Quantum sets

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We define and investigate a quantum generalization of the category of sets and binary relations within the framework of quantum mathematics, in the sense of noncommutative geometry. We show that the subcategory of quantum sets and quantum functions is dual to the category of hereditarily atomic von Neumann algebras and unital normal $\ast$-homomorphisms, and that it is a closed monoidal category that is complete and cocomplete. (This is an extended abstract for an arXiv preprint of the same title.)

1 Introduction

Formalizations of computation fall on a spectrum of abstraction: on one extreme are formalizations specific to individual machines, and on the other extreme are formalizations in terms of abstract objects convenient to mathematical thought. Quantum computation poses challenges across the length of this spectrum. At the low level of the spectrum, we are confronted with engineering challenges, at the mid level, with fundamental questions about the limitations of quantum computation and the nature of quantum information, and at the high level, with the alien nature of quantum mathematical structures that may be implemented in quantum computation. Quantum mathematics in the sense of noncommutative geometry is a robust framework for analyzing quantum structures by comparison with ordinary mathematical structures. We define a category of quantum sets and quantum functions within this framework.

Dagger-compact categories are a well-established formalization of quantum processes, roughly at the mid level of the spectrum outlined above. The motivating example of a dagger-compact category is the category $\text{FdHilb}$ of finite-dimensional Hilbert spaces and linear operators; the standard second example is the category $\text{Rel}$ of sets and binary relations. Any quantum generalization of the latter category within the framework of quantum mathematics in the sense of noncommutative geometry should include $\text{Rel}$ as a full subcategory. We define a category $\text{qRel}$ closely related to $\text{FdHilb}$ that does have this property. Its objects may be characterized as sets of nonzero finite-dimensional Hilbert spaces, which we term “quantum sets”, or contravariantly, as hereditarily atomic von Neumann algebras, in other words, von Neumann algebras all of whose von Neumann subalgebras are atomic.

Much of the usual structure on the category $\text{Rel}$ generalizes canonically to $\text{qRel}$. In particular, $\text{qRel}$ is a dagger-compact category, whose morphism sets are partially ordered. The usual characterization of functions in $\text{Rel}$ in terms of this structure is also our definition of quantum functions in $\text{qRel}$. The resulting monoidal category $\text{qFun}$ is

1. finitely complete,
2. finitely cocomplete,
3. closed,
4. has a terminal monoidal unit $1$, and
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5. has, for every monic \( Z \rightarrow \mathcal{X} \), a unique “classical” quantum function from \( \mathcal{X} \) to \( 2 = 1 + 1 \) making the following diagram into a pullback square:

\[
\begin{array}{ccc}
Z & \longrightarrow & 1 \\
\downarrow & & \downarrow \tau \\
\mathcal{X} & \longrightarrow & 2
\end{array}
\]

Intuitively, a quantum function is “classical” if it can be applied to a quantum system without disturbing that quantum system. Formally, a morphism \( F : \mathcal{X} \rightarrow \mathcal{2} \) in a symmetric monoidal category with terminal unit is “classical” just in case there is morphism \( \tilde{F} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{2} \) whose first component is the identity on \( \mathcal{X} \), and whose second component is \( F \). Here, \( \times \) is the monoidal product.

Observe that any symmetric monoidal category satisfying the properties listed above is a topos, if its monoidal structure is equivalent to the usual category-theoretic product. Conversely, any topos is a symmetric monoidal category satisfying the listed properties, if it is equipped with its category-theoretic product. Thus, we have a class of categories that generalizes topoi in the spirit of quantum mathematics. It is not yet clear what other properties of \( \text{qFun} \) are necessary to a quantum generalization of topoi in this sense. I expect to include conditions on the functor \( F^* = (F^\dagger)^* \).

Effectus theory [1] offers a broader generalization of topoi to the quantum setting. Our definition of quantum binary relations is essentially that of Weaver [5]. Our definition of quantum sets is also just a variation on the approach in Weaver’s *Mathematical Quantization* [4], which takes Hilbert spaces to be the quantum analogs of sets. The duality between quantum functions and unital normal \( \dagger \)-homomorphisms is due to the author [2]; the properties of \( \text{qFun} \) are also established just as they were for the opposite of the category of von Neumann algebras and unital normal \( \dagger \)-homomorphisms [3].

2 Definitions.

**Definition 1.** A quantum set is a set of nonzero finite-dimensional Hilbert spaces. For each ordinary set \( M \), define the corresponding quantum set \( 'M = \{ \mathbb{C}^{(m)} \mid m \in M \} \). For all quantum sets \( \mathcal{X} \) and \( \mathcal{2} \) define:

\[
\mathcal{X} \cup \mathcal{2} = \{ H \mid H \in \mathcal{X} \text{ or } H \in \mathcal{2} \}
\]

\[
\mathcal{X} \times \mathcal{2} = \{ X \otimes Y \mid X \in \mathcal{X} \text{ and } Y \in \mathcal{2} \}
\]

\[
\mathcal{X} + \mathcal{2} = \mathcal{X} \times ' \{ 0 \} \cup \mathcal{2} \times ' \{ 1 \}
\]

**Definition 2.** A quantum binary relation from a quantum set \( \mathcal{X} \) to quantum set \( \mathcal{2} \) is an ordinary function that assigns to each pair \((X, Y) \in \mathcal{X} \times \mathcal{2}\) a subspace \( R(X, Y) \) of linear operators from \( X \) to \( Y \). For each ordinary binary relation \( R \) from an ordinary set \( M \) to an ordinary set \( N \), define a quantum binary relation \( 'R \) from \( 'M \) to \( 'N \) by

\[
'R(\mathbb{C}^{(m)}, \mathbb{C}^{(n)}) = \begin{cases} 
L(\mathbb{C}^{(m)}, \mathbb{C}^{(n)}) & (m, n) \in R \\
0 & (m, n) \notin R.
\end{cases}
\]

For all quantum binary relations \( R \) from \( \mathcal{X} \) to \( \mathcal{2} \), and \( S \) from \( \mathcal{2} \) to \( \mathcal{3} \), define:

\[
(S \circ R)(X, Z) = \text{span}\{ sr \mid \exists Y \in \mathcal{2} : s \in S(Y, Z) \text{ and } r \in R(X, Y) \}.
\]
We write $\text{qRel}$ for the category of quantum sets and quantum binary relations. The identity quantum binary relation $I_X$ on a quantum set $X$, the partial order $\leq$ on quantum binary relations, the product $\times$ on quantum binary relations, and the functors $(-)^*$ and $(-)^\dagger$ are defined in the obvious way.

**Definition 3.** A quantum function from a quantum set $X$ to a quantum set $Y$ is a quantum binary relation $F$ from $X$ to $Y$ such that $F^\dagger \circ F \geq I_X$ and $F \circ F^\dagger \leq I_Y$.

We write $\text{qFun}$ for the category of quantum sets and quantum functions; it is a subcategory of $\text{qRel}$.

**Definition 4.** A quantum function $F$ from a quantum set $X$ to a quantum set $Y$ is classical just in case there is a quantum function $\tilde{F}$ from $X$ to $X \times Y$ making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{I_X} & X \\
\downarrow{\tilde{F}} & & \downarrow{F} \\
X \times Y & \xrightarrow{I_X \times I_Y} & 1 \times Y \\
\end{array}
\]

**Definition 5.** A von Neumann algebra $A$ is hereditarily atomic just in case every von Neumann subalgebra of $A$ is atomic.

Recall that a von Neumann algebra is said to be atomic if every nonzero projection is above a minimal projection.

### 3 Results

The functor from $\text{Rel}$ to $\text{qRel}$ taking each ordinary set $M$ to the quantum set $\hat{M}$, and each ordinary binary relation $R$ to the quantum binary relation $\hat{R}$ is full, faithful, and essentially surjective onto the full subcategory of quantum sets whose elements are all 1-dimensional. This equivalence is monoidal for the Cartesian product structure on $\text{Rel}$, and it likewise preserves order and dagger-compact structure.

**Theorem 6.** The structure $(\text{qRel}, \times, 1, (-)^*, (-)^\dagger)$ is a dagger-compact category.

**Theorem 7.** There is a monoidal equivalence between the category $\text{qFun}$ of quantum sets and quantum functions with the product $\times$, and the category of hereditarily atomic von Neumann algebras and unital ultraweakly-continuous $\dagger$-homomorphisms with the spatial tensor product.

**Theorem 8.** The symmetric monoidal category $\text{qFun}$ of quantum sets and quantum functions with the product $\times$ is complete, cocomplete, and closed. Furthermore, for every monic quantum function $\hat{Z} \hookrightarrow X$, there exists a unique classical quantum function from $X$ to $2 = 1 + 1$ making the following diagram into a pullback square:

\[
\begin{array}{ccc}
\hat{Z} & \rightarrow & 1 \\
\downarrow & & \downarrow{T} \\
X & \rightarrow & 2 \\
\end{array}
\]

**Proposition 9.** The subobjects of a quantum set $X$ in the category $\text{qFun}$ are exactly the subsets of $X$.

**Proposition 10.** Let $F$ be a quantum function from a quantum set $X$ to a quantum set $Y$. Then, $F$ is monic iff $F^\dagger \circ F = 1_X$, and $F$ is epic iff $F \circ F^\dagger = 1_Y$. 
4 Bibliography

References


