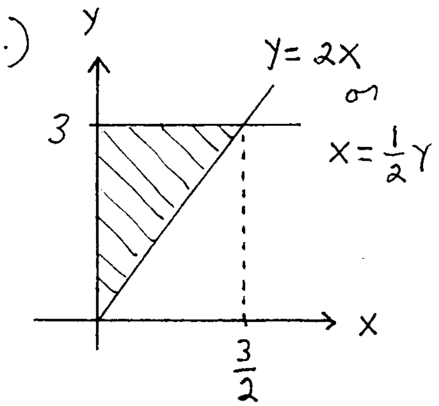
ESP
Kouba
Worksheet 14 Solutions

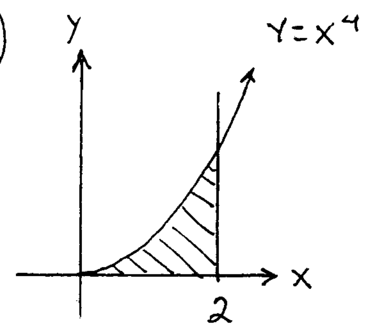
1.) a.)



$$\bar{x} = \frac{\int_0^{3/2} x(3-2x)\,dx}{\int_0^{3/2} (3-2x)\,dx}$$ and

$$\bar{y} = \frac{\int_0^3 y\left(\frac{1}{2}y\right)dy}{\int_0^3 \frac{1}{2}y\,dy}.$$

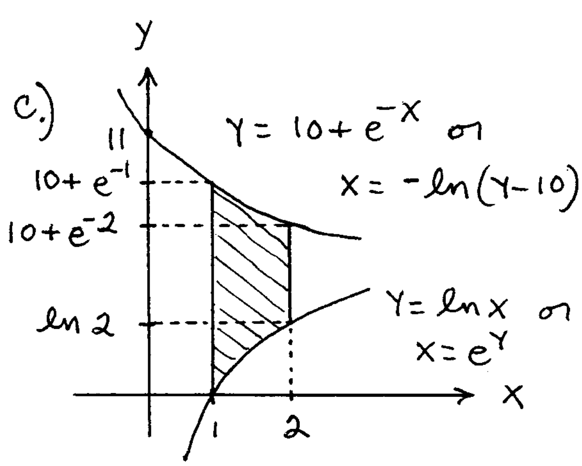
b.)



$$\bar{x} = \frac{\int_0^2 x(x^4)\,dx}{\int_0^2 x^4\,dx}$$ and

$$\bar{y} = \frac{\int_0^2 \frac{1}{2}(x^4)^2\,dx}{\int_0^2 x^4\,dx}.$$

c.)

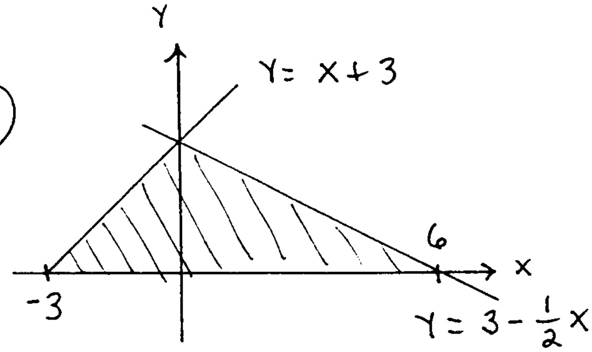


$$\bar{x} = \frac{\int_1^2 x\left(10 + e^{-x} - \ln x\right)dx}{\int_1^2 \left(10 + e^{-x} - \ln x\right)dx}$$

and

$$\bar{y} = \frac{\int_0^{\ln 2} y(e^y - 1)\,dy + \int_{\ln 2}^{10+e^{-2}} y(2-1)\,dy + \int_{10+e^{-2}}^{10+e^{-1}} y\left(-\ln(y-10) - 1\right)dy}{\int_1^2 \left(10 + e^{-x} - \ln x\right)dx}$$

d.)

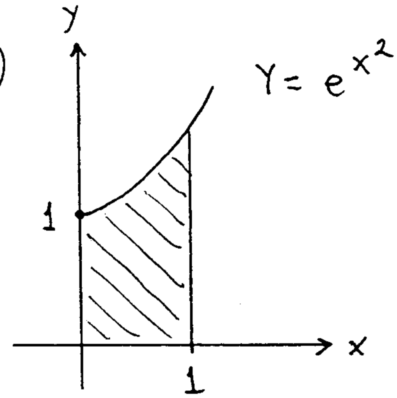


$$\bar{X} = \frac{\int_{-3}^{0} x(x+3)\,dx + \int_{0}^{6} x\left(3-\tfrac{1}{2}x\right) dx}{\int_{-3}^{0} (x+3)\,dx + \int_{0}^{6} \left(3-\tfrac{1}{2}x\right) dx}$$

and

$$\bar{Y} = \frac{\int_{-3}^{0} \frac{1}{2}(x+3)^2\,dx + \int_{0}^{6} \frac{1}{2}\left(3-\frac{1}{2}x\right)^2 dx}{\int_{-3}^{0} (x+3)\,dx + \int_{0}^{6} \left(3-\frac{1}{2}x\right) dx} .$$

e.)

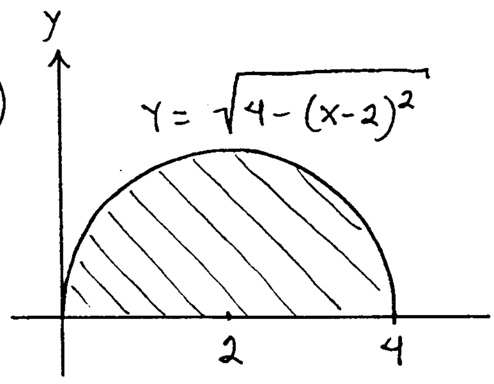


$$\bar{X} = \frac{\int_{0}^{1} x \cdot e^{x^2}\,dx}{\int_{0}^{1} e^{x^2}\,dx}$$

and

$$\bar{Y} = \frac{\int_{0}^{1} \frac{1}{2}\left(e^{x^2}\right)^2 dx}{\int_{0}^{1} e^{x^2}\,dx} .$$

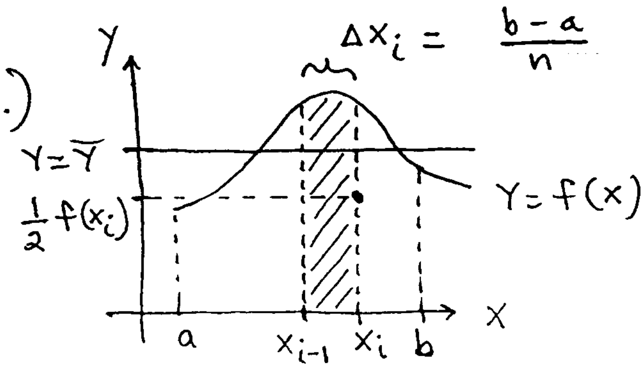
f.)



$\bar{X} = 2$ and

$$\bar{Y} = \frac{\int_{0}^{4} \frac{1}{2}\left(\sqrt{4-(x-2)^2}\right)^2 dx}{\frac{1}{2}\pi(2)^2}$$

2.)



assume that the mass of the "strip" is centered at the "midpoint" of the

(assume density $\sigma = 1$)

"strip", i.e., at the point $(x_i, \frac{1}{2} f(x_i))$. Thus, an estimate for the moment of the "strip" about the line $Y = \bar{Y}$ is

(mass) (signed distance)

$$= ((\Delta x_i) f(x_i) \cdot \sigma) \left(\frac{1}{2} f(x_i) - \bar{Y}\right)$$

$$= \left(\frac{1}{2} (f(x_i))^2 - \bar{Y} f(x_i)\right) \cdot \Delta x_i$$

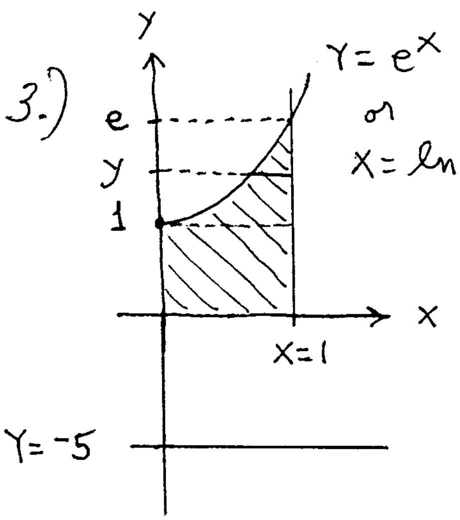
so that the total moment about $Y = \bar{Y}$ is

$$M_{\bar{Y}} = \int_a^b \left(\frac{1}{2} (f(x))^2 - \bar{Y} f(x)\right) dx$$

. at the centroid $M_{\bar{Y}} = 0$ so ~~that~~

$$\int_a^b \left(\frac{1}{2} (f(x))^2 - \bar{Y} f(x)\right) dx = 0, \text{ i.e., } \bar{Y} = \frac{\int_a^b \frac{1}{2} (f(x))^2 \, dx}{\int_a^b f(x) \, dx} .$$

3.)



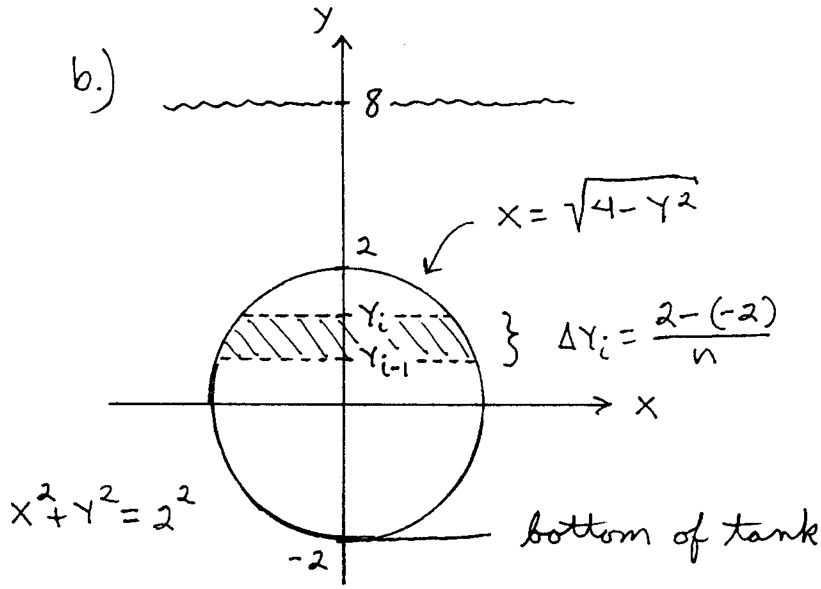
a.) $$Vol = \pi \int_0^1 (e^x + 5)^2 \, dx - \pi \int_0^1 (5)^2 \, dx$$

b.) $$Vol = 2\pi \int_1^e (Y+5)(1 - \ln Y) \, dy + 2\pi \int_0^1 (Y+5)(1) \, dy$$

4.) a.) $\text{Force} = \underbrace{\pi(2)^2}_{\text{area}} \cdot \underbrace{(10)}_{\text{depth}} \cdot \underbrace{(62.4)}_{\text{specific wt. of } H_2O}$

$= 2496\pi$ lbs.

b.)



An estimate of the force on the "strip" is

$$\underbrace{(\Delta y_i) \cdot 2\sqrt{4-y_i^2}}_{\text{area}} \cdot \underbrace{(8-y_i)}_{\text{depth}} \cdot \underbrace{(62.4)}_{\text{specific wt. of } H_2O}$$

so total force is

$$\text{Force} = \int_{-2}^{2} (124.8)(8-y)\sqrt{4-y^2}\, dy$$

$$= 998.4 \int_{-2}^{2} \sqrt{4-y^2}\, dy - 124.8 \int_{-2}^{2} y(4-y^2)^{1/2}\, dy$$

$$= (998.4)\left(\tfrac{1}{2}\pi(2)^2\right) - (124.8)\left(\tfrac{-1}{3}\right)(4-y^2)^{3/2}\Big|_{-2}^{2}$$

$$= 1996.8\pi \text{ lbs.}$$

5.) a.) $\int \sqrt{1+\sqrt{x}}\, dx$ (let $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow$
$2u\, du = dx$)

$= \int \sqrt{1+u} \cdot 2u\, du$ (let $v = 1+u \Rightarrow u = v-1$ and
$dv = du$)

$= \int \sqrt{v} \cdot 2(v-1)\, dv = 2\int (v^{3/2} - v^{1/2})\, dv$

$= 2\left(\frac{2}{5} v^{5/2} - \frac{2}{3} v^{3/2}\right) + c = \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + c$

b.) $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx$ (let $u = 1+\sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\, dx$,
$2\, du = \frac{1}{\sqrt{x}}\, dx$)

$= \int 2\sqrt{u}\, du = 2 \cdot \frac{2}{3} u^{3/2} + c = \frac{4}{3}(1+\sqrt{x})^{3/2} + c$

c.) $\int \frac{1}{\sqrt{2-x^2}}\, dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\, dx$ (let $u = \frac{x}{\sqrt{2}}$,
$du = \frac{1}{\sqrt{2}}\, dx$, $\sqrt{2}\, du = dx$)

$= \frac{1}{\sqrt{2}} \int \sqrt{2} \cdot \frac{1}{\sqrt{1-u^2}}\, du = \arcsin u + c = \arcsin\left(\frac{x}{\sqrt{2}}\right) + c$.

d.) $\int \frac{1}{\sqrt{2-x}\,\sqrt{x-1}}\, dx$ (let $u = \sqrt{2-x} \Rightarrow u^2 = 2-x \Rightarrow$
$x = 2-u^2$, $dx = -2u\, du$)

$= \int \frac{-2\cancel{u}\, du}{\cancel{u}\sqrt{1-u^2}} = -2\arcsin u + c = -2\arcsin\sqrt{2-x} + c$

e.) $\displaystyle\int \frac{1}{\sqrt{x}\,\sqrt{1-(\sqrt{x})^2}}\,dx$ (let $u=\sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $2\,du = \frac{1}{\sqrt{x}}\,dx$)

$$= \int \frac{2}{\sqrt{1-u^2}}\,du = 2\arcsin u + C = 2\arcsin\sqrt{x} + C$$

f.) $\displaystyle\int \sin(\ln x)\,dx$ (let $u=\ln x$, $x=e^u$, $du = \frac{1}{x}\,dx \Rightarrow e^u\,du = dx$)

$$= \int \underbrace{\sin u \cdot e^u}\,du \quad (w=e^u,\ dv=\sin u\,du,\ dw=e^u\,du,\ v=-\cos u)$$

$$= -e^u\cos u + \int \cos u \cdot e^u\,du \quad (w=e^u,\ dv=\cos u\,du,\ dw=e^u\,du,\ v=\sin u)$$

$$= -e^u\cos u + e^u\sin u - \int \underbrace{\sin u \cdot e^u}\,du$$ so that

$$2\int \sin u\cdot e^u\,du = e^u\sin u - e^u\cos u + C,$$ i.e.,

$$\int \sin(\ln x)\,dx = \int \sin u\cdot e^u\,du = \tfrac{1}{2}e^u\sin u - \tfrac{1}{2}e^u\cos u + C.$$

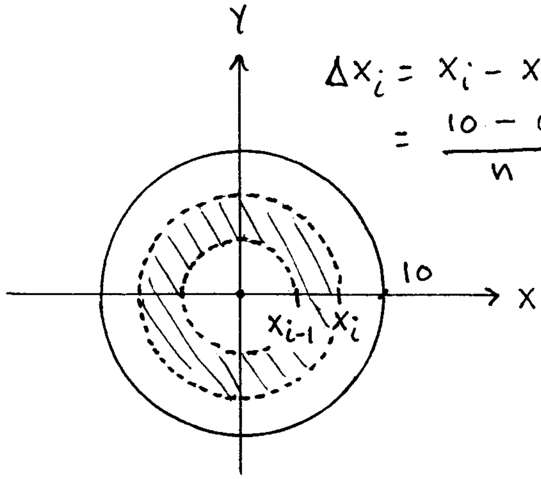
g.) $\displaystyle\int \tan^2 x\,\sec^2 x\,dx = \frac{1}{3}\tan^3 x + C$

h.) $\displaystyle\int \frac{x^2}{(2x^3+1)(x^3+1)}\,dx$ (let $u=x^3$, $du = 3x^2\,dx$, $\frac{1}{3}\,du = x^2\,dx$)

$$= \frac{1}{3}\int \frac{1}{(2u+1)(u+1)}\,du = \frac{1}{3}\int\left[\frac{2}{2u+1} + \frac{-1}{u+1}\right]du$$

$$= \frac{1}{3}\Big[\ln|2u+1| - \ln|u+1|\Big] + C = \frac{1}{3}\Big[\ln|2x^3+1| - \ln|x^3+1|\Big] + C$$

6.)



$$\Delta x_i = x_i - x_{i-1} = \frac{10 - 0}{n}$$

The mass of the "strip" is approximately

$$\underbrace{(2\pi\, x_i)\cdot \Delta x_i}_{\text{area}} \cdot \underbrace{e^{-x_i}\sqrt{x_i^2+1}}_{\text{density}} \quad gm.$$

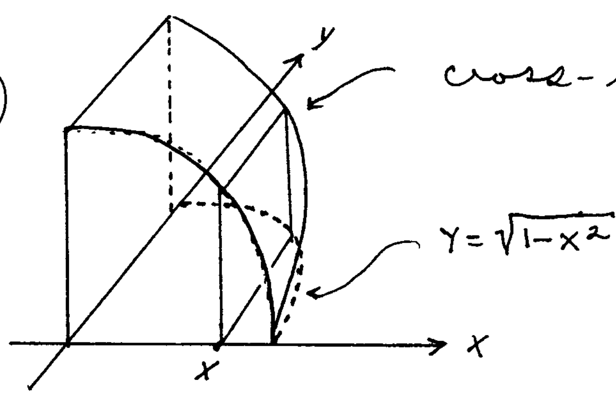
and velocity is approximately $(2\pi x_i)(5) = 10\pi x_i$ ft./min.
so that kinetic energy of "strip" is approximately

$$\frac{1}{2} m v^2 = \frac{1}{2}(2\pi x_i)\cdot(\Delta x_i)\cdot e^{-x_i}\sqrt{x_i^2+1}\cdot(10\pi x_i)^2 \text{ ; thus,}$$

$$K.E. = \int_0^{10} \frac{1}{2}(2\pi x)\, e^{x}\sqrt{x^2+1}\,(10\pi x)^2\, dx$$

$$= 100\pi^3 \int_0^{10} x^3 e^{x}\sqrt{x^2+1}\, dx .$$

7.)



cross-sectional slice is a square so area is

$$A(x) = \left(\sqrt{1-x^2}\right)^2 = 1 - x^2 .$$

Thus, volume is

$$Vol = \int_0^1 A(x)\, dx = \int_0^1 (1-x^2)\, dx = \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 2/3$$