Definition**

Let $K$ be a metrizable compact convex subset of a locally convex space. Then $f : K \to \mathbb{C}$ is strongly affine in case $f(x_0) = \int f(x) \, dm$ whenever $m$ is a probability measure on $K$ with barycenter $x_0$.

**Theorem**

*Let $A$ be a unital separable C*-algebra, and let $x \in \gamma(A)''$. Then $x \in V^*(A)$ iff $\hat{x} : S(A) \to \mathbb{C}$ is strongly affine.*

Recall that $\hat{x}(\varphi) = \langle \xi_\varphi | \gamma_S(x) \xi_\varphi \rangle$. 
The atomic representation

Let $A$ be an abstract C*-algebra, and let $\gamma_{\partial S}$ be its atomic representation:

$\gamma_{\partial S} : a \mapsto \bigoplus_{\varphi \in \partial S} \gamma_{\varphi}(a)$

**Theorem**

*If $A$ is separable, then $V^*(A) \cong \overline{\gamma_{\partial S}(A)}^{cw}$.***
Let $A$ be a separable C*-algebra. Then $A$ is type I iff $\hat{A} \subsetneq \mathbb{R}$.

Thus, a C*-algebra that is not type I has more than continuum many irreducible representations.
The enveloping $V^*$-algebra of a type I $C^*$-algebra

**Theorem**

Let $A$ be a separable $C^*$-algebra of type I. Then

$$V^*(A) \cong \bigoplus_{\gamma \in \hat{A}} \gamma(A)'',$$

Thus, $V^*(A)$ is an $\ell^\infty$ direct sum of type I factors.

**Example**

The enveloping $V^*$-algebra $V^*(C^*(F_2))$ is not a von Neumann algebra.