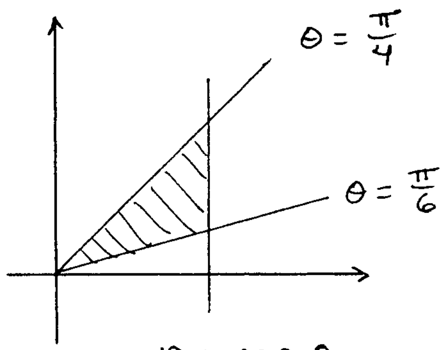
ESP

Kouba

Worksheet 16 Solutions

1.)



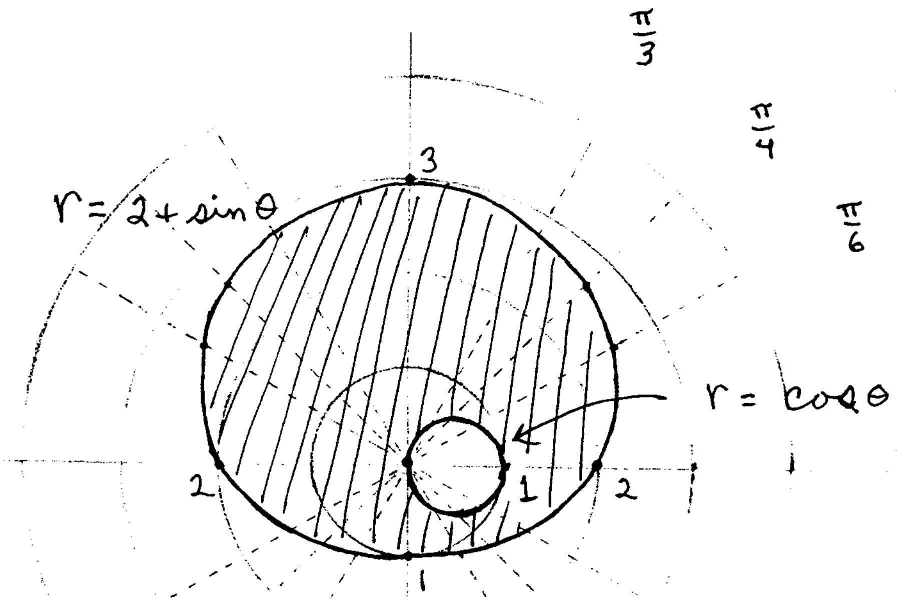
$r = \sec\theta$

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sec\theta)^2 \, d\theta$$

$$= \frac{1}{2}\tan\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}-1}{2\sqrt{3}}$$

2.)



$$\text{Area} = \int_0^{2\pi} \frac{1}{2}(2+\sin\theta)^2 \, d\theta - \int_0^{\pi} \frac{1}{2}(\cos\theta)^2 \, d\theta$$

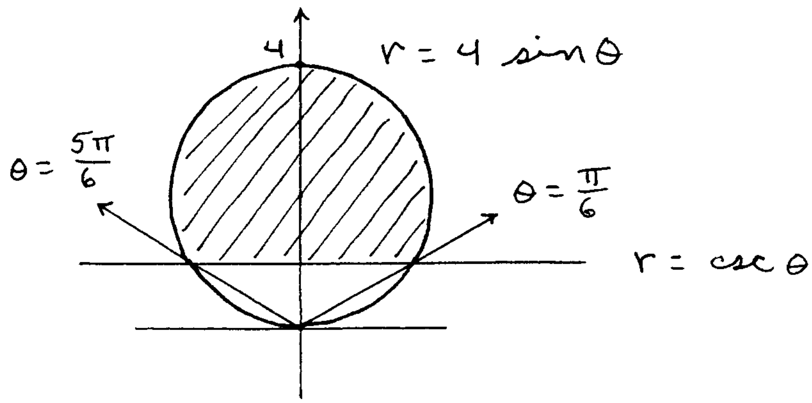
$$= \int_0^{2\pi} \frac{1}{2}(4 + 4\sin\theta + \sin^2\theta) \, d\theta - \int_0^{\pi} \frac{1}{4}(1+\cos 2\theta) \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(4 + 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right) d\theta - \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{9}{2}\theta - 4\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} - \frac{\pi}{4} = \frac{9\pi}{2} - \frac{\pi}{4} = \frac{17\pi}{4}$$

3.)

$$\left.\begin{aligned} r &= 4\sin\theta \\ r &= \csc\theta = \frac{1}{\sin\theta} \end{aligned}\right\} \quad 4\sin\theta = \frac{1}{\sin\theta} \Rightarrow$$



$\sin^2\theta = \frac{1}{4} \Rightarrow$

$\sin\theta = \frac{1}{2} \Rightarrow$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ so

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(4\sin\theta)^2\,d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(\csc\theta)^2\,d\theta$$

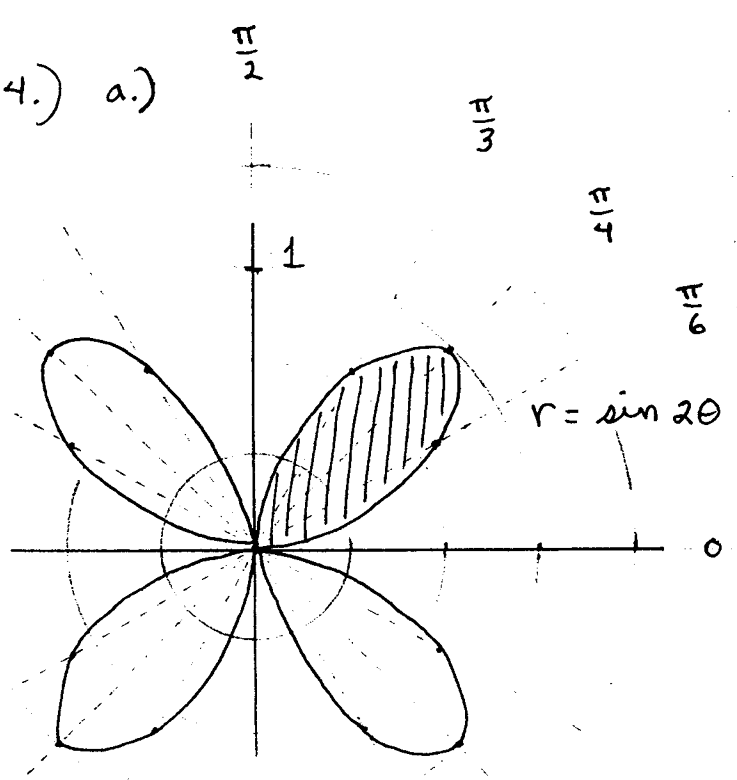
$$= 8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1-\cos 2\theta)\,d\theta + \frac{1}{2}\cot\theta\,\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= 4 \cdot \left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{1}{2}\cot\frac{5\pi}{6} - \frac{1}{2}\cot\frac{\pi}{6}$$

$$= 4\left(\frac{5\pi}{6} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) - 4\left(\frac{\pi}{6} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right) + \frac{1}{2}(-\sqrt{3}) - \frac{1}{2}(\sqrt{3})$$

$$= \frac{8\pi}{3} + \sqrt{3}$$

4.) a.)



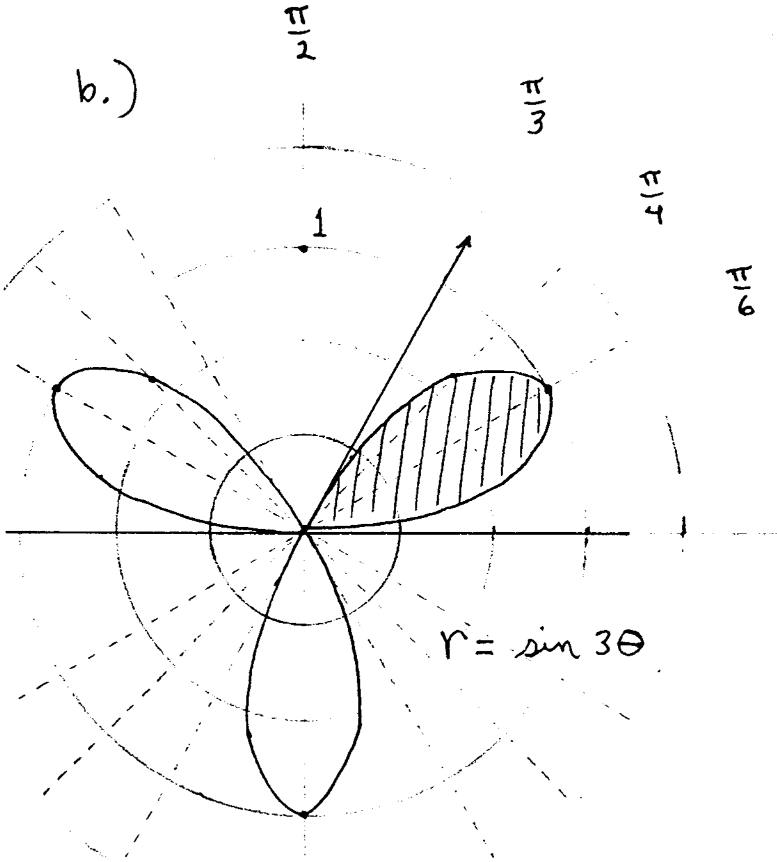
$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^2\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}$$

b.)



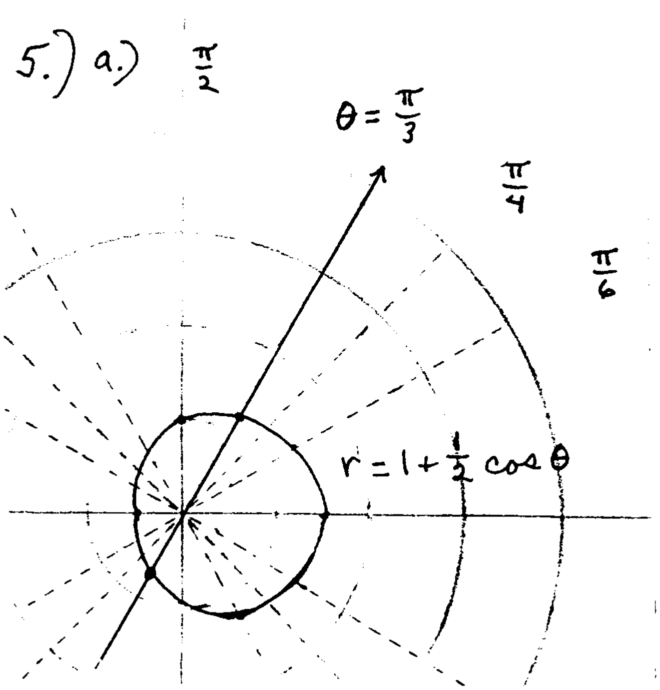
$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{3}} (\sin 3\theta)^2\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} (1 - \cos 6\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{12}.$$

5.) a.)



If $\theta = \frac{\pi}{3}$, then

$r = 1 + \frac{1}{2}\cos\left(\frac{\pi}{3}\right) = 5/4$;

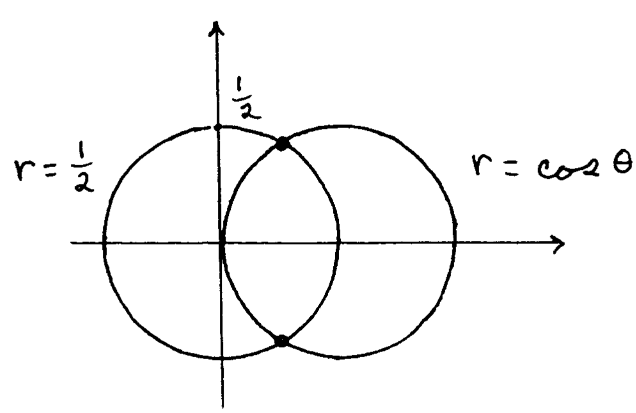
if $\theta = \frac{4\pi}{3}$, then

$r = 1 + \frac{1}{2}\cos\left(\frac{4\pi}{3}\right) = \frac{3}{4}$ so

pts. of $\cap$ are

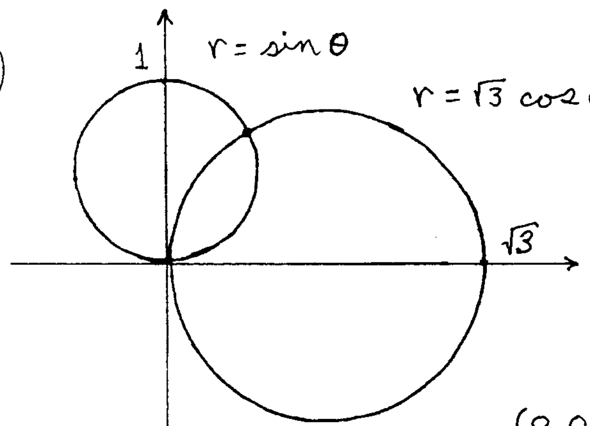
$\left(\frac{5}{4}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{4}, \frac{4\pi}{3}\right)$.

b.) $\left.\begin{array}{l} r = \frac{1}{2} \\ r = \cos\theta \end{array}\right\}$ $\cos\theta = \frac{1}{2}$



$\Rightarrow \quad \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$; so pts. of $\cap$ are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$.

c.)

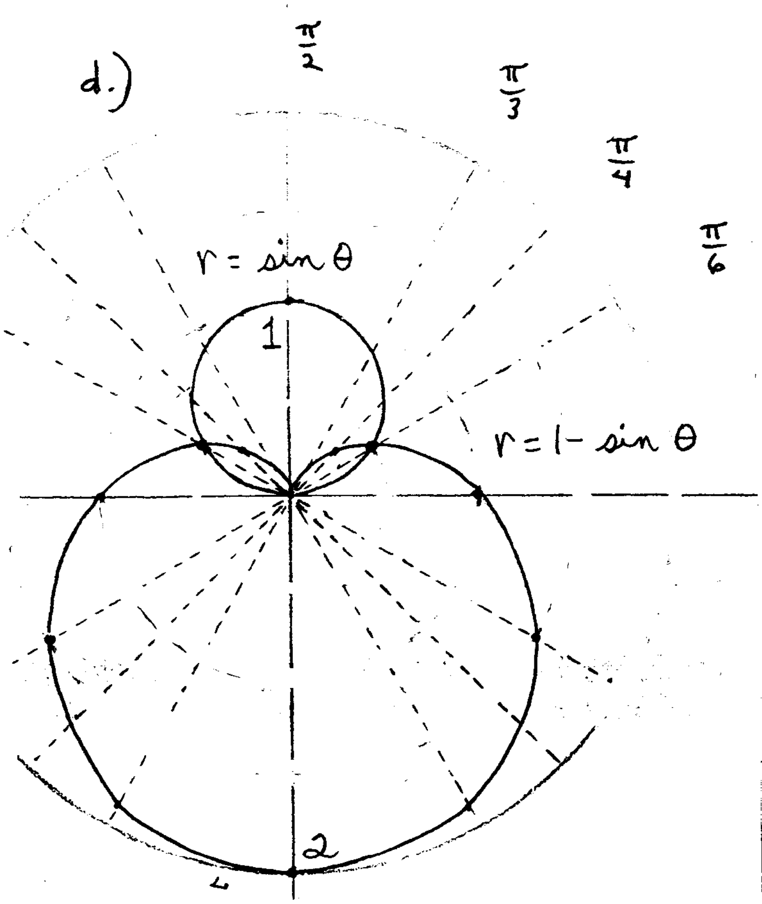


$$\left.\begin{array}{l} r = \sin\theta \\ r = \sqrt{3}\cos\theta \end{array}\right\} \sin\theta = \sqrt{3}\cos\theta \Rightarrow$$

$$\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

so pts. of $\cap$ are $(0,0)$ and $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.

d.)



$$\left.\begin{array}{l} r = 1 - \sin\theta \\ r = \sin\theta \end{array}\right\} 1 - \sin\theta = \sin\theta \Rightarrow$$

$$\sin\theta = \frac{1}{2} \Rightarrow$$

$$\theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6} ;$$

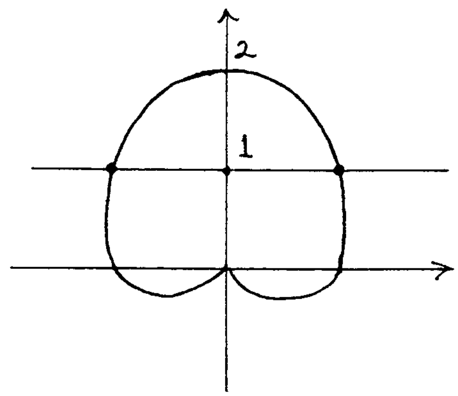
so pts. of $\cap$ are

$(0,0)$,

$\left(\frac{1}{2}, \frac{\pi}{6}\right)$, and

$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.

e.)



$$\left.\begin{array}{r} r = 1 + \sin\theta \\ r = \csc\theta \end{array}\right\} \quad 1 + \sin\theta = \csc\theta \Rightarrow$$

$$1 + \sin\theta = \frac{1}{\sin\theta} \Rightarrow$$

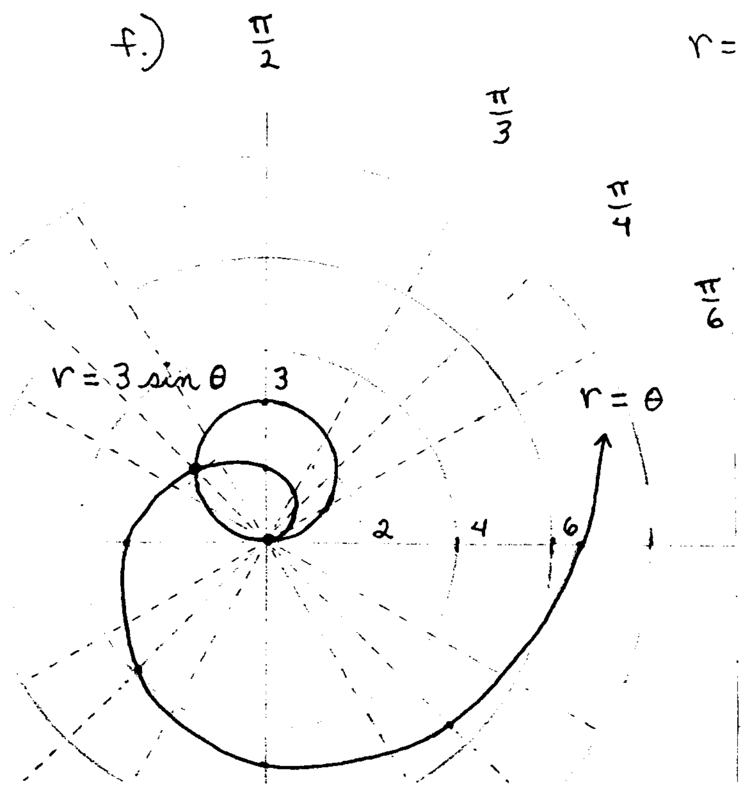
$$\sin^2\theta + \sin\theta - 1 = 0 \Rightarrow$$

$$\sin\theta = \frac{-1 \pm \sqrt{5}}{2} = \frac{-1+\sqrt{5}}{2} \quad \text{so} \quad \theta = \arcsin\left(\frac{-1+\sqrt{5}}{2}\right)$$

$$\approx .618 \text{ radians} \; ;$$

if $\theta = .618$, then $r \approx 1.726$ and
if $\theta = \pi - .618 \approx 2.524$, then $r \approx 1.726$ so
pts. of $\cap$ are $(1.726, .618)$ and $(1.726, 2.524)$.

f.)



$$\left.\begin{array}{r} r = 3\sin\theta \\ r = \theta \end{array}\right\} \quad 3\sin\theta = \theta \Rightarrow$$

$$f(\theta) = 3\sin\theta - \theta = 0$$

so use Newton's method :

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)}$$

$$= \theta_n - \frac{3\sin\theta_n - \theta_n}{3\cos\theta_n - 1}$$

$$= \frac{3 \cdot \theta_n \cdot \cos\theta_n - 3\sin\theta_n}{3\cos\theta_n - 1} \; ;$$

let $\theta_0 = 2.3$ radians then ...

n	θ_n
1	2.279030354
2	2.278862671
3	2.278862660

so $\theta \approx 2.279$;

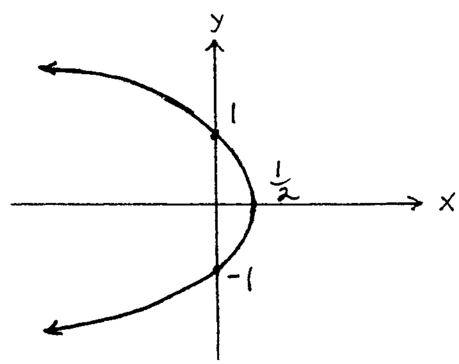
pts. of $\cap$ are $(0,0)$ and $(2.279, 2.279)$

6.) $r = \dfrac{1}{\cos\theta + 1} \Rightarrow r\cos\theta + r = 1 \Rightarrow x + r = 1 \Rightarrow$

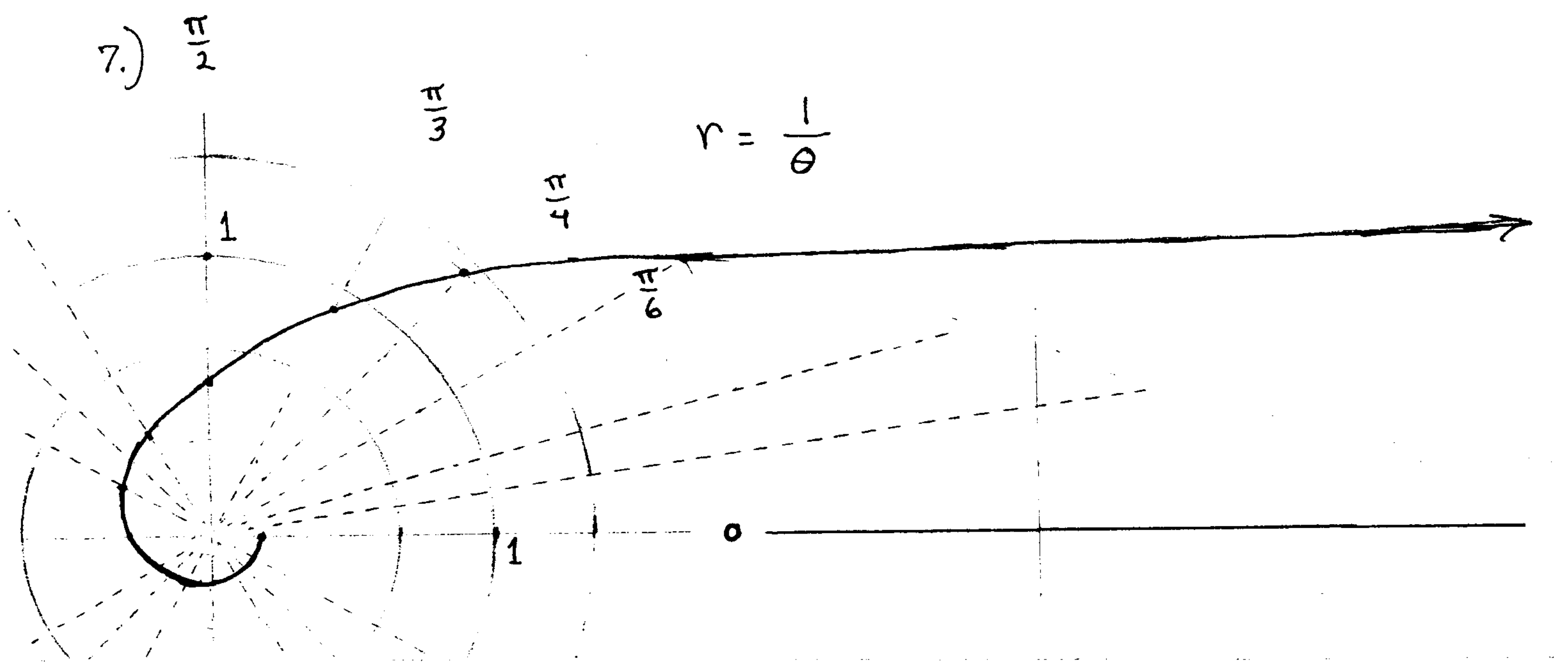
$r = 1 - x \Rightarrow r^2 = (1-x)^2 \Rightarrow \cancel{x^2} + y^2 = 1 - 2x + \cancel{x^2} \Rightarrow$

$2x = -y^2 + 1$

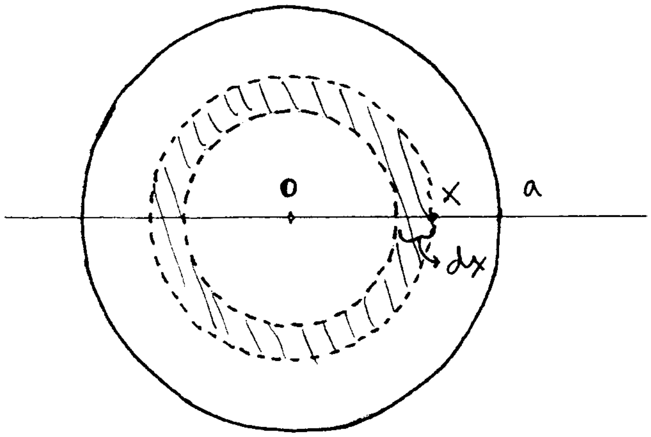
(a parabola)



7.)



8.) a.)



$\text{density} = \dfrac{M}{\pi a^2}$,

$\text{mass} = (\text{area})(\text{density})$

$= (2\pi x \, dx)\left(\dfrac{M}{\pi a^2}\right)$

$= \dfrac{2M}{a^2} x \, dx$,

$\text{velocity} = (2\pi x)(f) = (2\pi f)x$ so

$$K.E. = \int_0^a \frac{1}{2}\left(\frac{2M}{a^2}x\right)(2\pi f \cdot x)^2 dx$$

$$= \frac{4M\pi^2 f^2}{a^2}\int_0^a x^3 dx = \frac{4M\pi^2 f^2}{a^2}\cdot\frac{a^4}{4} = M\pi^2 f^2 a^2$$

b.) $K.E. = M\pi^2 f^2 a^2$

9.) a.) $\displaystyle\int_3^\infty \frac{1}{\sqrt{x-3}}\,dx = \int_3^4 \frac{1}{\sqrt{x-3}}\,dx + \int_4^\infty \frac{1}{\sqrt{x-3}}\,dx$

$$= \lim_{A\to 3^+} 2\sqrt{x-3}\,\Big|_A^4 + \lim_{B\to\infty} 2\sqrt{x-3}\,\Big|_4^B$$

$$= \lim_{A\to 3^+}\left(2-2\sqrt{A-3}\right) + \lim_{B\to\infty}\left(2\sqrt{B-3}-2\right) = 2+\infty = \infty .$$

so $\displaystyle\int_3^\infty \frac{1}{\sqrt{x-3}}\,dx$ diverges.

b.) $\displaystyle\int_0^2 \frac{2x}{x^2-1}\,dx = \int_0^1 \frac{2x}{x^2-1}\,dx + \int_1^2 \frac{2x}{x^2-1}\,dx = C+D$;

$$C = \lim_{A\to 1^-} \ln|x^2-1|\,\Big|_0^A = \lim_{A\to 1^-}\left(\ln|A^2-1| - \ln 1\right) = \infty$$

so $\int_0^2 \frac{2x}{x^2-1}\,dx$ diverges.

c.) $\int_0^{\pi} \tan x\,dx = \int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx$

$= C + D$; $C = \lim_{A \to \frac{\pi}{2}^-} \ln|\sec x| \Big|_0^A$

$= \lim_{A \to \frac{\pi}{2}^-} \left(\ln|\sec A| - \ln|\underbrace{\sec 0}_{1}|\right) = \infty$ so

$\int_0^{\pi} \tan x\,dx$ diverges.

d.) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{A \to 1^-} \arcsin x \Big|_0^A$

$= \lim_{A \to 1^-} (\arcsin A - \arcsin 0) = \frac{\pi}{2}$

e.) $\int_0^{\infty} \frac{x^2}{(x^3+1)^3}\,dx = \lim_{A \to \infty} \frac{-\frac{1}{6}}{(x^3+1)^2} \Big|_0^A$

$= \lim_{A \to \infty} \left(\frac{-\frac{1}{6}}{(A^3+1)^2} - \frac{-\frac{1}{6}}{1^2}\right) = \frac{1}{6}$.