ESP
Kouba
Worksheet 14 Solutions

1.) a.) $f'(x) = x^3 \cdot 2(3-5x)\cdot(-5) + 3x^2(3-5x)^2$
$= x^2(3-5x)\cdot[-10x + 3(3-5x)]$
$= x^2(3-5x)[9-25x] = 0 \rightarrow$

$x=0$, $x = 3/5$, or $x = 9/25$.

b.) $f'(x) = x \cdot \frac{1}{2}(x-1)^{-1/2} + \sqrt{x-1} = \dfrac{x}{2\sqrt{x-1}} + \sqrt{x-1}$

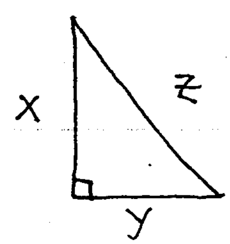
$= \dfrac{x + 2(x-1)}{2\sqrt{x-1}} = \dfrac{3x-2}{2\sqrt{x-1}} = 0 \rightarrow 3x - 2 = 0 \rightarrow$

$x = 2/3$ but $x = 2/3$ is not in the domain of f so there is no solution.

c.) $f'(x) = \dfrac{(x^2-1)3x^2 - x^3 \cdot 2x}{(x^2-1)^2} = \dfrac{x^2(x^2-3)}{(x^2-1)^2} = 0 \rightarrow$

$x = 0$ or $x = \pm\sqrt{3}$

2.) $\dfrac{dx}{dt} = -5$ in./sec. and
$\dfrac{dy}{dt} = 7$ in./sec.



a.) Find $\dfrac{dz}{dt}$ when $x = 4$ in. and $y = 3$ in. :
$x^2 + y^2 = z^2 \xrightarrow{D} 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt} \rightarrow$
$(4)(-5) + (3)(7) = (5)\dfrac{dz}{dt} \rightarrow \dfrac{dz}{dt} = \dfrac{1}{5}$ in./sec.

b.) Perimeter $P = x + y + z$, find $\dfrac{dP}{dt}$ when $x = 4$ in. and $y = 3$ in. :

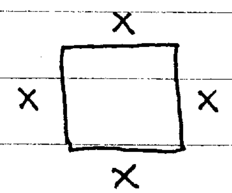
from part a.)

$$\frac{dP}{dt} = \frac{dx}{dt} + \frac{dY}{dt} + \frac{dz}{dt} = (-5) + (7) + \left(\frac{1}{5}\right) = \frac{11}{5} \text{ in./sec.}$$

c.) Area $A = \frac{1}{2}xY$, find $\frac{dA}{dt}$ when $x = 4$ in. and $Y = 3$ in. : $\frac{dA}{dt} = \frac{1}{2}x \cdot \frac{dY}{dt} + \frac{1}{2}\frac{dx}{dt} \cdot y$

$$= \frac{1}{2}(4)(7) + \frac{1}{2}(-5)(3) = \frac{13}{2} \text{ in./sec.}$$

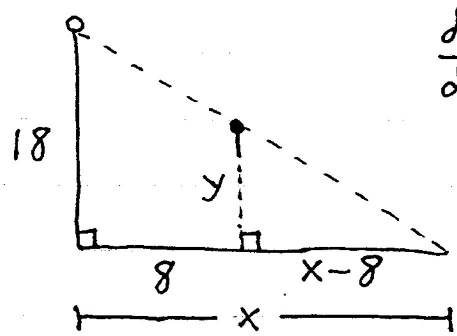
3.)



Perimeter $P = 4x$; $\frac{dP}{dt} = 24$ cm./min., find $\frac{dx}{dt}$ when $x = 10$ cm. :

$$\frac{dP}{dt} = 4 \cdot \frac{dx}{dt} \rightarrow 24 = 4 \cdot \frac{dx}{dt} \rightarrow \frac{dx}{dt} = 6 \text{ cm./min.}$$

4.)



$\frac{dy}{dt} = 3$ ft./sec.

a.) Find $\frac{dx}{dt}$ when $t = 4$ sec. :

By similar triangles

$$\frac{18}{x} = \frac{y}{x-8} \rightarrow \boxed{18x - 144 = xy} \xrightarrow{D}$$

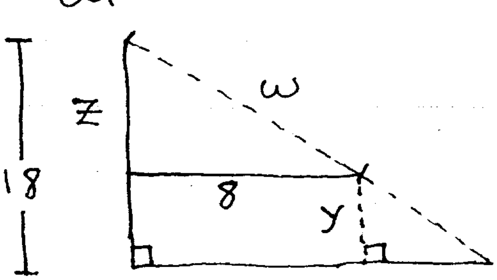
$$18\frac{dx}{dt} = x \cdot \frac{dY}{dt} + \frac{dx}{dt} \cdot y \quad (t = 4 \text{ sec.} \rightarrow$$

$$Y = 12 \text{ ft} \rightarrow 18x - 144 = 12x \rightarrow x = 24 \text{ ft.}) \rightarrow$$

$$(18)\frac{dx}{dt} = (24)(3) + (12)\frac{dx}{dt} \rightarrow (6)\frac{dx}{dt} = 72 \rightarrow$$

$$\frac{dx}{dt} = 12 \text{ ft./sec.}$$

b.)



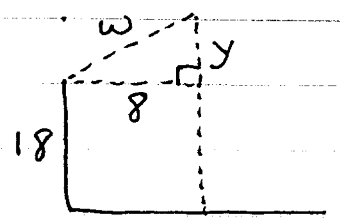
$\frac{dY}{dt} = 3$ ft./sec. so $\frac{dz}{dt} = -3$ ft./sec. ; find $\frac{d\omega}{dt}$:

$8^2 + z^2 = w^2 \xrightarrow{D} \cancel{2}z \cdot \frac{dz}{dt} = \cancel{2}w \cdot \frac{dw}{dt}$

i.) $t = 4$ sec. $\rightarrow$ $y = 12$ ft. $\rightarrow$ $z = 6$ ft.
$\rightarrow$ $w = 10$ ft. so $(6)(-3) = (10)\frac{dw}{dt} \rightarrow$
$\frac{dw}{dt} = -1.8$ ft./sec.

ii.) $t = 6$ sec. $\rightarrow$ $y = 18$ ft. $\rightarrow$ $z = 0$ ft.
$\rightarrow$ $w = 8$ ft. so $(0)(-3) = (8)\frac{dw}{dt} \rightarrow$
$\frac{dw}{dt} = 0$ ft./sec.

iii.)



$\frac{dy}{dt} = 3$ ft./sec., find $\frac{dw}{dt}$ when $t = 8$ sec.
($y = 6$ ft., $w = 10$ ft.):

$8^2 + y^2 = w^2 \rightarrow \cancel{2}y \cdot \frac{dy}{dt} = \cancel{2}w \cdot \frac{dw}{dt} \rightarrow$
$(6)(3) = (10)\frac{dw}{dt} \rightarrow \frac{dw}{dt} = 1.8$ ft./sec.

5.) $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$, $\frac{dS}{dt} = -2\pi$ ft.^{2}/sec.

a.) Find $\frac{dr}{dt}$ when $r = 4$ ft. : $S = 4\pi r^2$
$\xrightarrow{D} \frac{dS}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt} \rightarrow -2\pi = 8\pi(4)\frac{dr}{dt}$
$\rightarrow \frac{dr}{dt} = \frac{-1}{16}$ ft./sec.

b.) Find $\frac{dV}{dt}$ when $r = 2$ ft. : $V = \frac{4}{3}\pi r^3$
$\rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(2)^2 \frac{dr}{dt} = 16\pi \frac{dr}{dt}$
(Find $\frac{dr}{dt}$: $\frac{dS}{dt} = 8\pi r \frac{dr}{dt} \rightarrow -2\pi = 16\pi \cdot \frac{dr}{dt}$
$\rightarrow \frac{dr}{dt} = \frac{-1}{8}$ ft/sec.) so $\frac{dV}{dt} = 16\pi \cdot \left(\frac{-1}{8}\right) = -2\pi$ ft.^{3}/sec.

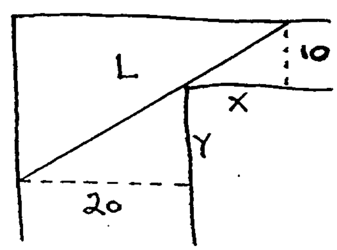
c.) $\frac{dV}{dt} = -6\pi\ ft^3/sec.$ and $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \rightarrow$

$-6\cancel{\pi} = 4\cancel{\pi} r^2 \cdot \frac{dr}{dt}$; and $\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} \rightarrow$

$-2\pi = 8\pi r \cdot \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{-1}{4r}$ (substitute)

$-6 = \cancel{4} r^2 \cdot \left(\frac{-1}{\cancel{4}r}\right) \rightarrow r = 6\ ft.$

6.) <u>Method 1</u> :



By similar triangles

$$\frac{Y}{20} = \frac{10}{X} \rightarrow Y = \frac{200}{X}$$ and minimize length

$$L = \sqrt{20^2 + Y^2} + \sqrt{X^2 + 10^2} = \sqrt{20^2 + \left(\frac{200}{X}\right)^2} + \sqrt{X^2 + 10^2}$$

$$= \frac{\sqrt{400X^2 + 200^2}}{X} + \sqrt{X^2 + 10^2} \longrightarrow$$

$$L' = \frac{X \cdot \frac{1}{2}\left(400X^2 + 200^2\right)^{-\frac{1}{2}} \cdot 800X - \sqrt{400X^2 + 200^2}}{X^2} + \frac{1}{2}\left(X^2 + 10^2\right)^{-\frac{1}{2}} \cdot 2X$$

$$= \cdots = \frac{-200^2}{X^2\sqrt{400X^2 + 200^2}} + \frac{X}{\sqrt{X^2 + 10^2}} = 0 \rightarrow$$

$$\frac{x}{\sqrt{x^2+10^2}} = \frac{200^2}{x^2\sqrt{400x^2+200^2}} \rightarrow \frac{x^2}{x^2+10^2} = \frac{200^4}{x^4(400x^2+200^2)} \rightarrow$$

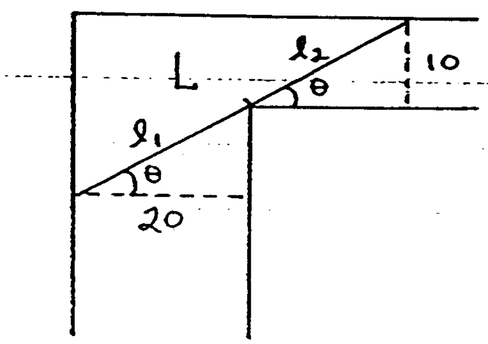
$$\cdots \rightarrow 0 = x^8 + 100x^6 - 4{,}000{,}000x^2 - 400{,}000{,}000 \rightarrow$$

(by calculator) $x \approx 12.6$ ft. $\rightarrow$ length of plank is $L = 41.6$ ft. ;

$$\begin{array}{ccc} - & 0 & + \end{array} \quad L'$$

$x = 12.6$ ft.
$Y = 15.9$ ft.

Method 2 : Write length L as a function of θ:



$\cos\theta = \frac{20}{l_1} \rightarrow l_1 = 20\sec\theta$ and

$\sin\theta = \frac{10}{l_2} \rightarrow l_2 = 10\csc\theta$ so

minimize length

$$L = l_1 + l_2 = 20\sec\theta + 10\csc\theta \rightarrow$$

$$L' = 20\sec\theta\tan\theta - 10\csc\theta\cot\theta = 0 \rightarrow$$

$$2\sec\theta\tan\theta = \csc\theta\cot\theta \rightarrow$$

$$2\frac{\sin\theta}{\cos^2\theta} = \frac{\cos\theta}{\sin^2\theta} \rightarrow 2\sin^3\theta = \cos^3\theta \rightarrow$$

$$2^{\frac{1}{3}}\sin\theta = \cos\theta \rightarrow \tan\theta = \frac{1}{2^{1/3}} \rightarrow \theta \approx .67 \text{ radians}$$

$\rightarrow$ length $L = 41.6$ ft.

$$\begin{array}{ccc} - & 0 & + \end{array} \quad L'$$

$\theta = 0.67$ rad.

7.) x: # groups of 7 trees

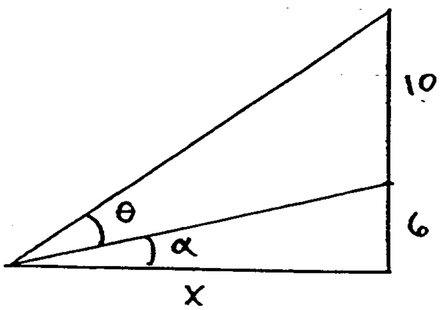
maximize production

$P = $ (# trees) (bushels/tree)

$= (100 + 7x)(55 - 2x) \rightarrow$

$$P' = (100+7x)(-2) + (7)(55-2x)$$
$$= -200 - 14x + 385 - 14x$$
$$= 185 - 28x = 0 \rightarrow x = 6.6$$ so plant
$$100 + 7x \approx 146 \text{ trees}$$

8.)



$$\tan\alpha = \frac{6}{x}, \quad \tan(\alpha+\theta) = \frac{16}{x}$$ so

$$\frac{\tan\alpha + \tan\theta}{1-\tan\alpha\tan\theta} = \frac{16}{x} \rightarrow$$

$$\frac{\frac{6}{x} + \tan\theta}{1 - \frac{6}{x}\cdot\tan\theta} = \frac{16}{x} \rightarrow$$

$$\frac{6}{x} + \tan\theta = \frac{16}{x} - \frac{96}{x^2}\tan\theta \rightarrow \tan\theta = \frac{\frac{10}{x}}{1+\frac{96}{x^2}} = \frac{10x}{x^2+96} \rightarrow$$

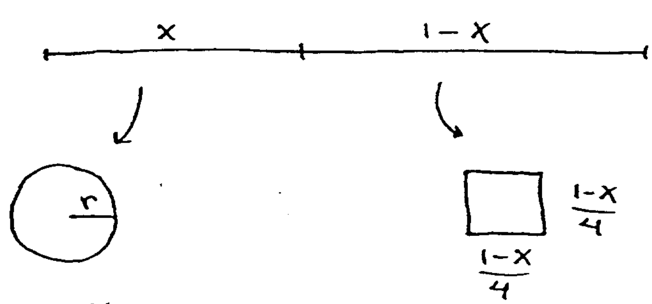
$$\sec^2\theta\cdot\theta' = \frac{(x^2+96)(10) - 10x(2x)}{(x^2+96)^2} = \frac{10(96-x^2)}{(x^2+96)^2} = 0 \rightarrow$$

$$x = \sqrt{96} \approx 9.8 \text{ ft.}$$

$+ \quad 0 \quad -$ θ'

$x = 9.8$ ft.
max. $\alpha = 0.55$ rad.
$\theta = 0.47$ rad.

9.)



maximize total area

$2\pi r = x \rightarrow$
$r = \frac{x}{2\pi}$

$$A = A_{\circ} + A_{\square} \rightarrow$$

$$A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{1-x}{4}\right)^2 \rightarrow$$
$$= \frac{1}{4\pi}x^2 + \frac{1}{16}(1-x)^2 \rightarrow$$

$$A' = \frac{1}{2\pi}x - \frac{1}{8}(1-x) = \frac{1}{2\pi}x - \frac{1}{8} + \frac{1}{8}x = 0 \rightarrow$$

$$\left(\frac{1}{2\pi}+\frac{1}{8}\right)x = \frac{1}{8} \rightarrow x = \frac{\frac{1}{8}}{\frac{1}{2\pi}+\frac{1}{8}} = \frac{\pi}{4+\pi} \approx .44 \rightarrow$$

$-$ 0 $+$ A'

$x=0$	$x=.44$	$x=1$
$A=.0625$	$A=.035$	$A=.08$
	minimum !!	maximum

10.) $x''(t) = -32$

$x'(t) = -32t + C \rightarrow x'(0) = 100$ mph. $= 146.67$ ft./sec.

$\rightarrow \boxed{x'(t) = -32t + 146.67} \rightarrow$

$x(t) = -16t^2 + 146.67t + C \rightarrow x(0) = 0 \rightarrow$

$\boxed{x(t) = -16t^2 + 146.67t}$; at its highest

point $x'(t) = 0 \rightarrow -32t + 146.67 = 0 \rightarrow$

$t = 4.58$ sec. ;

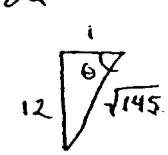
$x(4.58) = 336.12$ ft.

11.) a.) $\tan\theta = \frac{x}{1} \rightarrow x = \tan\theta$

b.) i.) $\frac{d\theta}{dt} = 10 \frac{\text{rev.}}{\text{min.}} = 20\pi \frac{\text{radians}}{\text{min.}}$

ii.) $x = \tan\theta \rightarrow \frac{dx}{dt} = \sec^2\theta \cdot \frac{d\theta}{dt} = 20\pi \sec^2\theta$

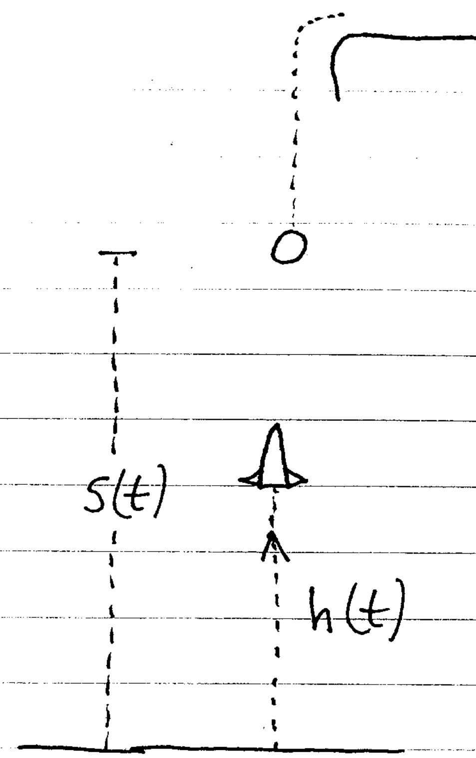
c.) For $\theta = 0 \rightarrow \frac{dx}{dt} = 20\pi \sec^2 0 = 20\pi \frac{\text{mi.}}{\text{min.}} = 3770$ mph.

d.) For $x = 12 \rightarrow$ 

$\rightarrow \sec\theta = \sqrt{145} \rightarrow$

$\frac{dx}{dt} = 20\pi\left(\sqrt{145}\right)^2 = 9111 \frac{\text{mi.}}{\text{min.}} = 546,637$ mph.

12.) Let $s(t)$ be height (ft.) of egg above ground at time t (seconds); $h(t)$ is height (ft.) of rocket above ground at time t (seconds). Then



acc. $s''(t) = -32$ ft./sec.^{2}
$\rightarrow$ $s'(t) = -32t + c$
$(s'(0) = 0 \rightarrow 0 = 0 + c \rightarrow c = 0)$
$\rightarrow$ vel. $\underline{s'(t) = -32t}$
$\rightarrow$ $s(t) = -16t^2 + c$ $(s(0) = 256 \rightarrow 256 = 0 + c$
$\rightarrow c = 256) \rightarrow$ ht. $\underline{s(t) = 256 - 16t^2}$;

vel. $\underline{h'(t) = 96}$ ft./sec. $\rightarrow$
$h(t) = 96t + c$ $(h(0) = 0 \rightarrow 0 = 0 + c \rightarrow c = 0)$
$\rightarrow$ ht. $\underline{h(t) = 96t}$;

a.) collide : $s(t) = h(t) \rightarrow 256 - 16t^2 = 96t \rightarrow$
$0 = 16t^2 + 96t - 256 = 16(t^2 + 6t - 16)$
$= 16(t-2)(t+8) \longrightarrow t = 2$ sec.

b.) egg's velocity : $s'(2) = -64$ ft./sec.,
rocket's velocity : $h'(2) = 96$ ft./sec.;
so distance between them is changing at the rate of $96 - (-64) = 160$ ft./sec.