ESP
Kouba
Worksheet 6 Solutions

-12

1.) a.) $\lim\limits_{x \to 0} \left(\frac{4}{3}x + \frac{2}{3} \cdot \frac{\sin 2x}{2x} \right) = 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$

b.) $\lim\limits_{x \to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{(3x-9)} = 1 \cdot \frac{-2}{9} = \frac{-2}{9}$

c.) $\lim\limits_{x \to \pi} \frac{1}{2+2\cos x} = \text{"}\frac{1}{0^+}\text{"} = +\infty$

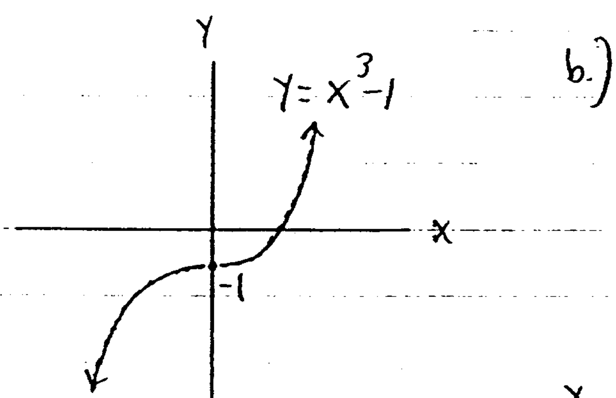
d.) $\lim\limits_{x \to 3^-} \frac{(x-1)(x-2)}{(x+3)(x-3)} = \text{"}\frac{(2)(1)}{(6)(0^-)}\text{"} = \text{"}\frac{2}{0^-}\text{"} = -\infty$

e.) $\lim\limits_{x \to -\infty} \frac{\sqrt{2x^2+1}}{x+2} = \lim\limits_{x \to -\infty} \frac{\sqrt{2x^2+1}}{-\sqrt{(x+2)^2}}$

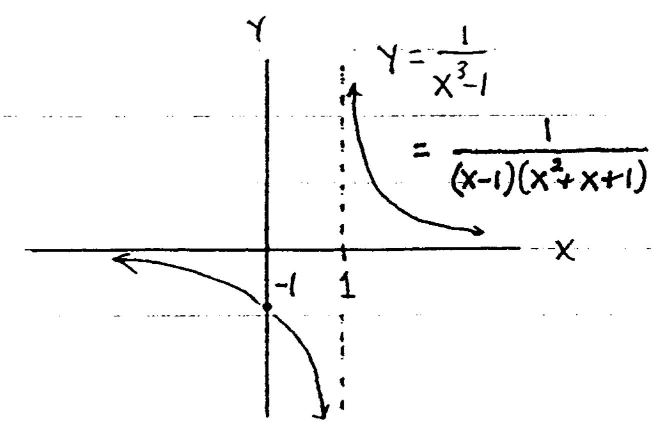
$= \lim\limits_{x \to -\infty} -\sqrt{\frac{2x^2+1}{x^2+4x+4}} = \lim\limits_{x \to -\infty} -\sqrt{\frac{2 + 1/x^2}{1 + 4/x + 4/x^2}} = -\sqrt{2}$

f.) $\lim\limits_{x \to 0} \frac{x^2 \cos x + \sin 2x - 2x}{x^2+1} = \frac{0+0-0}{0+1} = 0$

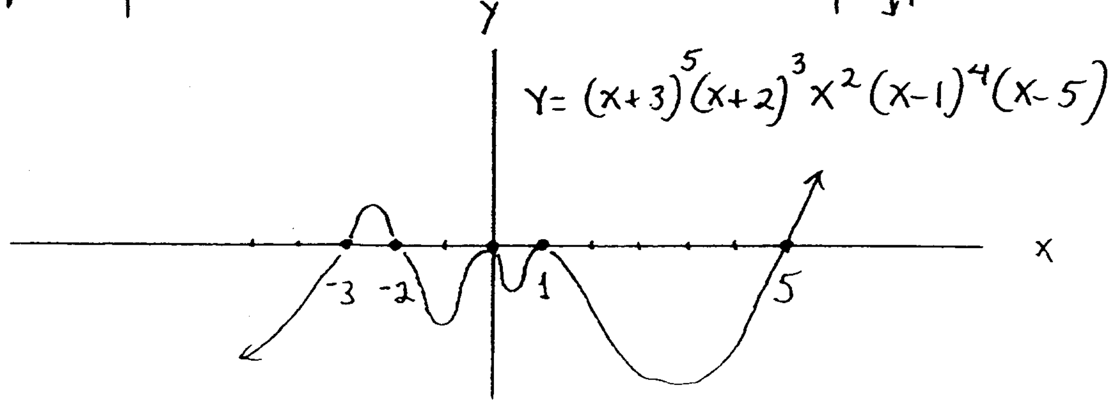
2.) a.)



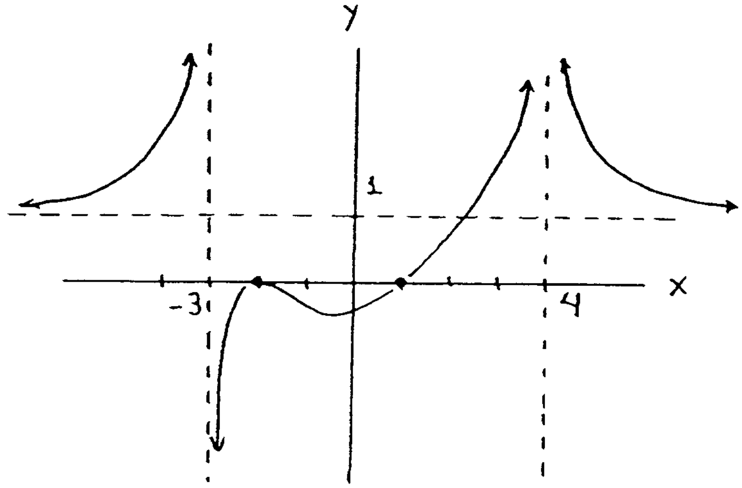
b.)



c.)

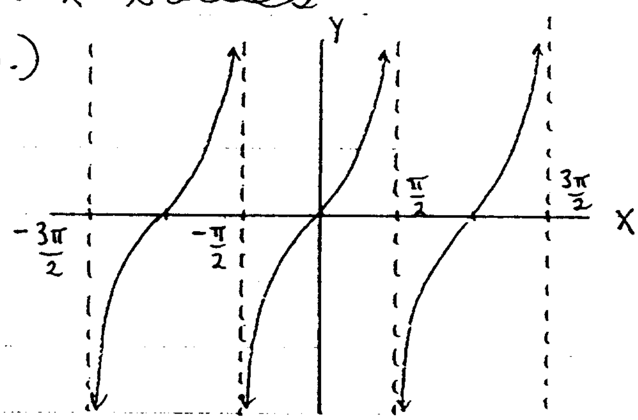


d.)



3.) a.) f is continuous for all x-values since $Y = X^3$, $Y = \sin 3X$, and $Y = |X|$ are continuous for all x-values

b.)



$Y = \tan x$

f is continuous for all x-values except

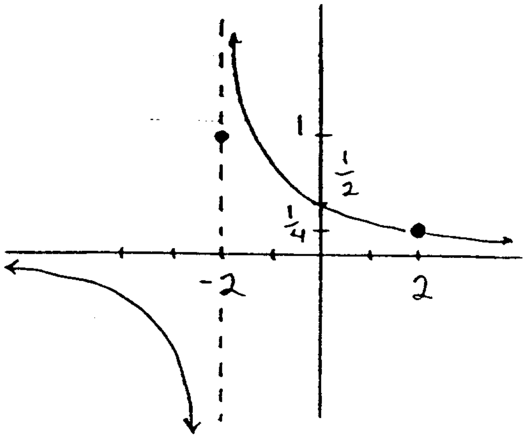
$x = \frac{\pi}{2} \pm n\pi$ for

$n = 0, 1, 2, 3, \ldots$

c.) $f(x) = \frac{x-1}{(x-1)(x+1)}$ is continuous for all x-values except $x = 1$, $x = -1$.

d.) $f(x) = \frac{1}{(x-10)(x+10)}$ is continuous for all x-values except $x = 10$, $x = -10$.

e.) $f(x) = \begin{cases} \frac{\cancel{x-2}}{(\cancel{x-2})(x+2)}, & x \neq 2, -2 \\ \frac{1}{4}, & x = 2 \\ 1, & x = -2 \end{cases} = \begin{cases} \frac{1}{x+2}, & x \neq 2, -2 \\ \frac{1}{4}, & x = 2 \\ 1, & x = -2 \end{cases}$



f is continuous for all x-values except $x = -2$.

4.) a.) $f(x) = mx^3 + 7$ is a polynomial for all values of m, and, hence, is continuous for all values of x for <u>all values of m</u>.

b.) <u>Impossible</u>. For each value of m, f is discontinuous at $x = 5$.

c.) If $m = 0$, then $f(x) = -x^2$ is continuous. If $m = -k < 0$, then

$$f(x) = \frac{-k + x^2}{-kx^2 - 1} = \frac{-(-k + x^2)}{kx^2 + 1}, \text{ where } kx^2 + 1 > 0.$$

If $m > 0$ then $f(x) = \dfrac{m + x^2}{(\sqrt{m}\,x - 1)(\sqrt{m}\,x + 1)}$ is discontinuous at $x = \frac{1}{\sqrt{m}}$ and $x = \frac{-1}{\sqrt{m}}$. Thus,

f is continuous for all x-values for $\boxed{m \leq 0}$

d.) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x^2 = 4$ and

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} e^{mx} = e^{2m}$, so for

continuity it must be that $4 = e^{2m} \Leftrightarrow$
$\ln 4 = 2m \Leftrightarrow \underline{m = \frac{1}{2}\ln 4}$.

e.) f is continuous at $x=0$ since
$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0 = \lim_{x\to 0^+} \sqrt{x} = \lim_{x\to 0^+} f(x)$;

and $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \sqrt{x} = 2$ and

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} m\cos[(x+1)\pi] = m\cos 5\pi = -m$.

For continuity we need $2=-m$ or $\boxed{m=-2}$.

f.) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2mx+n) = 4m+n = 6$ and
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (m-nx) = m-2n = 6$, then

$\left.\begin{array}{l} 4m+n=6 \\ m-2n=6 \end{array}\right\}$ $\left.\begin{array}{l} 8m+2n=12 \\ m-2n=6 \end{array}\right\}$ $9m = 18 \to \boxed{m=2 \text{ and } n=-2}$.

g.) $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (x+m) = 2+m$ and
$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \sqrt{mx^2+8} = \sqrt{4m+8}$, then

$2+m = \sqrt{4m+8} \Leftrightarrow (2+m)^2 = 4m+8 \Leftrightarrow$
$4+\cancel{4m}+m^2 = \cancel{4m}+8 \Leftrightarrow m^2=4 \Leftrightarrow$
$m=2$ or $m=-2$. But if $m=-2$

then $\sqrt{mx^2+8} = \sqrt{-2x^2+8} = \sqrt{-2(x^2-4)}$ is <u>not</u> defined for $x > 2$. Thus, only $\boxed{m=2}$ is a solution.

h.) $\lim_{x\to 1} h(x) = \lim_{x\to 1} \frac{x^3-x}{x^2-1} = \lim_{x\to 1} \frac{x\cancel{(x-1)}\cancel{(x+1)}}{\cancel{(x-1)}\cancel{(x+1)}} = 1$

so let $\boxed{m=1}$; $\lim_{x\to -1} h(x) = \lim_{x\to -1} \frac{x\cancel{(x-1)}\cancel{(x+1)}}{\cancel{(x-1)}\cancel{(x+1)}} = -1$

so let $\boxed{n=-1}$.

i.) $\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{\cancel{x+5}}{\cancel{(x+5)}(x-2)} = \text{"}\frac{1}{0^{\pm}}\text{"}$ so

does not exist. Thus <u>no</u> choice for m will make g continuous at $x=2$. <u>Impossible</u>.

(Note: $n = \frac{-1}{7}$ will make g continuous at $x=-5$.)

5.) One cup of K.J. is added to 80 cups (5 gal.) of P.J. creating a mixture in the pineapple bucket which is $\frac{80}{81}$ P.J. and $\frac{1}{81}$ K.J. Thus, the one cup mixture <u>returning to the kiwi bucket</u> is

$\boxed{\frac{80}{81} \text{ cup P. J.}}$ and $\frac{1}{81}$ cup K.

The amount of K.J. <u>left behind</u> in the pineapple bucket is

$$1 \text{ cup} - \frac{1}{81} \text{ cup} = \boxed{\frac{80}{81} \text{ cup K. J.}}$$

The amounts are equal.

6.) Let $f(x) = x^5 - x^2 - 2x + 17$, which is continuous for all x-values since it is a polynomial. Consider f on the interval $[-2, 0]$ and let $m = 0$:

$$f(-2) = -32 - 4 + 4 + 17 = -15 < 0 \quad \text{and}$$

$$f(0) = 17 > 0 .$$ Since $m=0$ is between the values -15 and 17,

by the IMVT there is at least one # c in $[-2, 0]$ satisfying $f(c) = 0$, i.e., solving $x^5 - x^2 - 2x + 17 = 0$, i.e., solving

$$x^5 - x^2 + 17 = 2x .$$

7.) a.) Given any $\varepsilon > 0$, determine $\delta > 0$ so that if $0 < |x-3| < \delta$, then $|f(x) - 7| < \varepsilon$, i.e.,

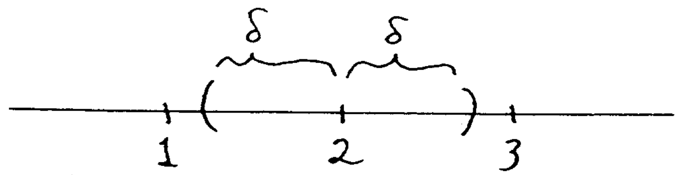
$$|(2x+1)-7| = |2x-6| = 2|x-3| < \varepsilon$$

Choose $\delta = \varepsilon/2$. If $0 < |x-3| < \delta = \varepsilon/2$, then $|x-3| < \varepsilon/2 \Rightarrow 2|x-3| < \varepsilon$. This completes the proof.

b.) Given any E, determine D so that if $x > D$, then $f(x) > E$, i.e.,

$$(1+\sqrt{x})^3 - 100 > E \iff (1+\sqrt{x})^3 > E + 100$$
$$\iff 1+\sqrt{x} > (E+100)^{1/3}$$
$$\iff \sqrt{x} > 1 + (E+100)^{1/3}$$
$$\iff x > \{1 + (E+100)^{1/3}\}^2$$

Choose $D = \{1 + (E+100)^{1/3}\}^2$. This completes the proof.



c.) Given any $\varepsilon > 0$, determine $\delta > 0$ so that if $0 < |x-2| < \delta$, then $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$, i.e.,

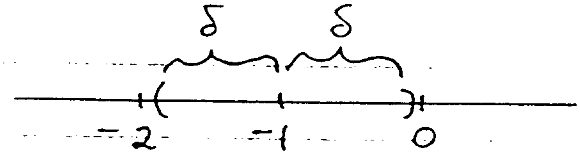
$\left|\frac{2-x}{2x}\right| < \varepsilon$, i.e., $\frac{1}{2}\frac{|x-2|}{|x|} < \varepsilon$. assume that $\delta \le 1$ so that $1 < x < 3$, $1 < |x| < 3$, and $\frac{1}{3} < \frac{1}{|x|} < 1$

so that $\frac{1}{2}\frac{|x-2|}{|x|} \le \frac{1}{2}\frac{|x-2|}{|1|} = \frac{1}{2}|x-2| < \varepsilon$.

Choose $\delta =$ minimum of $\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$ then $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{1}{2}\,\frac{|x-2|}{|x|} \le \frac{1}{2}\cdot|x-2| < \frac{1}{2}\cdot 2\varepsilon = \varepsilon$.

This completes the proof.

d.) Given any $\varepsilon > 0$, determine $\delta > 0$ so that if $0 < |x-(-1)| = |x+1| < \delta$, then $|x^2-1| < \varepsilon$, i.e., $|(x-1)(x+1)| = |x-1|\,|x+1| < \varepsilon$.



Assume that $\delta \le 1$ so that $-2 < x < 0$, $1 < |x-1| < 3$, and $|x-1|\,|x+1| \le 3\,|x+1| < \varepsilon$. Choose $\delta =$ minimum of $\{1, \varepsilon/3\}$. If $0 < |x+1| < \delta$ then $|x^2-1| = |x-1|\,|x+1| \le 3\,|x+1| < 3\cdot\frac{\varepsilon}{3} = \varepsilon$. This completes the proof.