ESP
Kouba
Worksheet 13 Solutions

1.) a.) False, since $D \ln(1+e^x) = \dfrac{e^x}{1+e^x}$.

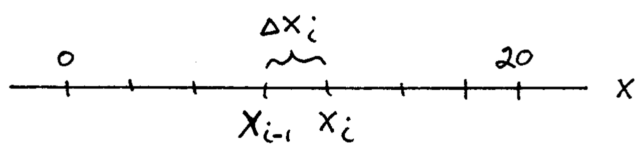
b.) True, since $D \arctan(e^x) = \dfrac{e^x}{1+(e^x)^2}$.

c.) True, since $D\, x \ln x = x \cdot \dfrac{1}{x} + \ln x$.

d.) $Y = x^x \Rightarrow \ln Y = x \ln x \Rightarrow \dfrac{1}{Y} Y' = 1 + \ln x \Rightarrow$

$Y' = Y(1+\ln x) = x^x(1+\ln x)$ so True .

2.) a.)

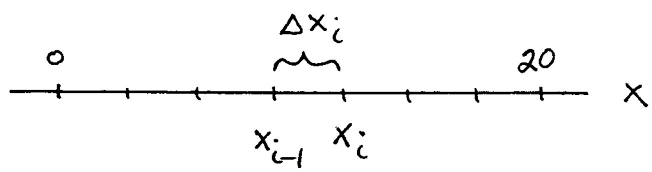


$\Delta x_i = \dfrac{20-0}{n}$, $f(x) = 3$ is density

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 3\cdot \Delta x_i = \int_0^{20} 3\,dx .$$

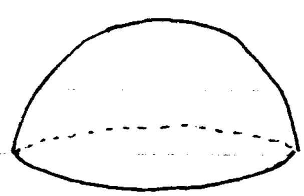
b.)



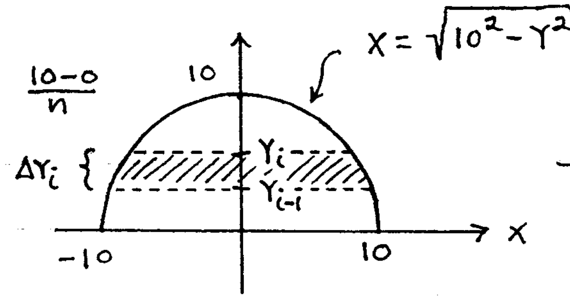
$\Delta x_i = \dfrac{20-0}{n}$, $f(x) = \dfrac{x}{1+x^2}$ is density

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i = \int_0^{20} \frac{x}{1+x^2}\,dx .$$

c.)



$\Delta Y_i = \dfrac{10-0}{n}$

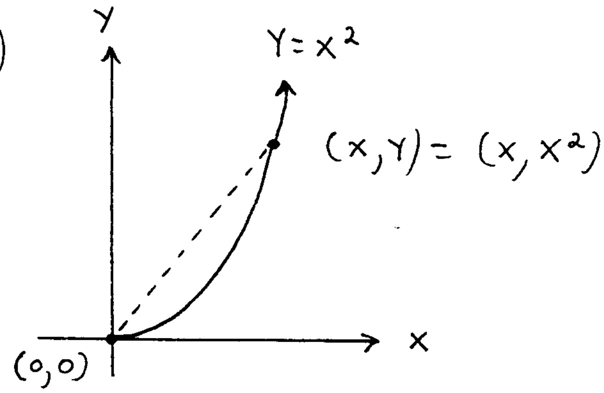


$f(Y) = Y^3 e^{-Y}$ is density

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} f(Y_i)\cdot \pi \left(\sqrt{10^2 - Y_i^2}\right)^2 \cdot \Delta Y_i$$

$$= \int_0^{10} y^3 e^{-y} \cdot \pi \left(100 - y^2\right) dy \quad .$$

3.)



$$f(x) = \sqrt{(x-0)^2 + (x^2-0)^2}$$
$$= \sqrt{x^2 + x^4} = x\sqrt{1+x^2} \quad \text{so}$$

$$AVE = \frac{1}{\sqrt{15}-0} \int_0^{\sqrt{15}} x\left(1+x^2\right)^{1/2} dx$$

$$= \frac{1}{\sqrt{15}} \cdot \frac{1}{3} \left(1+x^2\right)^{3/2} \Big|_0^{\sqrt{15}} = \frac{1}{3\sqrt{15}}\left(64-1\right) = \frac{21}{\sqrt{15}} \quad .$$

4.) a.) $\int (1-x)^{-1/2} = -2(1-x)^{1/2} + c$

b.) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

c.) $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$ $\left(\text{let } u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow \right.$
$\left. 2\,du = \frac{1}{\sqrt{x}} dx \text{ and } 1-x = 1-u^2 \right)$

$$= \int \frac{2}{\sqrt{1-u^2}} du = 2 \arcsin u + c = 2 \arcsin \sqrt{x} + c$$

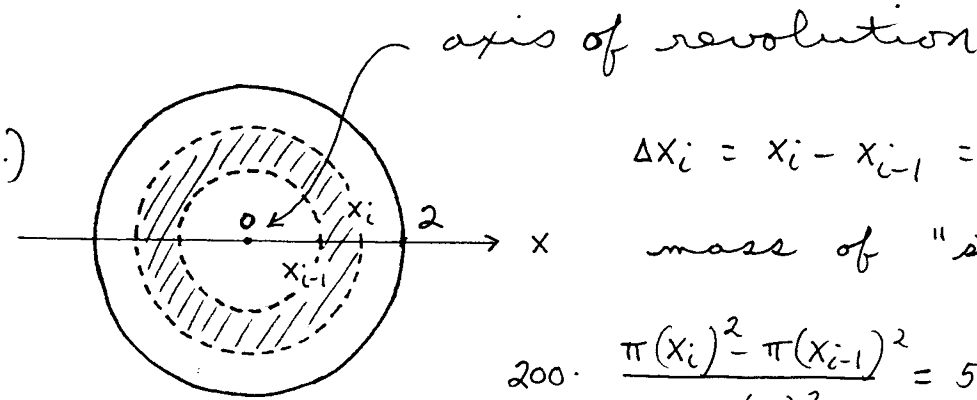
d.) $\int_{-1}^{1} \sqrt{x^2}\, dx = \int_{-1}^{1} |x|\, dx = \int_{-1}^{0} -x\, dx + \int_0^1 x\, dx$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} + \frac{1}{2} = 1 \quad .$$

5.) $\text{velocity} = 3\underbrace{\left(2\pi(2)\right)}_{\text{circumference}} = 12\pi$ ft./sec. so

$$K.E. = \frac{1}{2} m v^2 = \frac{1}{2}(200)(12\pi)^2 = 14{,}400\pi^2 \ \frac{\text{gm. ft.}^2}{\text{sec}^2} \quad .$$

6.) a.)



$\Delta x_i = x_i - x_{i-1} = \frac{2-0}{n}$

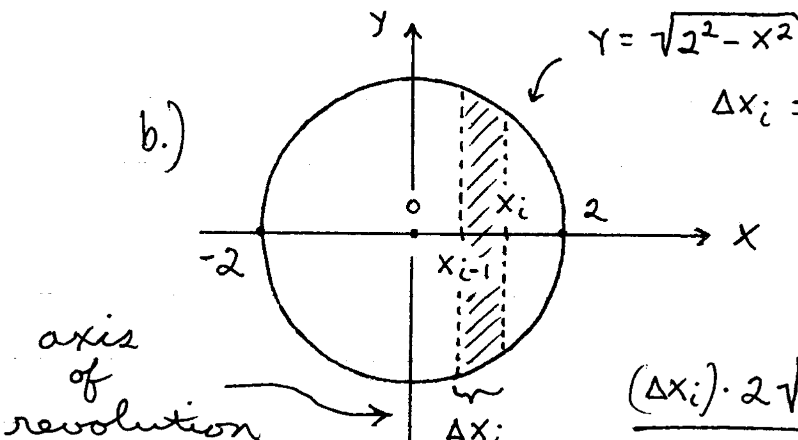
mass of "strip" is

$$200 \cdot \frac{\pi(x_i)^2 - \pi(x_{i-1})^2}{\pi(2)^2} = 50(x_i + x_{i-1})(x_i - x_{i-1})$$
$$= 50(x_i + x_{i-1}) \cdot \Delta x_i$$

and velocity of "strip" is approximately
$3 \cdot \underbrace{(2\pi x_i)}_{\text{circumference}} \cdot = 6\pi x_i$ ft./sec. so

$$K.E. = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} m v^2$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} \{50(x_i + x_{i-1}) \cdot \Delta x_i\} \cdot (6\pi x_i)^2$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} 900\pi^2 (x_i + x_{i-1}) \cdot (x_i)^2 \cdot \Delta x_i$$
$$= 900\pi^2 \int_0^2 (x+x)x^2\,dx$$
$$= 1800\pi^2 \int_0^2 x^3\,dx = 7200\pi^2 \quad \frac{\text{gm. ft.}^2}{\text{sec.}^2}$$

b.)



$\Delta x_i = x_i - x_{i-1} = \frac{2-(-2)}{n}$

mass of "strip" is approximately

$$\frac{(\Delta x_i) \cdot 2\sqrt{2^2 - (x_i)^2}}{\pi(2)^2} \cdot 200$$

$= \frac{100}{\pi} \sqrt{4 - (x_i)^2} \cdot \Delta x_i$ and velocity of

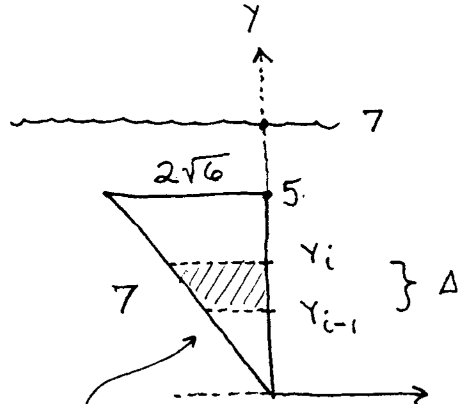
"strip" is approximately
$3(2\pi x_i) = 6\pi x_i$ ft./sec. so

$$K.E. = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} m v^2$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{100}{\pi} \sqrt{4-(x_i)^2} \cdot \Delta x_i \right\} (6\pi x_i)^2$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} 1800\pi x_i^2 \sqrt{4-x_i^2} \cdot \Delta x_i$$
$$= 1800\pi \int_{-2}^{2} x^2 \sqrt{4-x^2}\, dx$$

(Let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$ and $x: -2 \to 2$ so $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$)

$$= 1800\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin\theta)^2 \sqrt{4-(2\sin\theta)^2} \cdot 2\cos\theta\, d\theta$$
$$= 28{,}800\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$
$$= 28{,}800\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$$
$$= 28{,}800\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} \sin 2\theta\right)^2 d\theta$$
$$= 7200\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta$$
$$= 7200\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 4\theta)\, d\theta$$
$$= 3600\pi \left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= 3600\pi^2 \quad \frac{\text{gm. ft.}^2}{\text{sec.}^2}$$

7.)



$\} \; \Delta Y_i = \frac{5-0}{n}$

$Y = \frac{-5}{2\sqrt{6}} X$ or $X = \frac{-2\sqrt{6}}{5} Y$

Water weighs 62.4 lbs./ft.^{3} so force on "strip" is approximately

$$\underbrace{(\Delta Y_i) \cdot \left| \frac{-2\sqrt{6}}{5} Y_i \right|}_{\text{volume}} (7 - Y_i) \cdot (62.4)$$

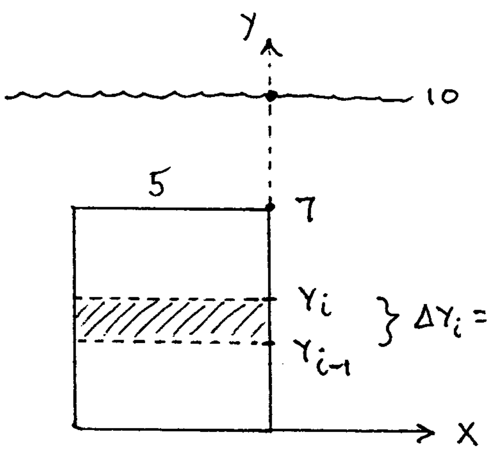
$$\approx (61.14)(7Y_i - Y_i^2) \cdot (\Delta Y_i) \quad \text{so}$$

$$\text{Force} = \lim_{n \to \infty} \sum_{i=1}^{n} (61.14)(7Y_i - Y_i^2)(\Delta Y_i)$$

$$= 61.14 \int_0^5 (7Y - Y^2)\, dy$$

$$= 61.14 \left(\tfrac{7}{2} Y^2 - \tfrac{1}{3} Y^3 \right) \Big|_0^5 = 2802.25 \text{ lbs.}$$

8.)



$\} \; \Delta Y_i = \frac{7-0}{n}$

a.) Force on "strip" is approximately

$$\underbrace{(\Delta Y_i)(5)}_{\text{volume}} (10 - Y_i) \cdot (62.4)$$

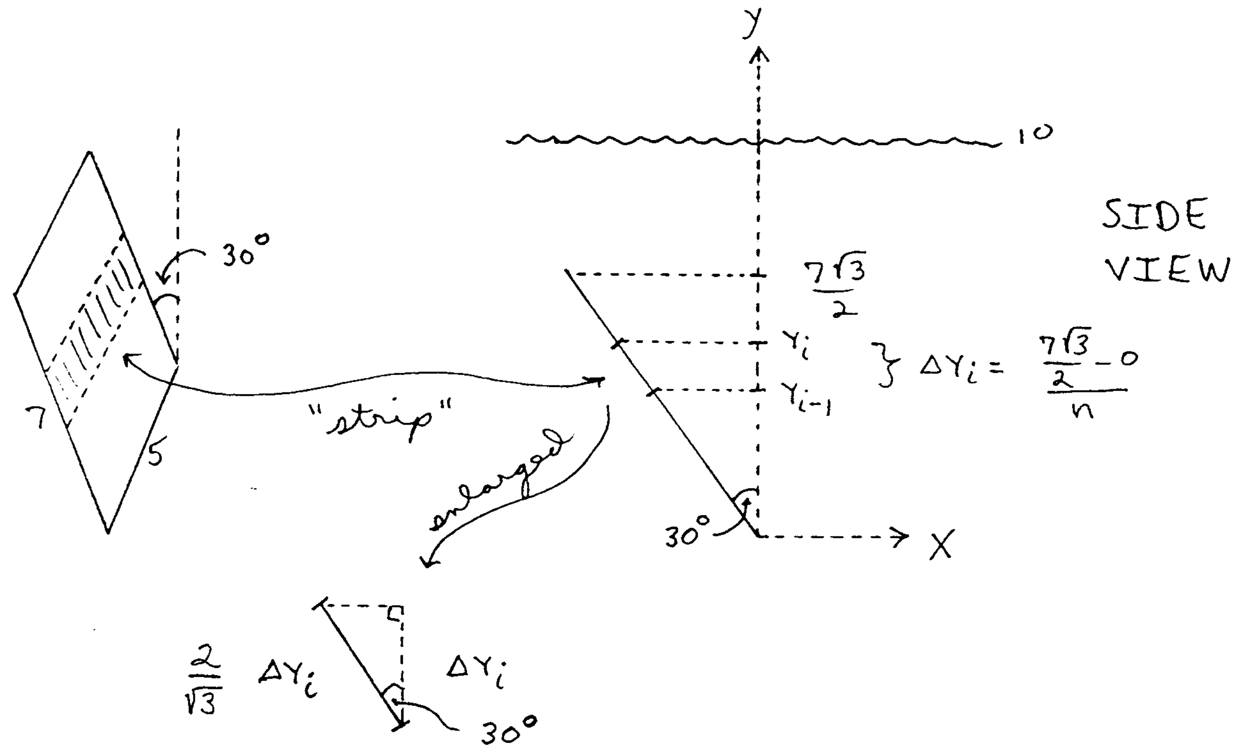
$$= 312 \, (10 - Y_i)(\Delta Y_i) \quad \text{so}$$

$$\text{Force} = \lim_{n \to \infty} \sum_{i=1}^{n} 312 \, (10 - Y_i)(\Delta Y_i)$$

$$= 312 \int_0^7 (10 - Y)\, dy$$

$$= 312 \left(10Y - \frac{Y^2}{2} \right) \Big|_0^7 = 14{,}196 \text{ lbs.}$$

b.)



Force on "strip" is approximately

$$\underbrace{\left(\frac{2}{\sqrt{3}} \Delta Y_i\right)(5)(10 - Y_i)}_{\text{volume}} \cdot (62.4) \approx (360.3)(10 - Y_i)(\Delta Y_i)$$

so Force $= \lim_{n \to \infty} \sum_{i=1}^{n} (360.3)(10 - Y_i)(\Delta Y_i)$

$$= 360.3 \int_0^{\frac{7\sqrt{3}}{2}} (10 - Y)\, dy$$

$$= 360.3 \left(10Y - \frac{Y^2}{2}\right) \Big|_0^{\frac{7\sqrt{3}}{2}}$$

$\approx$ 15,221.5 lbs.