ESP
Kouba
Worksheet 10 Solutions

-20

1.) a.) $y' = 7\left(\dfrac{3x-1}{\tan 5x}\right)^6 \cdot \dfrac{\tan 5x \cdot 3 - (3x-1)\cdot \sec^2 5x \cdot 5}{\tan^2 5x}$

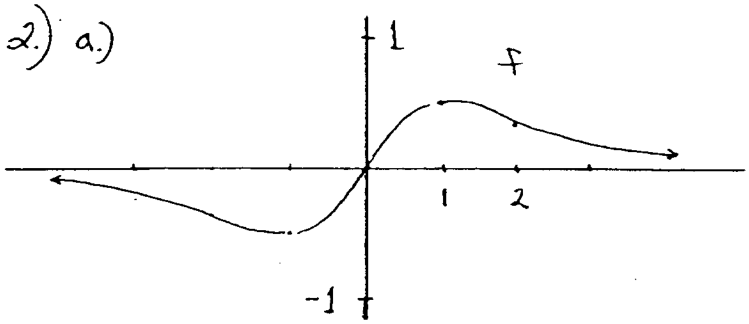
b.) $f'(x) = \sec^2\left\{\dfrac{\sin(3x+7)}{\cos(7x-3)}\right\} \cdot \dfrac{\cos(7x-3)\cdot \cos(3x+7)\cdot 3 - \sin(3x+7)\cdot -\sin(7x-3)\cdot 7}{\cos^2(7x-3)}$

c.) $g'(x) = \cos\left(\sin\left(\sin\left(\sin\left(\frac{1}{x}\right)\right)\right)\right)\cdot \cos\left(\sin\left(\sin\left(\frac{1}{x}\right)\right)\right)\cdot$

$\longrightarrow \cos\left(\sin\left(\frac{1}{x}\right)\right)\cdot \cos\left(\frac{1}{x}\right)\cdot \dfrac{-1}{x^2}$

d.) $y' = 3\csc^2\sqrt{1-x^2}\cdot -\csc\sqrt{1-x^2}\cdot \cot\sqrt{1-x^2}\cdot \frac{1}{2}(1-x^2)^{-1/2}\cdot -2x$

2.) a.)



b.) $f'(x) = \dfrac{(x^2+1)(1) - x\cdot(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow$

$1 - x^2 = 0 \Rightarrow x = 1$ or $x = -1$.

c.) $\dfrac{1-x^2}{(x^2+1)^2} = \dfrac{1}{2} \Rightarrow 2(1-x^2) = (x^2+1)^2 \Rightarrow$

$2 - 2x^2 = x^4 + 2x^2 + 1 \Rightarrow 0 = x^4 + 4x^2 - 1 \Rightarrow$

$0 = (x^2)^2 + 4(x^2) - 1 \Rightarrow x^2 = \dfrac{-4 \pm \sqrt{16+4}}{2} = -2 + \sqrt{5} \Rightarrow$

$x = +\sqrt{-2+\sqrt{5}}$ or $x = -\sqrt{-2+\sqrt{5}}$.

d.) $\dfrac{1-x^2}{(x^2+1)^2} = \dfrac{-1}{10} \Rightarrow 10(1-x^2) = -(x^2+1)^2 \Rightarrow$

$10 - 10x^2 = -x^4 - 2x^2 - 1 \Rightarrow (x^2)^2 - 8(x^2) + 11 = 0 \Rightarrow$

$x^2 = \dfrac{8 \pm \sqrt{64-44}}{2} = 4 \pm \sqrt{5} \Rightarrow$

$x = +\sqrt{4+\sqrt{5}}$, $x = -\sqrt{4+\sqrt{5}}$, $x = +\sqrt{4-\sqrt{5}}$, or $x = -\sqrt{4-\sqrt{5}}$.

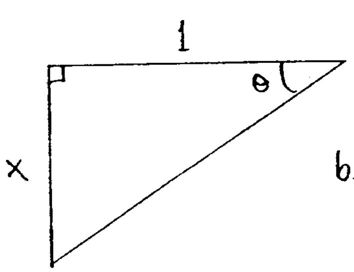
e.) $f'(x) = A \Rightarrow \dfrac{1-x^2}{(x^2+1)^2} = A \Rightarrow 1 - x^2 = A(x^4 + 2x^2 + 1) \Rightarrow$

$0 = A(x^2)^2 + (2A+1)(x^2) + (A-1) \Rightarrow$

$$x^2 = \frac{-(2A+1) \pm \sqrt{(2A+1)^2 - 4A(A-1)}}{2A} = \frac{-(2A+1) \pm \sqrt{8A+1}}{2A},$$

which is solvable if $8A + 1 \geq 0 \Rightarrow A \geq \frac{-1}{8}$, so the smallest value of $f'(x)$ is $A = -1/8$.

3.)



a.) $\tan\theta = \dfrac{x}{1}$ so $x = \tan\theta$

b.) i.) $\dfrac{d\theta}{dt} = \dfrac{10(2\pi) \text{ radians}}{1 \text{ min.}} = 20\pi \ \dfrac{\text{rad.}}{\text{min.}}$

ii) $x = \tan\theta \Rightarrow \dfrac{dx}{dt} = \sec^2\theta \cdot \dfrac{d\theta}{dt}$

$\Rightarrow \dfrac{dx}{dt} = 20\pi \sec^2\theta$

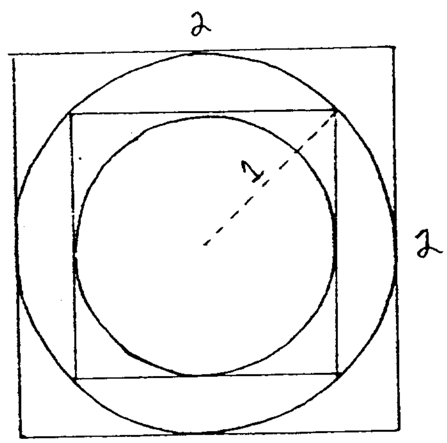
4.) X: liters of gooh
Y: liters of sludge

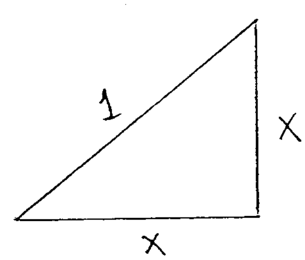
$$\left.\begin{array}{l} X + Y = 10 \\ 200X + 250Y = 238(10) \end{array}\right\} \quad \left.\begin{array}{l} Y = 10 - X \\ 200X + 250(10 - X) = 2380 \end{array}\right\}$$

$$200X + 2500 - 250X = 2380 \Rightarrow 120 = 50X \Rightarrow$$

$X = \frac{12}{5} = 2\frac{2}{5}\,l.$ and $Y = 7\frac{3}{5}\,l.$

5.)





$X^2 + X^2 = 1 \Rightarrow$
$2X^2 = 1 \Rightarrow$
$X = \sqrt{\frac{1}{2}}$

is radius of smaller circle so area $= \pi\left(\sqrt{\frac{1}{2}}\right)^2 = \frac{\pi}{2}$.

6.) a.) $f'(x) = 2X - 1$ and $\frac{f(2) - f(0)}{2 - 0} = \frac{2 - 0}{2 - 0} = 1 \Rightarrow$

$2x - 1 = 1 \Rightarrow 2X = 2 \Rightarrow X = 1$ so $c = 1$.

b.) $f'(x) = \frac{1}{3}x^{-2/3}$ and $\frac{f(0) - f(-2)}{0 - (-2)} = \frac{1 - ((-2)^{1/3} + 1)}{2} = \frac{1}{2^{2/3}} \Rightarrow$

$\frac{1}{3}\,\frac{1}{X^{2/3}} = \frac{1}{2^{2/3}} \Rightarrow X^{2/3} = \frac{2^{2/3}}{3} \Rightarrow X^2 = \left(\frac{2^{2/3}}{3}\right)^3 = \frac{2^2}{3^3}$

$\Rightarrow \; x = \pm\sqrt{\frac{4}{27}}$ so $c = -\frac{2}{3\sqrt{3}}$

c.) $f'(x) = 1 - \frac{1}{x^2}$ and $\frac{f(3) - f(\frac{1}{2})}{3 - \frac{1}{2}} = \frac{(3+\frac{1}{3}) - (\frac{1}{2}+2)}{\frac{5}{2}} = \frac{1}{3}$

$\Rightarrow \; 1 - \frac{1}{x^2} = \frac{1}{3} \Rightarrow \frac{2}{3} = \frac{1}{x^2} \Rightarrow x^2 = \frac{3}{2} \Rightarrow$

$x = \pm\sqrt{\frac{3}{2}}$ so $c = +\sqrt{\frac{3}{2}}$.

d.) $f'(x) = 15 \cos 3x$ and $\frac{f(\pi) - f(0)}{\pi - 0} = 0 \Rightarrow$

$15 \cos 3x = 0 \Rightarrow 3x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots \Rightarrow$

$x = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}, \pm\frac{7\pi}{6}, \ldots$ so $c = \frac{\pi}{6}, \frac{\pi}{2},$ or $\frac{5\pi}{6}$.

e.) $f'(x) = -2 \sin(2x+1)$ and $\frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} = \frac{\cos(\pi+1) - \cos 1}{\frac{\pi}{2}}$

$\approx -.6879 \Rightarrow -2 \sin(2x+1) = -.6879 \Rightarrow$

$\sin(2x+1) = .344 \Rightarrow 2x+1 = .35, 2.79, .35+2\pi,$

$2.79 + 2\pi, .35 + 4\pi, 2.79 + 4\pi, \ldots$

$= .35, 2.79, 6.63, 9.07, \ldots \Rightarrow$

$2x = -.65, 1.79, 5.63, 8.07, \ldots \Rightarrow$

$x = -.325, .895, 2.815, 4.035, \ldots$ so $c = .895$

7.) Function d is continuous on $[0, 2]$ and differentiable on $(0, 2)$ so by MVT there is a number c in $(0, 2)$ satisfying

$$d'(c) = \frac{d(2)-d(0)}{2-0} = \frac{46-0}{2-0} = 23 \text{ mph}, \text{ i.e.,}$$

at time $t=c$ speed $d'(c)$ is 23 mph.

8.) Function $h(x) = \dfrac{f(x)}{g(x)}$ is continuous on $[a,b]$

and differentiable on (a,b) with

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}. \text{ Also,}$$

$h(a) = \dfrac{f(a)}{g(a)} = 1$ and $h(b) = \dfrac{f(b)}{g(b)} = 1$ so

$\dfrac{h(b)-h(a)}{b-a} = \dfrac{1-1}{b-a} = 0$. Thus, by MVT there is

some number c satisfying $h'(c) = 0$, i.e.,

$$\frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2} = 0, \text{ i.e., } g(c)f'(c) = f(c)g'(c).$$

9.) a.) $f(x) = x^3 - x$

b.) $f(x) = \pi x + \frac{2}{3}x^{3/2}$

c.) $f(x) = \sin(5x-7)$

d.) $f(x) = x\tan x$

e.) $f(x) = \dfrac{\sin x}{x}$

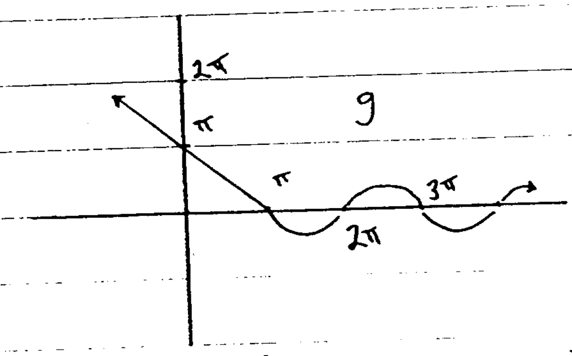
10.) $Y' = \dfrac{1}{2Y} \Rightarrow 2(Y)Y' = 1 \Rightarrow [(Y)^2]' = [x]' \Rightarrow$

$Y^2 = x \Rightarrow Y = \pm\sqrt{x}$ so $Y = \sqrt{x}$ works.

11.) Let $f(x) = x^5 - \sqrt{x} - x^{1/3} - 1$ on the interval $[0,2]$; f is continuous since all of its components are continuous. Let $m = 0$. Now $f(0) = -1$ and $f(2) = 32 - \sqrt{2} - 2^{1/3} - 1 > 0$, so by the IMVT there is a number c, $0 < c < 2$, satisfying $f(c) = 0$, i.e.,

$$c^5 - \sqrt{c} = c^{1/3} + 1 .$$

12.)



$$g'(\pi) = \lim_{h \to 0} \frac{g(\pi + h) - g(\pi)}{h} = \lim_{h \to 0} \frac{g(\pi + h)}{h} \ ;$$

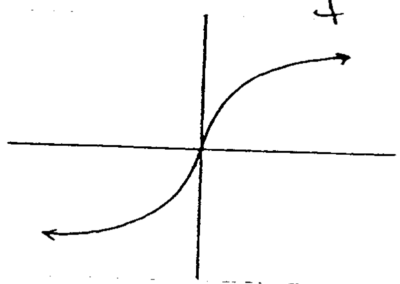
$$\lim_{h \to 0^+} \frac{g(\pi+h)}{h} = \lim_{h \to 0^+} \frac{\sin(\pi + h)}{h}$$

$$= \lim_{h \to 0^+} \frac{\overset{0}{\cancel{\sin \pi}} \cdot \cos h + \overset{-1}{\cancel{\cos \pi}} \sin h}{h} = \lim_{h \to 0^+} -\frac{\sin h}{h} = -1 \ ;$$

$$\lim_{h \to 0^-} \frac{g(\pi+h)}{h} = \lim_{h \to 0^-} \frac{\pi - (\pi + h)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1 ,$$

so $g'(\pi) = -1$.

13.)



$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/5}}{h} = \lim_{h \to 0} \frac{1}{h^{4/5}}$$ does not exist.

14.) x: time working together

$$\frac{x}{5} + \frac{x}{3} = 1 \Rightarrow 3x + 5x = 15 \Rightarrow 8x = 15 \Rightarrow$$

$$x = \frac{15}{8} = 1\tfrac{7}{8} \text{ hours.}$$

15.) a.) sequence:

0.8
0.836511642
0.829723987
0.830981657
0.830748482
0.830791708
0.830783695

The limit of the sequence appears to be about 0.8307

b.)

$$c^{c^{c^{c^{\cdot^{\cdot^{\cdot}}}}}} = 2 \Rightarrow c^{\left(c^{c^{c^{\cdot^{\cdot^{\cdot}}}}}\right)} = 2 \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}.$$