Reasoning about incomplete structures

Andre Kornell

UC Davis

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For motivation, I will argue that

ZFC is not justified for the universe \( U \) of all pure sets.

I will not argue
1. that ZFC is inconsistent, nor
2. that infinite sets do not exist.

intended picture

The standard model of ZFC is a transitive set \( V \subsetneq U \).
which classes form sets?

intuition
A class of objects is a set just in case all first-order formulas that quantify over that class are abstractly decidable (true or false).

Dummett’s definite. Weaver’s surveyable. Classically, completed.

argument sketch
Assume that all first-order formulas that quantify over the class $U$ of pure sets are abstractly decidable, i.e., that $U$ is a set. Contradiction (Russell’s paradox). Thus, some formulas that quantify over $U$ are not abstractly decidable, so classical reasoning is not justified.
potentialism in practice

ontological

\( V \) is a set in the metatheory

Indefinite extensibility! (Grothendieck universes)

epistemological

truth in \( V \) is a predicate in the metatheory

Indefinite expandability! (metamathematics)
Always more sets; always more words.

goal

to reason about the \textit{whole} hierarchy

Gödel’s program concerns the whole set-theoretic universe \( U \).
Mathematical practice presumes a single semantics for mathematical formulas, not a hierarchy of semantics.
Let $\tau$ be a functional vocabulary consisting of those constructions on sets that are deemed primitive and possible in principle, e. g., $\emptyset$, ${\cdot, \cdot}$, $\bigcup$, $\wp$.

premise

A formula $\phi$ is meaningful in the full universe $U$ just in case it is $\Sigma(\tau)$.

(We confine negation to literals: $\in$ and $\notin$ are distinct predicate symbols.)

Each $\Sigma(\tau)$ formula may be viewed as a nondeterministic transfinite process. Each existential quantifier correspond to the nullary process $E$, which has arbitrary outputs. (Example: $\exists x : \emptyset \in x$.) Each bounded quantifier corresponds to something like a for loop. (Example: $\forall x \in \wp(\wp(\emptyset)) : \emptyset \in x \lor \emptyset = x$.)
A positivistic proof is a finite sequence of $\Sigma(\tau)$ formulas: $\phi_0, \phi_1, \ldots, \phi_n$.

Each rule of inference $\phi_{i-1} \vdash \phi_i$ expresses the intuition that if the process $\phi_{i-1}$ halts, then so does the process $\phi_i$. The proof itself supports the intuition that if the process $\phi_0$ halts, then so does $\phi_n$.

However, the proof does not establish that the formula $\phi_0 \Rightarrow \phi_n$ is true, or even meaningful. Consider the distinction between the inference rule modus ponens, and the formula $(\phi \land (\phi \Rightarrow \psi)) \Rightarrow \psi$. 
logical axioms of positivistic propositional logic

positive connectives

\[ \land \quad \lor \quad T \quad \bot \]

\[ \bot \Rightarrow \psi \quad \phi \Rightarrow T \]

\[ \phi \land \psi \Rightarrow \phi \quad \phi \land \psi \Rightarrow \psi \quad \phi \Rightarrow \phi \land \phi \]

\[ \psi \lor \psi \Rightarrow \psi \quad \phi \Rightarrow \phi \lor \psi \quad \psi \Rightarrow \phi \lor \psi \]

\[ (\phi \lor \psi) \land \chi \Rightarrow (\phi \land \chi) \lor (\psi \land \chi) \]

deep inference

The implication symbol \( \Rightarrow \) indicates that these rules of inference may be applied to subformulas.
Same notion of proof, but a new conceptual question: what is the interpretation of the free variable in a proof $\phi(x) \vdash \psi(x)$?

The intuition supported by a proof $\phi() \vdash \psi()$ already has a modal character because the processes these formulas correspond to are generally nondeterministic. If we imagine this scenario in terms of a register machine, our intuition is that we are confident that process $\psi()$ will halt after we witness $\phi()$ halt, however that occurred. In the case of a proof $\phi(x) \vdash \psi(x)$, the process $\psi(x)$ is simply sensitive to the contents of one of the registers, left over from the process $\exists x: \phi(x)$.

Thus, the intuition supported by a proof $\phi(x) \vdash \psi(x)$ is that we are confident the process $\psi(x)$ will halt on an object, after we witness $E$ produce an object and $\phi(x)$ halt on it.
logical axioms of positivistic predicate logic

positivistic quantifiers

\[\exists v: \quad \forall v \in t: \]

\[\phi(t) \Rightarrow \exists v: \phi(v) \quad t \in s \land \forall v \in s: \phi(v) \Rightarrow \phi(t)\]

\[\exists w: \psi \Rightarrow \psi \quad \psi \Rightarrow \forall w \in z: (\psi \land w \in z)\]

\[\psi \land \exists w: \phi \Rightarrow \exists w: (\psi \land \phi) \quad \forall w \in z: (\psi \lor \phi) \Rightarrow \psi \lor (\forall w \in z: \phi)\]

\[\top \Rightarrow s \in t \lor s \not\in t \quad s \in t \land s \not\in t \Rightarrow \bot\]

*w* not free in \(\psi\)

completeness (K.)

Let \(T\) be a set of implications between \(\Sigma(\tau)\) formulas that includes the logical axioms. Let \(\phi\) and \(\psi\) be \(\Sigma(\tau)\) formulas. The theory \(T\) logically implies \(\phi \Rightarrow \psi\) iff there is a positivistic proof of \(\psi\) from \(\phi\) that uses the substitution instances of elements in \(T\) as (deep) rules of inference.
Primitive recursive set functions generalize primitive recursive functions on the natural numbers (Jensen & Karp). Rathjen introduced a parallel generalization of the theory PRA, which he called PRS. The (reformulated) axioms of PRS include:

1. \((\forall x \in s : x \in t) \land (\forall x \in t : x \in s) \Rightarrow s = t\)
2. \(\exists x : x \in t \Rightarrow \exists x \in t : \forall y \in t : y \notin x\)
3. \(u \in \{s, t\} \iff u = s \lor u = t\)
4. \(s \in \bigcup t \iff \exists x \in t : s \in x\)
5. \(s \in \{x \in t \mid \phi(x, \bar{u})\} \iff s \in t \land \phi(s, \bar{u})\) \(\phi\) a \(\Delta_0\) formula
6. \(s \in \{F(x, \bar{u}) \mid x \in t\} \iff \exists x \in t : s = F(x, \bar{u})\)

We can relativize this notion of primitive recursion by including the axiom:

- \(s \in \varphi(t) \iff \forall x \in s : x \in t\)

(PR5 also has axioms for equality, inequality, and primitive recursion.)

We define PRS\(^+\) by including the \(\Delta_0\)-collection axiom schema, and WOP.
The theory PRS$^+$ and its natural extensions prove Tarski’s truth axioms for a $\Sigma(\tau)$ predicate $T$:

1. $T(\phi \land \psi) \vdash T(\phi) \land T(\psi)$
2. $T(\phi \lor \psi) \vdash T(\phi) \lor T(\psi)$
3. $T(\exists v : \psi(v)) \vdash \exists a : T(\psi(a))$
4. $T(\forall v \in b : \phi(v)) \vdash \forall a \in b : T(\phi(a))$
5. $T(P(t_1, \ldots, t_n)) \vdash \exists a_1 : \ldots \exists a_n : P(a_1, \ldots, a_n) \land T(a_1 = t_1) \land \ldots \land T(a_n = t_n)$
6. $T(a = F(t_1, \ldots, t_n)) \vdash \exists b_1 : \ldots \exists b_n : a = F(b_1, \ldots, b_n) \land T(b_1 = t_1) \land \ldots \land T(b_n = t_n)$
Theories that prove their own correctness

The $T$ be the theory $PRS^+$, or one of its natural extensions. We can formalize the unary predicate $I_T(x)$ that expresses that $x$ is an inference rule of $T$. Let $T'$ be $T$ together with the axiom

$$T(\phi) \land I_T(\phi \vdash \psi) \Rightarrow T(\psi).$$

**Correctness (K.)**

The theory $T'$ proves its own correctness: $T(\phi) \land I_{T'}(\phi \vdash \psi) \vdash T(\psi)$.

The symbols $\phi$ and $\psi$ are variables. Thus if $\phi$ and $\psi$ are produced by $E$, and it is verified that $\phi$ halts and that $\phi \vdash \psi$ is an inference rule of $T$, we are confident that $\psi$ will halt. Colloquially, we may say that $T(\phi) \land I_T(\phi \vdash \psi)$ implies $T(\psi)$ for any $\phi$ and $\psi$. 
Introduction Rules for \( \land \) and \( \lor \)

\[
\begin{align*}
(\exists \phi \in K) & \quad \Gamma, \phi \vdash \Delta \\
\hline
\Gamma, \land K \vdash \Delta
\end{align*}
\]

\[
\begin{align*}
(\forall \phi \in K) & \quad \Gamma \vdash \Delta, \phi \\
\hline
\Gamma \vdash \Delta, \land K
\end{align*}
\]

\[
\begin{align*}
(\forall \phi \in K) & \quad \Gamma, \phi \vdash \Delta \\
\hline
\Gamma, \lor K \vdash \Delta
\end{align*}
\]

\[
\begin{align*}
(\exists \phi \in K) & \quad \Gamma \vdash \Delta, \phi \\
\hline
\Gamma \vdash \Delta, \lor K
\end{align*}
\]

Cut Rule

\[
\begin{align*}
\Gamma \vdash \Delta, \phi & \quad \Gamma', \phi \vdash \Delta' \\
\hline
\Gamma, \Gamma' \vdash \Delta, \Delta'
\end{align*}
\]
model completeness principle

Let $T$ be a set of $\mathcal{L}_{\infty\omega}$ formulas. Conclude that $T$ is inconsistent for $\mathcal{L}_{\infty\omega}$ or that $T$ has a model.

set completeness principle

Let $A$ be a transitive set. Let $T$ be a set of $\mathcal{L}_{\infty\omega}$ formulas in the vocabulary $\{=, \in, S\}$ and parameters from $A$. Assume that $T$ includes basic axioms describing the structure $(A, =, \in)$. Conclude that $T$ is inconsistent for $\mathcal{L}_{\infty\omega}$ or that there exists a set $B \subseteq A$ such that $(A, =, \in, B)$ is a model of $T$.

theorem (K.)

Work in $\text{PRS}^+$ together with cut-elimination* for $\mathcal{L}_{\infty\omega}$. Each may be proved using the other as an axiom:

1. the model completeness principle
2. the set completeness principle together with the axiom of infinity