

ESP

Kouba

## Worksheet 17 Solutions

$$1.) \text{ If } Y = \left(\sin\left(\frac{x}{2}\right)\right)^x \text{ then } \ln Y = x \cdot \ln\left(\sin\left(\frac{x}{2}\right)\right) \rightarrow$$

$$\frac{1}{Y} Y' = x \cdot \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{1}{2} + \ln\left(\sin\left(\frac{x}{2}\right)\right) \rightarrow$$

$$Y' = \left(\sin\left(\frac{x}{2}\right)\right)^x \cdot \left[\frac{1}{2}x \cot\left(\frac{x}{2}\right) + \ln\left(\sin\left(\frac{x}{2}\right)\right)\right] \text{ so for}$$

$$Y = \left(\sin\left(\frac{x}{2}\right)\right)^x + 5^x \rightarrow$$

$$Y' = \left(\sin\left(\frac{x}{2}\right)\right)^x \cdot \left[\frac{1}{2}x \cot\left(\frac{x}{2}\right) + \ln\left(\sin\left(\frac{x}{2}\right)\right)\right] + 5^x \ln 5 \text{ at } x = \pi$$

$$\rightarrow Y' = (1)^\pi \cdot [0 + \ln 1] + 5^\pi \ln 5 = 5^\pi \ln 5$$

$$2.) Y^3 + XY = 3Y^2 \rightarrow 3Y^2 Y' + XY' + Y = 6YY' \rightarrow$$

$$Y' = \frac{-Y}{(3Y^2 + X - 6Y)} \text{ at } (0, 3) \rightarrow Y' = \frac{-3}{9} = \frac{-1}{3};$$

$$Y'' = \frac{(3Y^2 + X - 6Y)(-Y') - (-Y)(6YY' + 1 - 6Y')}{(3Y^2 + X - 6Y)^2} \text{ at } x=0, Y=3$$

$$Y'' = \frac{(9)\left(\frac{1}{3}\right) + (3)\left(18\left(\frac{-1}{3}\right) + 1 + 2\right)}{9^2} = \frac{-2}{27}$$

$$3.) a.) Y' = \sec^2 x + \frac{1}{1+x^2}$$

$$b.) Y' = \cos \sqrt{x} \cdot \frac{1}{2\sqrt{x}} - \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$c.) Y' = -\csc^2(\sin(5x)) \cdot \cos(5x) \cdot 5$$

$$+ \frac{1}{|\csc x| \sqrt{\csc^2 x - 1}} \cdot -\csc x \cdot \cot x$$

$$d.) Y' = \frac{1}{\arctan(\ln x)} \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x}$$

$$e.) Y = \log_4 x + 3x \cdot \log_4 5 -$$

$$Y' = \frac{1}{x} \cdot \log_4 e + 3 \cdot \log_4 5$$

$$f.) Y' = \frac{1}{x^2 + e^{-x}} \cdot \{2x - e^{-x}\} \cdot \log_3 e$$

$$g.) \ln Y = (5+x) \{ \ln(x+1) - \ln(3x-2) \} \rightarrow$$

$$\frac{1}{Y} Y' = (5+x) \left\{ \frac{1}{x+1} - \frac{3}{3x-2} \right\} + \{ \ln(x+1) - \ln(3x-2) \} \rightarrow$$

$$Y' = \left( \frac{x+1}{3x-2} \right)^{5+x} \left[ (5+x) \left\{ \frac{1}{x+1} - \frac{3}{3x-2} \right\} + \ln \left( \frac{x+1}{3x-2} \right) \right]$$

$$h.) Y = x^{e^x} \rightarrow \ln Y = e^x \ln x \rightarrow$$

$$\frac{1}{Y} Y' = e^x \cdot \frac{1}{x} + e^x \ln x \rightarrow Y' = x^{e^x} \cdot \left\{ \frac{e^x}{x} + e^x \ln x \right\}$$

$$i.) x^2 \ln(xY) = xY^3 \ln(\tan Y) \rightarrow$$

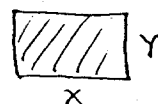
$$x^2 \cdot \frac{1}{xY} \cdot (xY' + Y) + 2x \ln(xY) = xY^3 \cdot \frac{1}{\tan Y} \cdot \sec^2 Y \cdot Y' + (x \cdot 3Y^2 Y' + Y^3) \ln(\tan Y) \rightarrow$$

$$\frac{x^2}{Y} Y' + x + 2x \ln(xY) = xY^3 \cdot \frac{\sec^2 Y}{\tan Y}$$

$$+ 3XY^2 \ln(\tan Y) \cdot Y' + Y^3 \ln(\tan Y) \rightarrow$$

$$Y' = \frac{xY^3 \cdot \frac{\sec^2 Y}{\tan Y} + Y^3 \ln(\tan Y) - 2x \ln(xY)}{\frac{x^2}{Y} - 3XY^2 \ln(\tan Y)}$$

$$4.) a.) \text{max. area } A = xY = x\sqrt{4-x}$$



$$\rightarrow A' = x \cdot \frac{1}{2} (4-x)^{-\frac{1}{2}} (-1) + \sqrt{4-x} = 0 \rightarrow$$

$$\sqrt{4-x} = \frac{x}{2\sqrt{4-x}} \rightarrow 2(4-x) = x \rightarrow 8 = 3x \rightarrow$$

$$x = \frac{8}{3} \text{ and } y = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$\begin{array}{c} + \quad 0 \quad - \\ | \\ x = \frac{8}{3} \end{array} A'$$

max.  $A = \frac{16}{3\sqrt{3}}$

b.) max. perimeter  $P = 2x + 2y = 2x + 2\sqrt{4-x} \rightarrow$

$$P' = 2 - (4-x)^{-\frac{1}{2}} = 0 \rightarrow 2 = \frac{1}{\sqrt{4-x}} \rightarrow 4-x = \frac{1}{4} \rightarrow$$

$$x = \frac{15}{4} \text{ and } y = \frac{1}{2}$$

$$\begin{array}{c} \text{///} \quad + \quad 0 \quad - \quad \text{///} \\ | \\ x=0 \quad x = \frac{15}{4} \quad x=4 \end{array} P'$$

max.  $P = 17/2$

c.) max.  $S = xy + 2x + 2y = x\sqrt{4-x} + 2x + 2\sqrt{4-x} \rightarrow$

$$S' = x \cdot \frac{1}{2}(4-x)^{-\frac{1}{2}} \cdot (-1) + \sqrt{4-x} + 2 - (4-x)^{-\frac{1}{2}} \rightarrow$$

$$= \frac{-x}{2\sqrt{4-x}} + \sqrt{4-x} + 2 - \frac{1}{\sqrt{4-x}} = \frac{6-3x}{2\sqrt{4-x}} + 2 = 0 \rightarrow$$

$$6-3x = -4\sqrt{4-x} \rightarrow 36-36x+9x^2 = 16(4-x) \rightarrow$$

$$9x^2 - 20x - 28 = 0 \rightarrow$$

$$x = \frac{20 \pm \sqrt{1408}}{18} \approx 3.2$$

$$\begin{array}{c} \text{///} \quad + \quad 0 \quad - \quad \text{///} \\ | \\ x=0 \quad x=3.2 \quad x=4 \end{array}$$

max.  $S \approx 11.03$

and  $y = .89$

5.) a.)  $\lim_{n \rightarrow -\infty} \left( \frac{n+2-1}{n+2} \right)^{7n} = \lim_{n \rightarrow -\infty} \left[ \left( 1 + \frac{1}{-(n+2)} \right)^{-(n+2)} \right]^{\frac{7n}{-(n+2)}}$

$$= e^{-7}$$

b.)  $\lim_{n \rightarrow +\infty} \frac{1}{\left( \frac{n^3+1}{n^3} \right)^n} = \lim_{n \rightarrow +\infty} \frac{1}{\left[ \left( 1 + \frac{1}{n^3} \right)^{n^3} \right]^{\frac{1}{n^2}}} = \frac{1}{e^0} = 1$

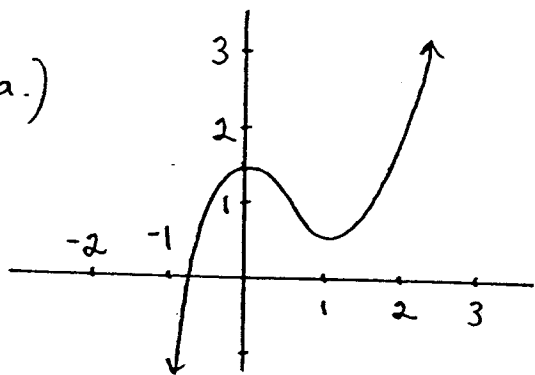
6.)  $x''(t) = -32 \rightarrow$   
 $x'(t) = -32t + c$  and  $x'(0) = 0 \rightarrow c = 0 \rightarrow$   
 $x'(t) = -32t \rightarrow$   
 $x(t) = -16t^2 + c$  and  $x(0) = 5280 \text{ ft} \rightarrow c = 5280 \rightarrow$   
 $x(t) = -16t^2 + 5280$  ;

a.) hit ground :  $x(t) = 0 \rightarrow -16t^2 + 5280 = 0 \rightarrow$   
 $t = 18.17 \text{ sec.}$

b.) distance =  $(100 \text{ mph})(18.17 \text{ sec.})$   
 $= (147 \frac{\text{ft.}}{\text{sec.}})(18.17 \text{ sec.})$   
 $= 2665 \text{ ft.}$

c.)  $x'(18.17) = -581.44 \text{ ft./sec.} = -396.4 \text{ mph.}$

7.) a.)



b.)  $f(x) = x^3 - 2x^2 + \frac{3}{2}$  is continuous with  $f(0) = \frac{3}{2}$  and  $f(-1) = -\frac{3}{2}$  so by IMVT there is some  $\# r, -1 < r < 0$ , satisfying  $f(r) = 0$ .

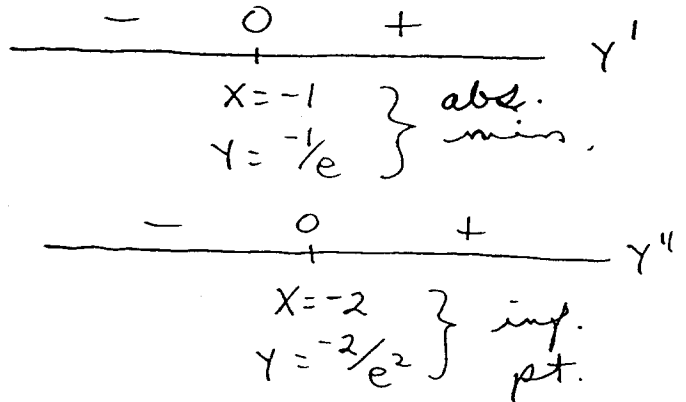
8.)  $f(x) = \ln(x^4 + 1)$  is continuous on  $[3, 4]$  and differentiable on  $(3, 4)$  so by MVT there

is some #  $c$ ,  $3 < c < 4$ , satisfying

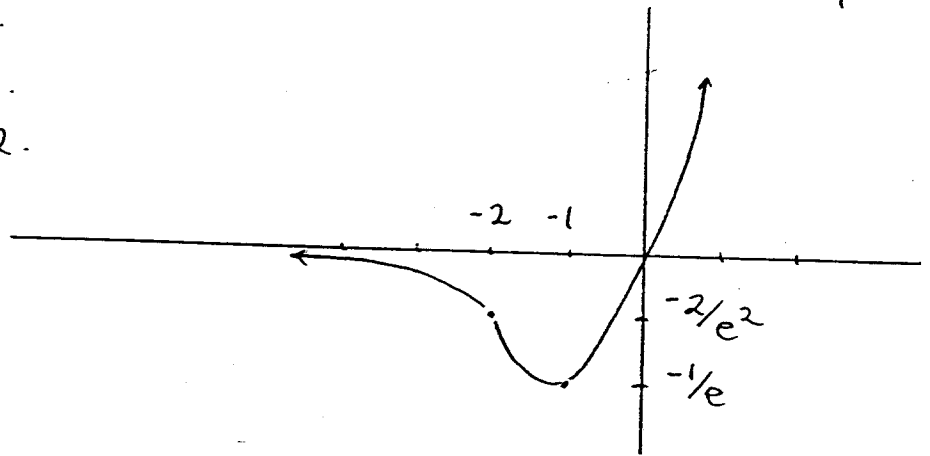
$$f'(c) = \frac{f(4) - f(3)}{4 - 3}, \text{ i.e., } \frac{4e^3}{c^4 + 1} = \ln 257 - \ln 82 = \ln\left(\frac{257}{82}\right)$$

9.) a.)  $y = xe^x \rightarrow$   
 $y' = xe^x + e^x = e^x(x+1) = 0$

$$y'' = e^x + e^x(x+1) = e^x(x+2) = 0$$

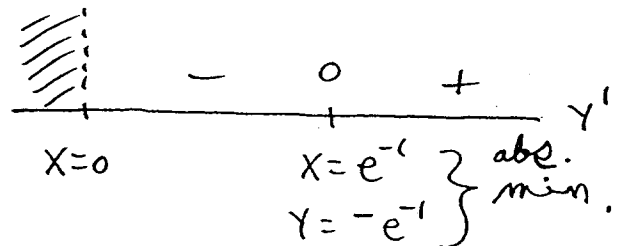


$y$  is  $\uparrow$  for  $x > -1$ .  
 $y$  is  $\downarrow$  for  $x < -1$ .  
 $y$  is  $\cup$  for  $x > -2$ .  
 $y$  is  $\cap$  for  $x < -2$ .

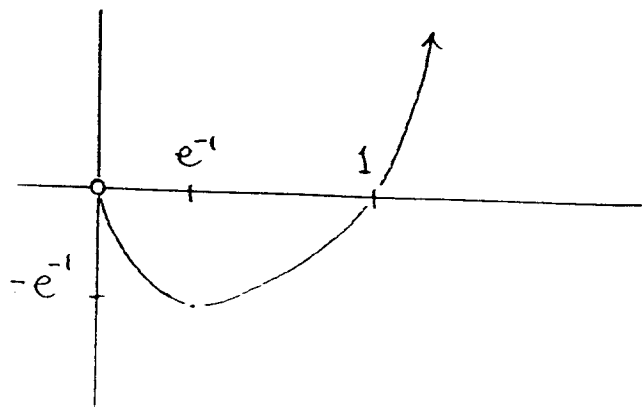


b.)  $y = x \ln x \rightarrow$   
 $y' = x \cdot \frac{1}{x} + \ln x = 1 + \ln x = 0$

$$y'' = \frac{1}{x}$$



$y$  is  $\uparrow$  for  $x > e^{-1}$   
 $y$  is  $\downarrow$  for  $0 < x < e^{-1}$   
 $y$  is  $\cup$  for  $x > 0$



$$c.) \quad y = e^x + e^{-x} \rightarrow$$

$$y' = e^x - e^{-x} = 0$$

-	0	+	
			$y'$
$x=0$ } abs. $y=2$ } min.			

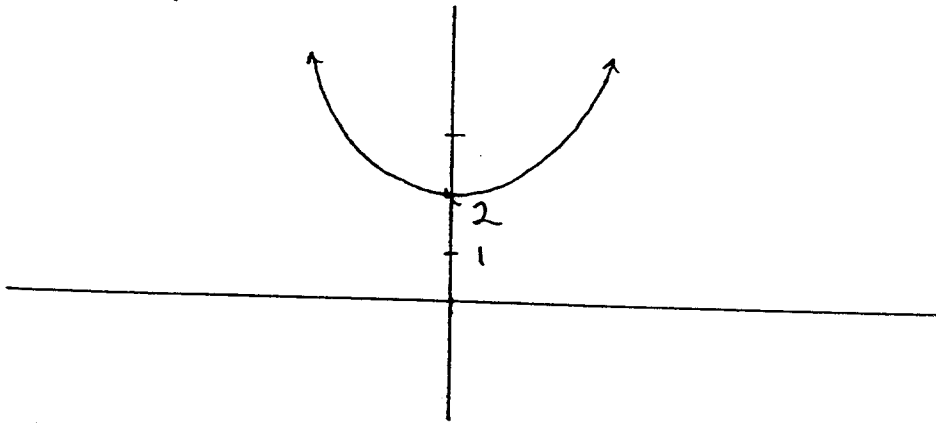
$$y'' = e^x + e^{-x} > 0$$

+	+	+	+	
				$y''$

$y$  is  $\uparrow$  for  $x > 0$ .

$y$  is  $\downarrow$  for  $x < 0$ .

$y$  is U for all  $x$ -values.



$$10.) a.) \lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

$$b.) \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 1} \frac{2x}{1} = 2$$

$$c.) \lim_{x \rightarrow 2} \frac{x^4 - 16}{\sqrt{x} - \sqrt{2}} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 2} \frac{4x^3}{\frac{1}{2\sqrt{x}}} = \frac{32}{\frac{1}{2\sqrt{2}}} = 64\sqrt{2}$$

$$d.) \lim_{x \rightarrow 0} \frac{\tan x}{x + \sin x} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{\sec^2 x}{1 + \cos x} = \frac{1}{2}$$

$$e.) \lim_{x \rightarrow 1} \frac{e^{x-1} - 2^{x-1}}{x^2 - x} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 1} \frac{e^{x-1} - 2^{x-1} \cdot \ln 2}{2x - 1} = 1 - \ln 2$$

$$f.) \lim_{x \rightarrow 0} \frac{x^2 \sin x + x \sin x}{x + 1 - \cos x} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{x^2 \cos x + 2x \sin x + x \cos x + \sin x}{1 + \sin x} = 0$$

$$g.) \lim_{x \rightarrow 1} \frac{x \ln x + 1 - x}{(x-1)^2} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 1} \frac{x \cdot \frac{1}{x} + \ln x - 1}{2(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{\ln x}{2(x-1)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{2} = \frac{1}{2}$$

$$h.) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{1 + \sec x} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec^2 x}{\sec x \cdot \tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} = 1$$

$$i.) \lim_{x \rightarrow +\infty} \frac{2^x + 2x}{5^x} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow +\infty} \frac{2^x \cdot \ln 2 + 2}{5^x \cdot \ln 5}$$

$$\stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow +\infty} \frac{2^x (\ln 2)^2}{5^x (\ln 5)^2} = \lim_{x \rightarrow +\infty} \left(\frac{2}{5}\right)^x \cdot \frac{(\ln 2)^2}{(\ln 5)^2} = 0$$

$$j.) \lim_{x \rightarrow +\infty} \frac{x^3}{10^x} \stackrel{''\infty/\infty''}{=} \lim_{x \rightarrow +\infty} \frac{3x^2}{10^x \cdot \ln 10} \stackrel{''\infty/\infty''}{=} \lim_{x \rightarrow +\infty} \frac{6x}{10^x (\ln 10)^2}$$

$$\stackrel{''\infty/\infty''}{=} \lim_{x \rightarrow +\infty} \frac{6}{10^x (\ln 10)^3} = 0$$

$$k.) \lim_{x \rightarrow 0} \frac{x e^x \cos^2 6x}{e^{2x} - 1}$$

$$\stackrel{''0/0''}{=} \lim_{x \rightarrow 0} \frac{e^x \cos^2 6x + x e^x \cos^2 6x - x e^x \cdot 2 \cos 6x \cdot \sin 6x \cdot 6}{2e^{2x}} = \frac{1}{2}$$

$$l.) \lim_{x \rightarrow +\infty} \frac{e^x - \frac{1}{x}}{e^x + \frac{1}{x}} \stackrel{''\infty/\infty''}{=} \lim_{x \rightarrow +\infty} \frac{x e^x - 1}{x e^x + 1} \stackrel{''\infty/\infty''}{=} \lim_{x \rightarrow +\infty} \frac{x e^x + e^x}{x e^x + e^x}$$

$$= \lim_{x \rightarrow +\infty} 1 = 1$$

$$m.) \lim_{x \rightarrow 0} \frac{\arcsin x}{\arctan 2x} \stackrel{''0/0''}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{1+4x^2}} = \frac{1}{2}$$

$$n.) \lim_{x \rightarrow 0} \left\{ \frac{1}{1-\cos x} - \frac{2}{x^2} \right\} = \lim_{x \rightarrow 0} \frac{x^2 - 2 + 2 \cos x}{x^2 (1-\cos x)}$$

$$\stackrel{''0/0''}{=} \lim_{x \rightarrow 0} \frac{2x - 2 \sin x}{x^2 \sin x + 2x (1-\cos x)}$$

$$\stackrel{''0/0''}{=} \lim_{x \rightarrow 0} \frac{2 - 2 \cos x}{x^2 \cos x + 2x \sin x + 2x \sin x + 2(1-\cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{2(1-\cos x)}{x^2 \cos x + 4x \sin x + 2(1-\cos x)}$$

$$\stackrel{''0/0''}{=} \lim_{x \rightarrow 0} \frac{2 \sin x}{-x^2 \sin x + 2x \cos x + 4x \cos x + 4 \sin x + 2 \sin x}$$

$$\stackrel{''0/0''}{=} \lim_{x \rightarrow 0} \frac{2 \cos x}{-x^2 \cos x - 2x \sin x + 6x \sin x + 6 \cos x + 6 \cos x} = \frac{1}{6}$$



$$0.) \lim_{x \rightarrow 0} \frac{\sin^2 x - x^2}{(e^{x^2} - 1)^2} \stackrel{0/0}{=} \lim_{x \rightarrow 0} \frac{\overbrace{2 \sin x \cos x}^{\sin 2x} - 2x}{2(e^{x^2} - 1)e^{x^2} \cdot 2x}$$

$$\stackrel{0/0}{=} \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{4(e^{x^2} - 1)e^{x^2} + 4x \cdot e^{x^2} \cdot 2x \cdot e^{x^2} + 4x(e^{x^2} - 1) \cdot e^{x^2} \cdot 2x}$$

$$= \lim_{x \rightarrow 0} \frac{2(\cos 2x - 1)}{2x e^{x^2} \cdot [4x^2 e^{x^2} + e^{x^2} - 2x^2 - 1]} = \frac{0}{-4} = 0$$

$$P.) Y = (\ln x)^{\frac{1}{x}} \rightarrow \ln Y = \frac{1}{x} \ln(\ln x) = \frac{\ln(\ln x)}{x} \quad \text{so}$$

$$\lim_{x \rightarrow +\infty} \ln Y = \lim_{x \rightarrow +\infty} \frac{\ln(\ln x)}{x} \stackrel{0/0}{=} \lim_{x \rightarrow +\infty} \frac{1}{\ln x} \cdot \frac{1}{x} = 0$$

$$\text{so if } \lim_{x \rightarrow +\infty} \ln Y = \ln \left( \lim_{x \rightarrow +\infty} Y \right) = 0$$

$$\text{then } \lim_{x \rightarrow +\infty} Y = 1$$

$$Q.) Y = (\sin x)^{\frac{1}{x}} \rightarrow \ln Y = \frac{1}{x} \ln(\sin x) = \frac{\ln(\sin x)}{x}$$

$$\text{so } \lim_{x \rightarrow 0^+} \ln Y = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{x} = \frac{-\infty}{0^+} = -\infty$$

$$\text{then } \ln \left( \lim_{x \rightarrow 0^+} Y \right) = -\infty \quad \text{and}$$

$$\lim_{x \rightarrow 0^+} Y = 0$$

$$R.) Y = (1+x)^{\frac{1}{x}} \rightarrow \ln Y = \frac{\ln(1+x)}{x} \rightarrow$$

$$\lim_{x \rightarrow 0} \ln Y = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} \stackrel{0/0}{=} \lim_{x \rightarrow 0} \frac{1}{1+x} = 1 \quad \text{so}$$

$$\ln\left(\lim_{x \rightarrow 0} Y\right) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} Y = e$$

$$s.) \quad Y = \left(1 + \frac{5}{n}\right)^{5n} \rightarrow \ln Y = 5n \ln\left(1 + \frac{5}{n}\right) = \frac{\ln\left(1 + \frac{5}{n}\right)}{\frac{1}{5n}}$$

$$\lim_{n \rightarrow +\infty} \ln Y = \lim_{n \rightarrow +\infty} \frac{\ln\left(1 + \frac{5}{n}\right)}{\frac{1}{5n}} \stackrel{\text{"0/0"}}{=} \lim_{n \rightarrow +\infty} \left(\frac{1}{1 + \frac{5}{n}}\right) \cdot \frac{\frac{1}{5n} \cdot \frac{-5}{n^2}}{\frac{-1}{5n^2}} = 25$$

$$\text{so } \ln\left(\lim_{n \rightarrow +\infty} Y\right) = 25 \quad \text{and} \quad \lim_{n \rightarrow +\infty} Y = e^{25}$$

$$t.) \quad Y = (1+n)^{1/n} \rightarrow \ln Y = \frac{\ln(1+n)}{n} \rightarrow$$

$$\lim_{n \rightarrow +\infty} \ln Y = \lim_{n \rightarrow +\infty} \frac{\ln(1+n)}{n} \stackrel{\text{"0/0"}}{=} \lim_{n \rightarrow +\infty} \frac{1}{1+n} = 0 \quad \text{so}$$

$$\ln\left(\lim_{n \rightarrow +\infty} Y\right) = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} Y = 1$$

$$u.) \quad \lim_{x \rightarrow 0^+} x^2 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}} \stackrel{\text{"0/0"}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}}$$

$$= \lim_{x \rightarrow 0^+} \frac{-x^2}{2} = 0$$

$$v.) \quad Y = (\tan x)^{\sqrt{x/3}} \rightarrow \ln Y = \sqrt{\frac{x}{3}} \ln(\tan x) = \frac{\ln(\tan x)}{\frac{\sqrt{3}}{\sqrt{x}}} \rightarrow$$

$$\lim_{x \rightarrow 0^+} \ln Y = \lim_{x \rightarrow 0^+} \frac{\ln(\tan x)}{\frac{\sqrt{3}}{\sqrt{x}}} \stackrel{\text{"0/0"}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{\sec^2 x}{\tan x}}{\frac{-\sqrt{3}}{2x^{3/2}}}$$

$$= \lim_{x \rightarrow 0^+} \left( \sqrt{x} \cdot \frac{x}{\sin x} \cdot \sec x \cdot \frac{2}{-\sqrt{3}} \right) = 0 \cdot 1 \cdot 1 \cdot \frac{2}{\sqrt{3}} = 0 \quad \text{so}$$

$$\ln\left(\lim_{x \rightarrow 0^+} Y\right) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} Y = e^0 = 1$$