

ESP  
Kouba  
Worksheet 14 Solutions

$$1.) \text{ a.) } f'(x) = x^3 \cdot 2(3-5x) \cdot (-5) + 3x^2(3-5x)^2 \\ = x^2(3-5x) \cdot [-10x + 3(3-5x)] \\ = x^2(3-5x)[9-25x] = 0 \rightarrow$$

$$x=0, x=\frac{3}{5}, \text{ or } x=\frac{9}{25}$$

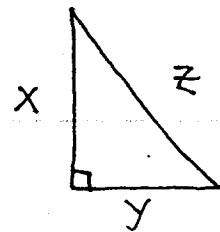
$$b.) f'(x) = x \cdot \frac{1}{2}(x-1)^{-\frac{1}{2}} + \sqrt{x-1} = \frac{x}{2\sqrt{x-1}} + \sqrt{x-1} \\ = \frac{x + 2(x-1)}{2\sqrt{x-1}} = \frac{3x-2}{2\sqrt{x-1}} = 0 \rightarrow 3x-2=0 \rightarrow$$

$x = \frac{2}{3}$  but  $x = \frac{2}{3}$  is not in the domain of  $f$  so there is no solution.

$$c.) f'(x) = \frac{(x^2-1)3x^2 - x^3 \cdot 2x}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2} = 0 \rightarrow$$

$$x=0 \text{ or } x = \pm\sqrt{3}$$

$$2.) \frac{dx}{dt} = -5 \text{ in./sec. and } \frac{dy}{dt} = 7 \text{ in./sec.}$$



$$a.) \text{ Find } \frac{dz}{dt} \text{ when } x=4 \text{ in. and } y=3 \text{ in. :} \\ x^2 + y^2 = z^2 \xrightarrow{D} 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \rightarrow \\ (4)(-5) + (3)(7) = (5) \frac{dz}{dt} \rightarrow \frac{dz}{dt} = \frac{1}{5} \text{ in./sec.}$$

$$b.) \text{ Perimeter } P = x + y + z, \text{ find } \frac{dP}{dt} \text{ when } x=4 \text{ in. and } y=3 \text{ in. :}$$

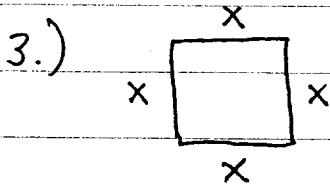
from part a.)

$$\frac{dP}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = (-5) + (7) + \left(\frac{1}{5}\right) = \frac{11}{5} \text{ in./sec.}$$

c.) Area  $A = \frac{1}{2}XY$ , find  $\frac{dA}{dt}$  when  $x = 4$  in.

and  $Y = 3$  in. :  $\frac{dA}{dt} = \frac{1}{2}X \cdot \frac{dY}{dt} + \frac{1}{2} \frac{dX}{dt} \cdot Y$

$$= \frac{1}{2}(4)(7) + \frac{1}{2}(-5)(3) = \frac{13}{2} \text{ in./sec.}$$

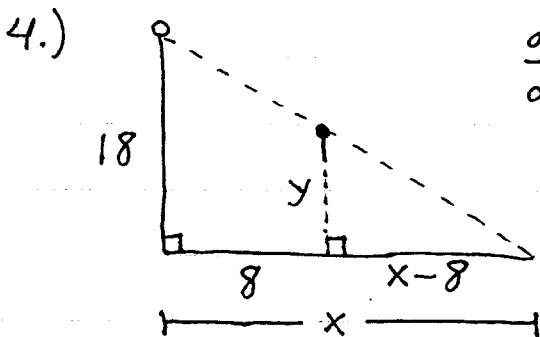


Perimeter  $P = 4X$  ;

$$\frac{dP}{dt} = 24 \text{ cm./min.}, \text{ find } \frac{dX}{dt}$$

when  $x = 10$  cm. :

$$\frac{dP}{dt} = 4 \cdot \frac{dX}{dt} \rightarrow 24 = 4 \cdot \frac{dX}{dt} \rightarrow \frac{dX}{dt} = 6 \text{ cm./min.}$$



$$\frac{dy}{dt} = 3 \text{ ft./sec.}$$

a.) Find  $\frac{dX}{dt}$  when  $t = 4$  sec. :

By similar triangles

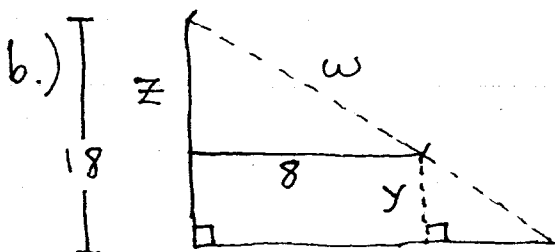
$$\frac{18}{x} = \frac{y}{x-8} \rightarrow \boxed{18x - 144 = xy} \xrightarrow{D}$$

$$18 \frac{dx}{dt} = x \cdot \frac{dy}{dt} + \frac{dx}{dt} \cdot y \quad (t = 4 \text{ sec.} \rightarrow$$

$$y = 12 \text{ ft} \rightarrow 18x - 144 = 12x \rightarrow x = 24 \text{ ft.}) \rightarrow$$

$$(18) \frac{dx}{dt} = (24)(3) + (12) \frac{dx}{dt} \rightarrow (6) \frac{dx}{dt} = 72 \rightarrow$$

$$\frac{dx}{dt} = 12 \text{ ft./sec.}$$



$$\frac{dy}{dt} = 3 \text{ ft./sec. so}$$

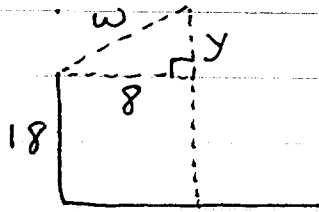
$$\frac{dz}{dt} = -3 \text{ ft./sec. ;}$$

find  $\frac{dw}{dt}$  :

$$8^2 + z^2 = w^2 \xrightarrow{D} 2z \cdot \frac{dz}{dt} = 2w \cdot \frac{dw}{dt}$$

i.)  $t = 4 \text{ sec.} \rightarrow y = 12 \text{ ft.} \rightarrow z = 6 \text{ ft.}$   
 $\rightarrow w = 10 \text{ ft.}$  so  $(6)(-3) = (10) \frac{dw}{dt} \rightarrow$   
 $\frac{dw}{dt} = -1.8 \text{ ft./sec.}$

ii.)  $t = 6 \text{ sec.} \rightarrow y = 18 \text{ ft.} \rightarrow z = 0 \text{ ft.}$   
 $\rightarrow w = 8 \text{ ft.}$  so  $(0)(-3) = (8) \frac{dw}{dt} \rightarrow$   
 $\frac{dw}{dt} = 0 \text{ ft./sec.}$

iii.)   $\frac{dy}{dt} = 3 \text{ ft./sec.}$ , find  $\frac{dw}{dt}$  when  $t = 8 \text{ sec.}$  ( $y = 6 \text{ ft.}$ ,  $w = 10 \text{ ft.}$ ):

$$8^2 + y^2 = w^2 \rightarrow 2y \cdot \frac{dy}{dt} = 2w \cdot \frac{dw}{dt} \rightarrow$$

$$(6)(3) = (10) \frac{dw}{dt} \rightarrow \frac{dw}{dt} = 1.8 \text{ ft./sec.}$$

5.)  $S = 4\pi r^2$  and  $V = \frac{4}{3}\pi r^3$ ,  $\frac{dS}{dt} = -2\pi \text{ ft.}^2/\text{sec.}$

a.) Find  $\frac{dr}{dt}$  when  $r = 4 \text{ ft.}$  :  $S = 4\pi r^2$   
 $\xrightarrow{D} \frac{dS}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt} \rightarrow -2\pi = 8\pi(4) \frac{dr}{dt}$   
 $\rightarrow \frac{dr}{dt} = \frac{-1}{16} \text{ ft./sec.}$

b.) Find  $\frac{dV}{dt}$  when  $r = 2 \text{ ft.}$  :  $V = \frac{4}{3}\pi r^3$   
 $\rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(2)^2 \frac{dr}{dt} = 16\pi \frac{dr}{dt}$   
 (Find  $\frac{dr}{dt}$  :  $\frac{dS}{dt} = 8\pi r \frac{dr}{dt} \rightarrow -2\pi = 16\pi \cdot \frac{dr}{dt}$   
 $\rightarrow \frac{dr}{dt} = \frac{-1}{8} \text{ ft./sec.}$ ) so  $\frac{dV}{dt} = 16\pi \cdot \left(\frac{-1}{8}\right) = -2\pi \text{ ft.}^3/\text{sec.}$

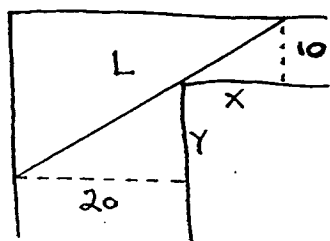
$$c.) \frac{dV}{dt} = -6\pi \text{ ft.}^3/\text{sec.} \quad \text{and} \quad \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \rightarrow$$

$$-6\pi = 4\pi r^2 \cdot \frac{dr}{dt} \quad ; \quad \text{and} \quad \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} \rightarrow$$

$$-2\pi = 8\pi r \cdot \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{-1}{4r} \quad (\text{substitute})$$

$$-6 = 4r^2 \cdot \left(\frac{-1}{4r}\right) \rightarrow r = 6 \text{ ft.}$$

6.) Method 1 :



By similar triangles

$$\frac{Y}{20} = \frac{10}{X} \rightarrow Y = \frac{200}{X} \quad \text{and minimize length}$$

$$L = \sqrt{20^2 + Y^2} + \sqrt{X^2 + 10^2} = \sqrt{20^2 + \left(\frac{200}{X}\right)^2} + \sqrt{X^2 + 10^2}$$

$$= \frac{\sqrt{400X^2 + 200^2}}{X} + \sqrt{X^2 + 10^2} \rightarrow$$

$$L' = \frac{X \cdot \frac{1}{2}(400X^2 + 200^2)^{-\frac{1}{2}} \cdot 800X - \sqrt{400X^2 + 200^2}}{X^2} + \frac{1}{2}(X^2 + 10^2)^{-\frac{1}{2}} \cdot 2X$$

$$= \dots = \frac{-200^2}{X^2 \sqrt{400X^2 + 200^2}} + \frac{X}{\sqrt{X^2 + 10^2}} = 0 \rightarrow$$

$$\frac{x}{\sqrt{x^2+10^2}} = \frac{200^2}{x^2 \sqrt{400x^2+200^2}} \rightarrow \frac{x^2}{x^2+10^2} = \frac{200^4}{x^4(400x^2+200^2)} \rightarrow$$

$$\dots \rightarrow 0 = x^8 + 100x^6 - 4,000,000x^2 - 400,000,000 \rightarrow$$

(by calculator)  $x \approx 12.6$  ft.  $\rightarrow$  length

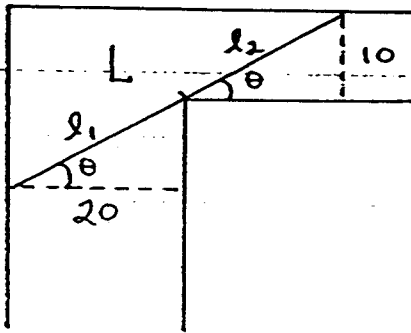
of plank is

$$L = 41.6 \text{ ft.}$$

$$; \quad \begin{array}{c} - \quad 0 \quad + \\ \hline x = 12.6 \text{ ft.} \\ y = 15.9 \text{ ft.} \end{array} L'$$

Method 2:

Write length  $L$  as a function of  $\theta$ :



$$\cos \theta = \frac{20}{l_1} \rightarrow l_1 = 20 \sec \theta \text{ and}$$

$$\sin \theta = \frac{10}{l_2} \rightarrow l_2 = 10 \csc \theta \text{ so}$$

minimize length

$$L = l_1 + l_2 = 20 \sec \theta + 10 \csc \theta \rightarrow$$

$$L' = 20 \sec \theta \tan \theta - 10 \csc \theta \cot \theta = 0 \rightarrow$$

$$2 \sec \theta \tan \theta = \csc \theta \cot \theta \rightarrow$$

$$2 \frac{\sin \theta}{\cos^2 \theta} = \frac{\cos \theta}{\sin^2 \theta} \rightarrow 2 \sin^3 \theta = \cos^3 \theta \rightarrow$$

$$2^{\frac{1}{3}} \sin \theta = \cos \theta \rightarrow \tan \theta = \frac{1}{2^{\frac{1}{3}}} \rightarrow \theta \approx .67 \text{ radians}$$

$$\rightarrow \text{length } L = 41.6 \text{ ft.}$$

$$\begin{array}{c} - \quad 0 \quad + \\ \hline \theta = 0.67 \text{ rad.} \end{array} L'$$

7.)  $x$ : # groups of 7 trees

maximize production

$$P = (\# \text{ trees}) (\text{bushels/tree})$$

$$= (100 + 7x)(55 - 2x) \rightarrow$$

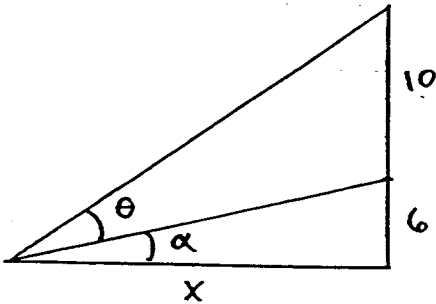
$$P' = (100 + 7x)(-2) + (7)(55 - 2x)$$

$$= -200 - 14x + 385 - 14x$$

$$= 185 - 28x = 0 \rightarrow x = 6.6 \quad \text{so plant}$$

$$100 + 7x \approx 146 \text{ trees}$$

8.)



$$\tan \alpha = \frac{6}{x}, \quad \tan(\alpha + \theta) = \frac{16}{x} \quad \text{so}$$

$$\frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} = \frac{16}{x} \rightarrow$$

$$\frac{\frac{6}{x} + \tan \theta}{1 - \frac{6}{x} \cdot \tan \theta} = \frac{16}{x} \rightarrow$$

$$\frac{6}{x} + \tan \theta = \frac{16}{x} - \frac{96}{x^2} \tan \theta \rightarrow \tan \theta = \frac{\frac{10}{x}}{1 + \frac{96}{x^2}} = \frac{10x}{x^2 + 96} \rightarrow$$

$$\sec^2 \theta \cdot \theta' = \frac{(x^2 + 96)(10) - 10x(2x)}{(x^2 + 96)^2} = \frac{10(96 - x^2)}{(x^2 + 96)^2} = 0 \rightarrow$$

$$x = \sqrt{96} \approx 9.8 \text{ ft.}$$

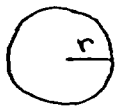
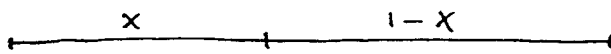
$$\begin{array}{c} + \quad 0 \quad - \\ \hline \theta' \end{array}$$

$$x = 9.8 \text{ ft.}$$

$$\alpha = 0.55 \text{ rad.}$$

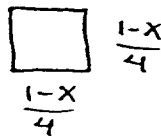
$$\text{max. } \theta = 0.47 \text{ rad.}$$

9.)



$$2\pi r = x \rightarrow$$

$$r = \frac{x}{2\pi}$$



maximize  
total area

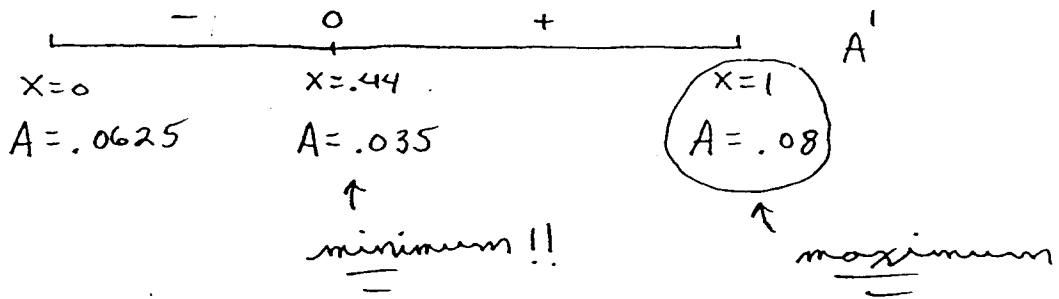
$$A = A_{\circ} + A_{\square} \rightarrow$$

$$A = \pi \left(\frac{x}{2\pi}\right)^2 + \left(\frac{1-x}{4}\right)^2 \rightarrow$$

$$= \frac{1}{4\pi} x^2 + \frac{1}{16} (1-x)^2 \rightarrow$$

$$A' = \frac{1}{2\pi} x - \frac{1}{8}(1-x) = \frac{1}{2\pi} x - \frac{1}{8} + \frac{1}{8}x = 0 \rightarrow$$

$$\left(\frac{1}{2\pi} + \frac{1}{8}\right)x = \frac{1}{8} \rightarrow x = \frac{\frac{1}{8}}{\frac{1}{2\pi} + \frac{1}{8}} = \frac{\pi}{4 + \pi} \approx .44 \rightarrow$$



10.)  $x''(t) = -32$

$x'(t) = -32t + c \rightarrow x'(0) = 100 \text{ mph.} = 146.67 \text{ ft./sec.}$

$\rightarrow \boxed{x'(t) = -32t + 146.67} \rightarrow$

$x(t) = -16t^2 + 146.67t + c \rightarrow x(0) = 0 \rightarrow$

$\boxed{x(t) = -16t^2 + 146.67t}$  ; at its highest

point  $x'(t) = 0 \rightarrow -32t + 146.67 = 0 \rightarrow$

$t = 4.58 \text{ sec.}$  ;

$x(4.58) = 336.12 \text{ ft.}$

11.) a.)  $\tan \theta = \frac{x}{r} \rightarrow x = r \tan \theta$

b.) i.)  $\frac{d\theta}{dt} = 10 \frac{\text{rev.}}{\text{min.}} = 20\pi \frac{\text{radians}}{\text{min.}}$

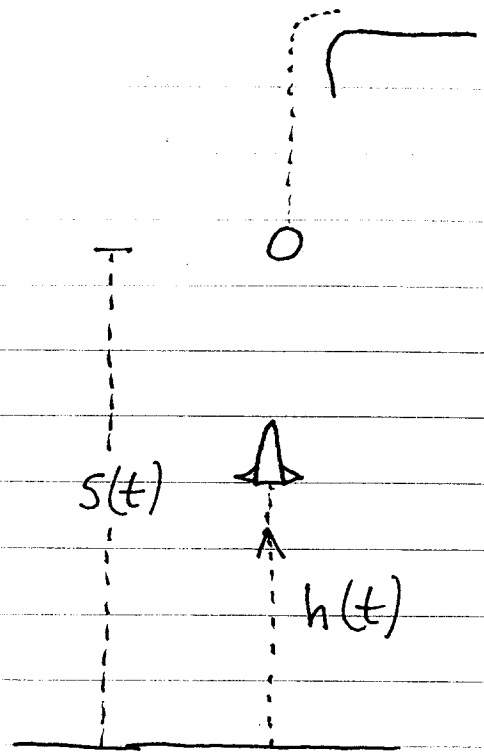
ii.)  $x = r \tan \theta \rightarrow \frac{dx}{dt} = \sec^2 \theta \cdot \frac{d\theta}{dt} = 20\pi \sec^2 \theta$

c.) For  $\theta = 0 \rightarrow \frac{dx}{dt} = 20\pi \sec^2 0 = 20\pi \frac{\text{mi.}}{\text{min.}} = 3770 \text{ mph.}$

d.) For  $x = 12 \rightarrow \begin{matrix} 1 \\ \theta \\ 12 \end{matrix} \sqrt{\frac{1}{145}} \rightarrow \sec \theta = \sqrt{145} \rightarrow$

$\frac{dx}{dt} = 20\pi (\sqrt{145})^2 = 9111 \frac{\text{mi.}}{\text{min.}} = 546,637 \text{ mph.}$

12.) Let  $s(t)$  be height<sup>(ft.)</sup> of egg above ground at time  $t$  (seconds);  $h(t)$  is height (ft.) of rocket above ground at time  $t$  (seconds). Then



acc.  $s''(t) = -32 \text{ ft./sec.}^2$

$\rightarrow s'(t) = -32t + c$

$(s'(0) = 0 \rightarrow 0 = 0 + c \rightarrow c = 0)$

$\rightarrow$  vel.  $s'(t) = -32t$

$\rightarrow s(t) = -16t^2 + c \quad (s(0) = 256 \rightarrow 256 = 0 + c$

$\rightarrow c = 256) \rightarrow$  ht.  $s(t) = 256 - 16t^2$  ;

vel.  $h'(t) = 96 \text{ ft./sec.} \rightarrow$

$h(t) = 96t + c \quad (h(0) = 0 \rightarrow 0 = 0 + c \rightarrow c = 0)$

$\rightarrow$  ht.  $h(t) = 96t$  ;

a.) collide :  $s(t) = h(t) \rightarrow 256 - 16t^2 = 96t \rightarrow$   
 $0 = 16t^2 + 96t - 256 = 16(t^2 + 6t - 16)$   
 $= 16(t-2)(t+8) \rightarrow t = 2 \text{ sec.}$

b.) egg's velocity :  $s'(2) = -64 \text{ ft./sec.}$ ,  
 rocket's velocity :  $h'(2) = 96 \text{ ft./sec.}$ ,  
 so distance between them is changing  
 at the rate of  $96 - (-64) = 160 \text{ ft./sec.}$