ESP
Kouba
Worksheet 11 Solutions

1.) a.) $f(x) = x^2(x-3)$

$\Rightarrow f'(x) = x^2 \cdot (1) + 2x \cdot (x-3)$

$= 3x^2 - 6x = 3x(x-2) = 0$

f': $+ \quad 0 \quad - \quad 0 \quad +$

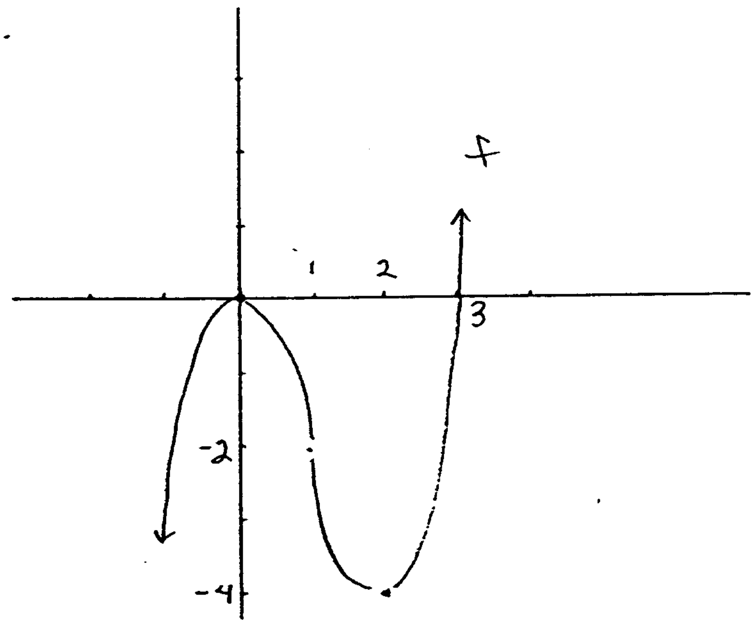
$x=0$, $y=0$: rel. max. $x=2$, $y=-4$: rel. min.

$\Rightarrow f''(x) = 6x - 6 = 6(x-1) = 0$

f'': $- \quad 0 \quad +$

$\left.\begin{array}{l} x=1 \\ y=-2 \end{array}\right\}$ infl. pt.

f is ↑ for $x<0$, $x>2$.
f is ↓ for $0<x<2$.
f is ∪ for $x>1$.
f is ∩ for $x<1$.



b.) $f(x) = \dfrac{x}{x^2+1} \Rightarrow f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$= \dfrac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1 - x^2 = 0$

$\Rightarrow x = \pm 1$

f': $- \quad 0 \quad + \quad 0 \quad -$

$x=-1$, $y=-1/2$: abs. min. $x=1$, $y=1/2$: abs. max.

$\Rightarrow f''(x) = \dfrac{(x^2+1)^2(-2x) - (1-x^2) \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$

$$= \frac{-2x(x^2+1)\cdot\left[(x^2+1)+2(1-x^2)\right]}{(x^2+1)^4}$$

$$= \frac{-2x(3-x^2)}{(x^2+1)^3} = 0 \Rightarrow$$

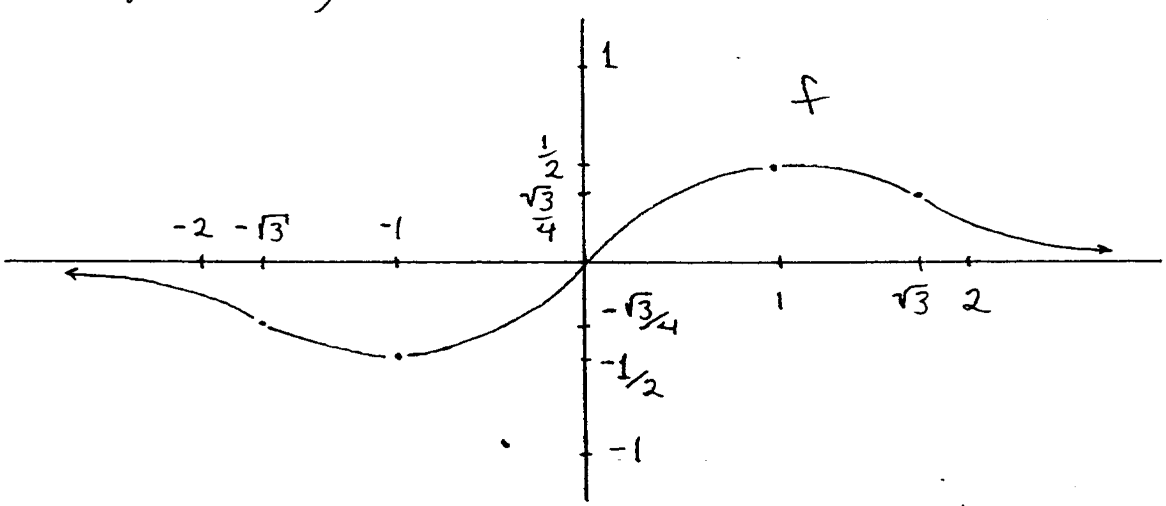
$$-2x(3-x^2)=0 \Rightarrow x=0,\ x=\pm\sqrt{3}$$

f'': $-$ 0 $+$ 0 $-$ 0 $+$

$x=-\sqrt{3}$, $y=-\frac{\sqrt{3}}{4}$; $x=0$, $y=0$; $x=\sqrt{3}$, $y=\frac{\sqrt{3}}{4}$

infl. pts.

f is $\uparrow$ for $-1<x<1$.
f is $\downarrow$ for $x<-1$, $x>1$.
f is $\cup$ for $-\sqrt{3}<x<0$, $x>\sqrt{3}$.
f is $\cap$ for $x<-\sqrt{3}$, $0<x<\sqrt{3}$.



c.) $f(x) = 2x - \frac{1}{x} \Rightarrow$

$f'(x) = 2 + \frac{1}{x^2} > 0 \Rightarrow$

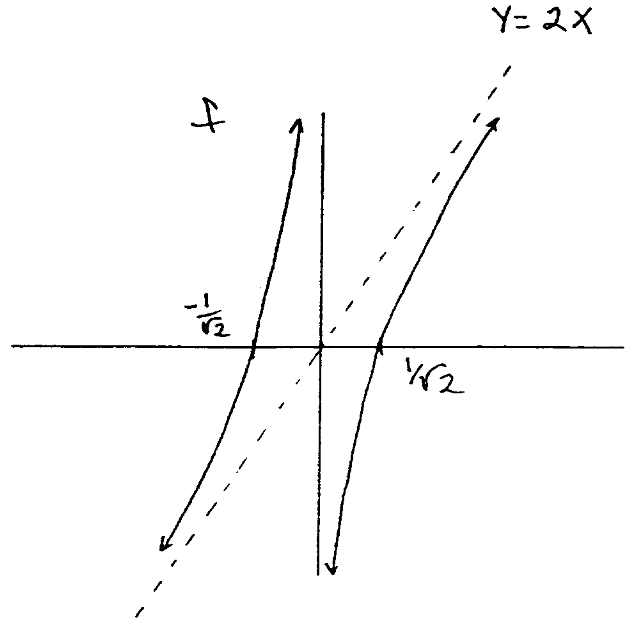
$f''(x) = \frac{-2}{x^3} \neq 0$

f': $+$ $+$ | $+$ $+$ ($x=0$)

f'': $+$ $+$ | $-$ $-$ ($x=0$)

f is $\uparrow$ for $x<0,\ x>0$.
f is $\downarrow$ for no x-values.
f is $\cup$ for $x<0$.
f is $\cap$ for $x>0$.

x-int: $+\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}$

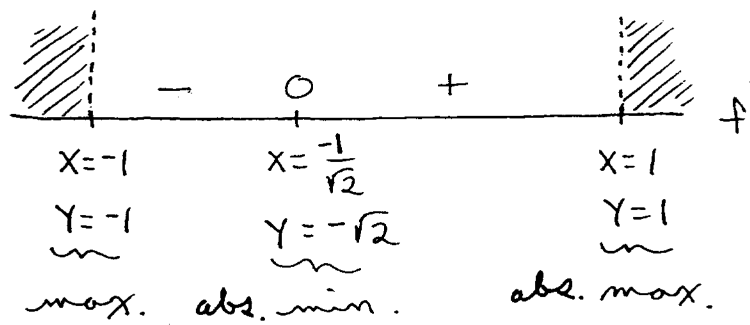


d.) $f(x) = x - \sqrt{1-x^2}$ for $-1 \le x \le 1 \Rightarrow$

$f'(x) = 1 - \frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) = 1 + \frac{x}{\sqrt{1-x^2}} = 0 \Rightarrow 1 = \frac{-x}{\sqrt{1-x^2}} \Rightarrow$

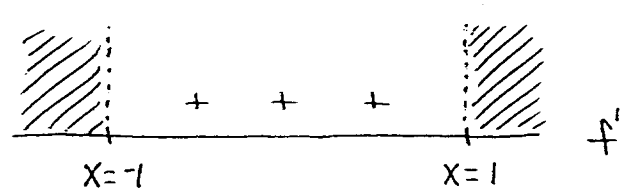
$\sqrt{1-x^2} = -x \Rightarrow 1-x^2 = x^2 \Rightarrow 1 = 2x^2 \Rightarrow x^2 = \frac{1}{2} \Rightarrow$

$x = \pm\frac{1}{\sqrt{2}} \Rightarrow x = -\frac{1}{\sqrt{2}}$



$$f''(x) = \frac{\sqrt{1-x^2}\cdot(1) - x\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)}{1-x^2} = \frac{\frac{\sqrt{1-x^2}}{1} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{1}{(1-x^2)^{3/2}} > 0$$



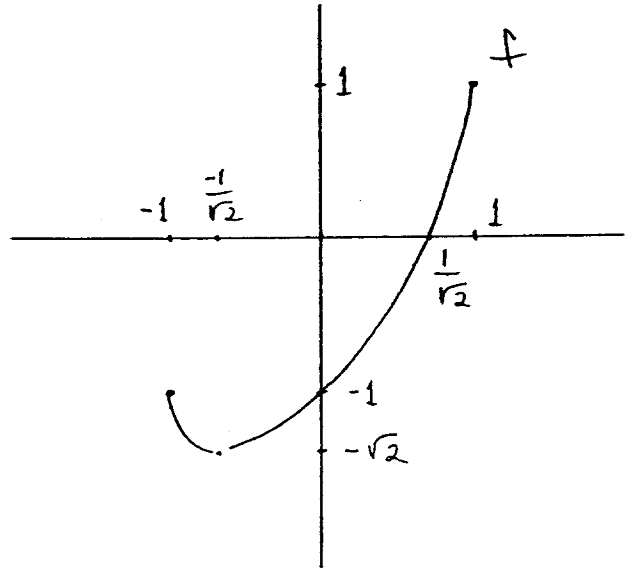
f is $\uparrow$ for $\frac{-1}{\sqrt{2}} < x \le 1$.

f is $\downarrow$ for $-1 \le x < \frac{-1}{\sqrt{2}}$.

f is $\cup$ for $-1 < x < 1$.

f is $\cap$ for no x-values.

x-int.: $+\frac{1}{\sqrt{2}}$



2.) a.) $f(x) = \pi^2 \cdot x - e^2 \cdot x$

b.) $f(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$

c.) $f(x) = \frac{1}{7}\tan 7x$

d.) $f(x) = (\sin x)^2$

e.) $f'(x) = 3(f(x))^2 \Rightarrow (f(x))^{-2} f'(x) = 3 \Rightarrow$

$-(f(x))^{-2} \cdot f'(x) = -3 \Rightarrow D\,(f(x))^{-1} = -3 \Rightarrow$

$(f(x))^{-1} = -3x \Rightarrow f(x) = \frac{-1}{3x}$.

3.) $f'(x) = \frac{(x+1)\cdot 3x^2 - x^3\cdot(1)}{(x+1)^2} = \frac{3x^2 + 2x^3}{(x+1)^2} = \frac{x^2(3+2x)}{(x+1)^2} = 0$

$\Rightarrow x^2(3+2x) = 0 \Rightarrow x = 0,\ x = \frac{-3}{2}$.

4.) X: amt. of A $\qquad$ $X+Y=2$

Y: amt. of B $\qquad$ $.35X+.60Y=.40(2)$

$$\left.\begin{array}{l} Y=2-X \\ 35X+60Y=80 \end{array}\right\} \quad 35X+60(2-X)=80 \Rightarrow$$

$$35X+120-60X=80 \Rightarrow 40=25X \Rightarrow$$

$$X=\frac{8}{5}=1\tfrac{3}{5}\,\ell. \quad , \quad Y=\frac{2}{5}\,\ell.$$

5.) $f(x)=g(h(x)) \Rightarrow f'(x)=g'(h(x))\cdot h'(x) \Rightarrow$

$$f''(x)=g'(h(x))\cdot h''(x)+g''(h(x))\cdot h'(x)\cdot h'(x)$$

a.) $f(1)=g(h(1))=g(2)=-3$

b.) $f'(1)=g'(h(1))\cdot h'(1)=g'(2)\cdot(-1)=\left(\frac{1}{2}\right)\cdot(-1)=\frac{-1}{2}$

c.) $f''(1)=g'(h(1))\cdot h''(1)+g''(h(1))\cdot(h'(1))^2$

$$=g'(2)\cdot(3)+g''(2)\cdot(-1)^2$$

$$=\frac{1}{2}\cdot 3+(-2)\cdot 1 \quad = \frac{-1}{2}$$

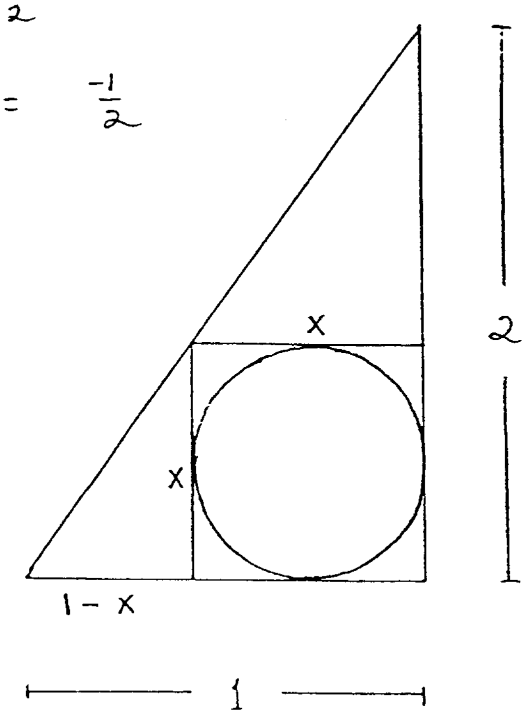
6.) By similar triangles

$$\frac{2}{1}=\frac{X}{1-X} \Rightarrow 2-2X=X \Rightarrow$$

$$2=3X \Rightarrow X=\frac{2}{3} \Rightarrow$$

circumference of circle is

$$\pi d=\frac{2}{3}\pi$$



7.) a.) $\lim_{x \to 2} (3x-1) = 5$: Let $\varepsilon > 0$ be given.

Determine δ so that if

$0 < |x-2| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$|(3x-1)-5| < \varepsilon \iff$$

$$3|x-2| < \varepsilon \iff |x-2| < \frac{\varepsilon}{3}.$$ Choose $\delta = \varepsilon/3$.

b.) $\lim_{x \to -\frac{1}{2}} (4+2x) = 3$: Let $\varepsilon > 0$ be given.

Determine δ so that if

$0 < |x+\frac{1}{2}| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$|(4+2x)-3| < \varepsilon \iff$$

$$|1+2x| < \varepsilon \iff 2\left|\tfrac{1}{2}+x\right| < \varepsilon \iff$$

$$|x+\tfrac{1}{2}| < \varepsilon/2.$$ Choose $\delta = \varepsilon/2$.

c.) $\lim_{x \to 1} (2x^2-3) = -1$: Let $\varepsilon > 0$ be given.

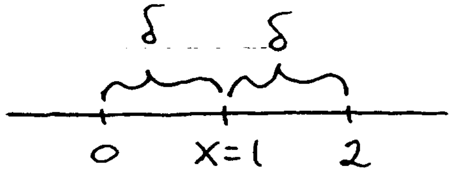
Determine δ so that if

$0 < |x-1| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$|(2x^2-3)-(-1)| < \varepsilon \iff$$

$$|2x^2-2| < \varepsilon \iff$$

$$2|x-1||x+1| < \varepsilon.$$ Assume $0 < \delta \leq 1$.



Then $1 < |x+1| < 3$ and

$$2|x-1||x+1| < 2|x-1|\cdot(3) < \varepsilon \iff$$
$$|x-1| < \varepsilon/6 .$$

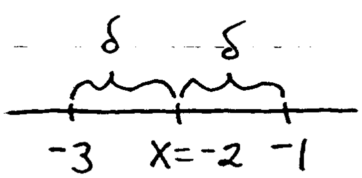
Choose $\delta = \min\{1, \varepsilon/6\}$.

d.) $\lim\limits_{x\to -2} (13-3x^2) = 1$: Let $\varepsilon > 0$ be given.

Determine δ so that if
$0 < |x+2| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$|(13-3x^2) - 1| < \varepsilon \iff$$
$$3\,|2-x||2+x| < \varepsilon \iff$$
$$3\,|x-2||x+2| < \varepsilon .$$

Assume $0 < \delta \leq 1$.



Then $3 < |x-2| < 5$ and

$$3|x-2||x+2| < 3\cdot(5)\cdot|x+2| < \varepsilon \iff$$
$$|x+2| < \varepsilon/15 .$$

Choose $\delta = \min\{1, \varepsilon/15\}$.

e.) $\lim\limits_{x\to 3} \dfrac{2}{x-1} = 1$: Let $\varepsilon > 0$ be given.

Determine δ so that if

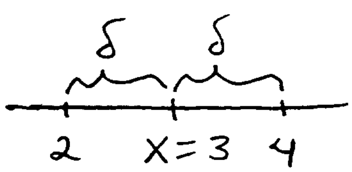
$0 < |x-3| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$\left| \frac{2}{x-1} - 1 \right| < \varepsilon \iff$$

$$\left| \frac{2}{x-1} - \frac{x-1}{x-1} \right| < \varepsilon \iff$$

$$\left| \frac{3-x}{x-1} \right| < \varepsilon \iff$$

$$\frac{|x-3|}{|x-1|} < \varepsilon .$$ Assume $0 < \delta \leq 1$.



Then $1 < |x-1| < 3$ and $\frac{1}{3} < \frac{1}{|x-1|} < 1$ so that

$$\frac{|x-3|}{|x-1|} < |x-3| \cdot (1) < \varepsilon \iff$$

$$|x-3| < \varepsilon .$$

Choose $\delta = \min\{1, \varepsilon\}$.

f.) $\lim_{x \to -\frac{1}{2}} \frac{x-1}{x+1} = -3$: Let $\varepsilon > 0$ be given.

Determine δ so that if

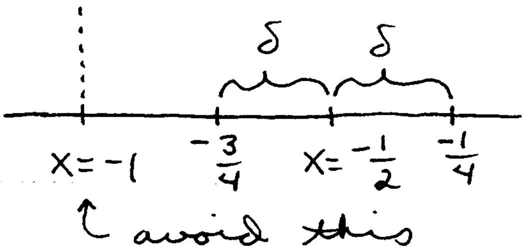
$0 < |x + \frac{1}{2}| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$\left| \frac{x-1}{x+1} - (-3) \right| < \varepsilon \iff$$

$$\left| \frac{x-1}{x+1} + \frac{3(x+1)}{x+1} \right| < \varepsilon \iff$$

$$\left| \frac{4x+2}{x+1} \right| < \varepsilon \iff$$

$$4 \frac{\left|x+\frac{1}{2}\right|}{|x+1|} < \varepsilon. \qquad \text{assume } 0<\delta \le \tfrac{1}{4} \text{ !!}$$



Then $\frac{1}{4} < |x+1| < \frac{3}{4}$ and

$\frac{4}{3} < \frac{1}{|x+1|} < 4$ so that

$$4 \frac{\left|x+\frac{1}{2}\right|}{|x+1|} < 4 \left|x+\tfrac{1}{2}\right| \cdot (4) < \varepsilon \iff$$

$$\left|x+\tfrac{1}{2}\right| < \frac{\varepsilon}{16}.$$

Choose $\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{16}\right\}$.

9.) $\lim_{x \to 4} (7 - 3\sqrt{x}) = 1$: Let $\varepsilon > 0$ be given.

Determine δ so that if

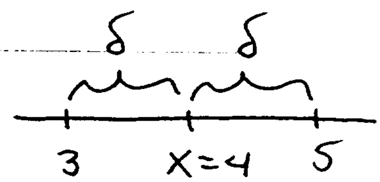
$0 < |x-4| < \delta$, then $|f(x) - L| < \varepsilon \iff$

$$\left|(7-3\sqrt{x}) - 1\right| < \varepsilon \iff$$

$$3\left|2-\sqrt{x}\right| < \varepsilon \iff$$

$$3\left|2-\sqrt{x}\right| \frac{|2+\sqrt{x}|}{|2+\sqrt{x}|} < \varepsilon \iff$$

$$3 \frac{|4-x|}{|2+\sqrt{x}|} < \varepsilon. \qquad \text{assume } 0<\delta\le 1.$$



Then $2+\sqrt{3} < |2+\sqrt{x}| < 2+\sqrt{5}$ and

$$\frac{1}{2+\sqrt{5}} < \frac{1}{|2+\sqrt{x}|} < \frac{1}{2+\sqrt{3}} \quad \text{so that}$$

$$3\,\frac{|x-4|}{|2+\sqrt{x}|} < 3\,|x-4| \cdot \frac{1}{2+\sqrt{3}} < \varepsilon \iff$$

$$|x-4| < \frac{2+\sqrt{3}}{3}\,\varepsilon \,. \quad \text{Choose } \delta = \min\left\{1, \frac{2+\sqrt{3}}{3}\,\varepsilon\right\}.$$

h.) $\displaystyle\lim_{x\to\frac{1}{2}} \frac{x^2}{2x+3} = \frac{1}{16}$. Let $\varepsilon > 0$ be given.

Determine δ so that if

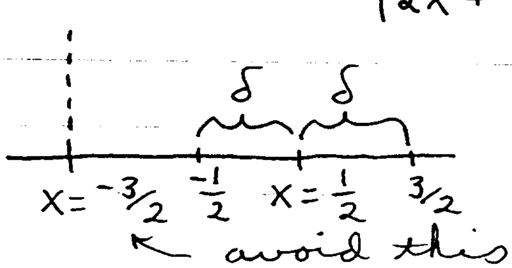
$0 < |x-\frac{1}{2}| < \delta$, then $|f(x)-L| < \varepsilon \iff$

$$\left|\frac{x^2}{2x+3} - \frac{1}{16}\right| < \varepsilon \iff$$

$$\left|\frac{16x^2-(2x+3)}{16(2x+3)}\right| < \varepsilon \iff$$

$$\frac{\cancel{16}}{\cancel{16}}\,\frac{|x^2-\frac{1}{8}x-\frac{3}{16}|}{|2x+3|} < \varepsilon \iff$$

$$\frac{|x-\frac{1}{2}|\,|x+\frac{3}{8}|}{|2x+3|} < \varepsilon \,. \quad \text{Assume } 0 < \delta \le 1 .$$



Then $\frac{1}{8} < |x+\frac{3}{8}| < \frac{15}{8}$,

$2 < |2x+3| < 6$ so that

$$\frac{1}{6} < \frac{1}{|2x+3|} < \frac{1}{2} \quad \text{and}$$

$$\frac{|x-\frac{1}{2}|\,|x+\frac{3}{8}|}{|2x-3|} < |x-\tfrac{1}{2}| \cdot \left(\tfrac{15}{8}\right) \cdot \left(\tfrac{1}{2}\right) < \varepsilon \iff$$

$$|x-\tfrac{1}{2}| < \tfrac{16}{15}\varepsilon \quad .$$

Choose $\delta = \min\{1, \frac{16}{15}\varepsilon\}$.