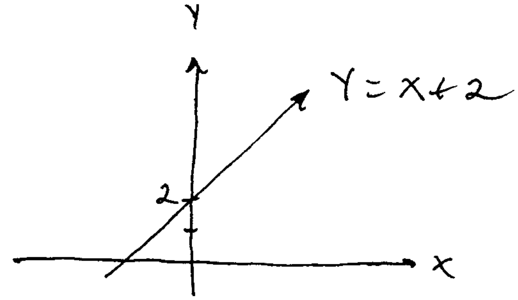
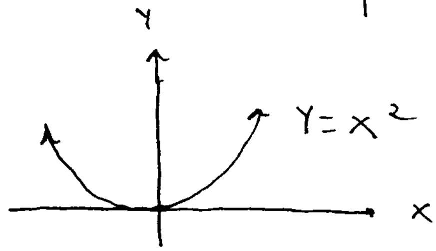
ESP
Kouba
Worksheet 17 Solutions

1.) a.) $\left.\begin{array}{l} X = t-1 \\ Y = t+1 \end{array}\right\}$ $t = X+1$, $Y = (X+1)+1$

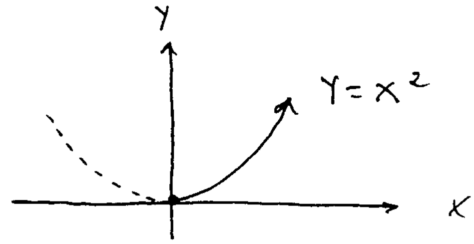
or $Y = X+2$



b.) $\left.\begin{array}{l} X = t \\ Y = t^2 \end{array}\right\}$ $Y = X^2$

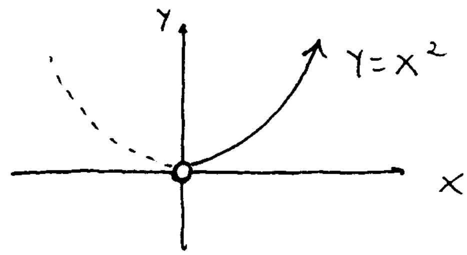


c.) $\left.\begin{array}{l} X = t^2 \geq 0 \\ Y = t^4 \end{array}\right\}$ $Y = X^2$



(right half only)

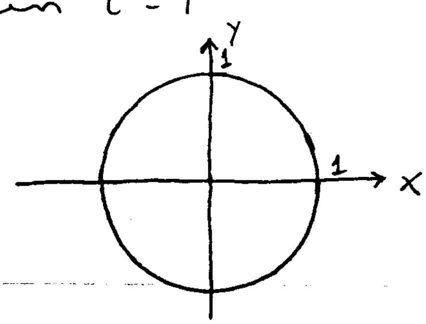
d.) $\left.\begin{array}{l} X = e^t > 0 \\ Y = (e^t)^2 \end{array}\right\}$ $Y = X^2$



(right half, not origin)

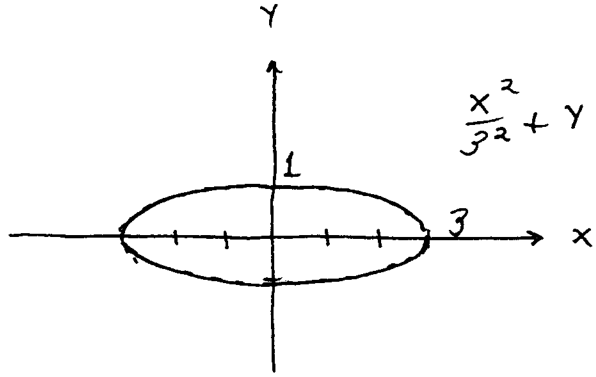
e.) $\left.\begin{array}{l} X = \cos t \\ Y = \sin t \end{array}\right\}$ $X^2 + Y^2 = \cos^2 t + \sin^2 t = 1$

or $X^2 + Y^2 = 1$



f.) $\left.\begin{array}{l} X = 3\cos t \\ Y = \sin t \end{array}\right\}$ $\left.\begin{array}{l} \frac{X}{3} = \cos t \\ Y = \sin t \end{array}\right\}$

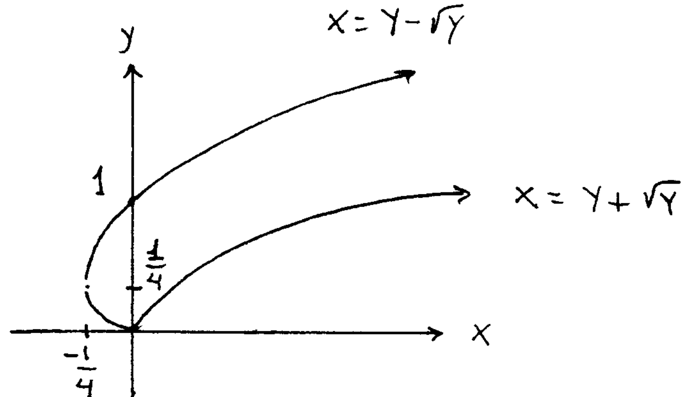
$\left(\frac{X}{3}\right)^2 + Y^2 = \cos^2 t + \sin^2 t = 1$ or $\frac{X^2}{3^2} + Y^2 = 1$



$$\frac{x^2}{3^2} + y^2 = 1$$

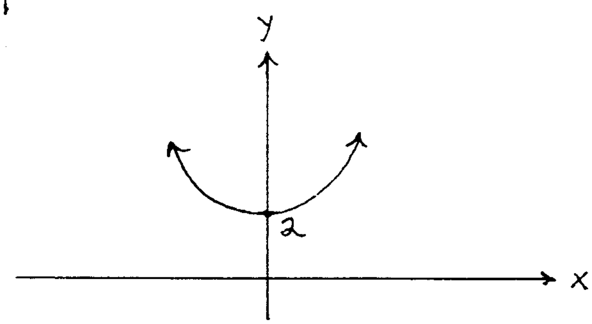
g.) $\left.\begin{array}{l} x = t^2 - t \\ y = t^2 \end{array}\right\}$ $\left.\begin{array}{l} t = \sqrt{y} \text{ for } t \geq 0 \\ t = -\sqrt{y} \text{ for } t < 0 \end{array}\right\}$ so

$x = y - \sqrt{y}$ for $t \geq 0$
$x = y + \sqrt{y}$ for $t < 0$



h.) $\left.\begin{array}{l} x = \ln t \\ y = t + \frac{1}{t} \end{array}\right\}$ $\begin{array}{l} t = e^x \\ y = e^x + \frac{1}{e^x} \end{array}$

or $y = e^x + e^{-x}$



2.) a.) $y = (\pi - \arctan x)^4 \rightarrow$

$y' = 4(\pi - \arctan x)^3 \cdot \frac{-1}{1+x^2}$ at $x = 1$ slope is

$y' = 4\left(\pi - \frac{\pi}{4}\right)^3 \cdot \frac{-1}{2} = 4\left(\frac{3}{4}\pi\right)^3 \cdot \frac{-1}{2} = -\frac{27}{32}\pi^3$

b.) $\left.\begin{array}{l} x = t^2 + 1 \\ y = e^{-t} + t \end{array}\right\}$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t} + 1}{2t}$ so at $t =$

slope is $y' = \dfrac{-e^{-1}+1}{2} = \dfrac{\frac{1}{e}+1}{2} = \dfrac{1+e}{2e}$.

c.) $\left.\begin{array}{l} x = r\cos\theta = (3+\sin\theta)\cos\theta = 3\cos\theta + \sin\theta\cdot\cos\theta \\ y = r\sin\theta = (3+\sin\theta)\sin\theta = 3\sin\theta + \sin^2\theta \end{array}\right\}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta + 2\sin\theta\cos\theta}{-3\sin\theta + \sin\theta\cdot(-\sin\theta) + \cos^2\theta}$$ so

at $\theta = \frac{\pi}{4}$ slope is

$$y' = \frac{3\left(\frac{\sqrt{2}}{2}\right) + 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{-3\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\frac{3\sqrt{2}}{2} + 1}{-\frac{3\sqrt{2}}{2}} = -1 - \frac{2}{3\sqrt{2}}.$$

3.) a.) $y = \dfrac{x}{x^2+1} \longrightarrow \dfrac{dy}{dx} = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \longrightarrow$

$$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2\cdot(-2x) - (1-x^2)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{(-2x)\cdot[3-x^2]}{(x^2+1)^3}$$

b.) $\left.\begin{array}{l} x = t + \sin t \\ y = e^{\tan t} - t \end{array}\right\}$ $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\sec^2 t\cdot e^{\tan t} - 1}{1+\cos t} \longrightarrow$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\sec^2 t\cdot e^{\tan t} - 1}{1+\cos t}\right)}{1+\cos t}$$

$$= \frac{(1+\cos t)\left[\sec^4 t\cdot e^{\tan t} + 2\sec^2 t\cdot\tan t\cdot e^{\tan t}\right] - \left(\sec^2 t\cdot e^{\tan t} - 1\right)(-\sin t)}{(1+\cos t)^3}$$

c.) $x = r\cos\theta = \theta\cos\theta$, $y = r\sin\theta = \theta\sin\theta$ $\Big\}$ $\quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$$= \frac{\theta\cos\theta + \sin\theta}{\theta\cdot(-\sin\theta) + \cos\theta} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta} \quad \text{and}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{d(y')}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right)}{\cos\theta - \theta\sin\theta}$$

$$= \frac{(\cos\theta - \theta\sin\theta)[\cos\theta + \theta(-\sin\theta) + \cos\theta] - (\sin\theta + \theta\cos\theta)[-\sin\theta - \theta\cos\theta - \sin\theta]}{(\cos\theta - \theta\sin\theta)^3}$$

$$= \frac{2 + \theta^2}{(\cos\theta - \theta\sin\theta)^3} .$$

d.) $x = r\cos\theta = \sin\theta\cos\theta$, $y = r\sin\theta = \sin^2\theta$ $\Big\}$ $\quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta(-\sin\theta) + \cos^2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

so $$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{d(y')}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(\tan 2\theta)}{\cos 2\theta}$$

$$= \frac{\sec^2 2\theta \cdot 2}{\cos 2\theta} = 2\sec^3 2\theta .$$

4.) $\text{Area} = \int_1^{1+e} Y(x)\, dx = \int_0^1 Y(x(t)) \cdot \frac{dx}{dt} \cdot dt$

$= \int_0^1 (t+e^t)(2t+e^t)\, dt = \int_0^1 [2t^2 + 3te^t + e^{2t}]\, dt$

$= \left[\frac{2}{3}t^3 + 3(te^t - e^t) + \frac{1}{2}e^{2t}\right]\Big|_0^1$

$= \left(\frac{2}{3} + \frac{1}{2}e^2\right) - \left(-3 + \frac{1}{2}\right) = \frac{19}{6} + \frac{1}{2}e^2$

5.) a.) $\text{Arc} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + \left(\frac{5}{4}x^{1/4}\right)^2}\, dx$

$= \int_0^1 \sqrt{1 + \frac{25}{16}\sqrt{x}}\, dx$ (Let $u = 1 + \frac{25}{16}\sqrt{x} \rightarrow \frac{16}{25}(u-1) = \sqrt{x} \rightarrow$

$\frac{256}{625}(u-1)^2 = x \rightarrow \frac{512}{625}(u-1)\, du = dx$ and

$x: 0 \rightarrow 1$ so $u: 1 \rightarrow \frac{41}{16}$)

$= \int_1^{41/16} \sqrt{u} \cdot \frac{512}{625}(u-1)\, du = \frac{512}{625} \int_1^{41/16} \left(u^{3/2} - u^{1/2}\right) du$

$= \frac{512}{625}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^{41/16} = \cdots \approx 1.423$

b.) $\text{Arc} = \int_2^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_2^3 \sqrt{1 + \left(\frac{-1}{x^3} + \frac{4x^3}{16}\right)^2}\, dx$

$= \int_2^3 \sqrt{1 + \left(\frac{x^6 - 4}{4x^3}\right)^2}\, dx = \int_2^3 \sqrt{\frac{x^{12} + 8x^6 + 16}{16x^6}}\, dx = \int_2^3 \sqrt{\frac{(x^6+4)^2}{16x^6}}\, dx$

$= \int_2^3 \frac{x^6 + 4}{4x^3}\, dx = \frac{1}{4}\int_2^3 (x^3 + 4x^{-3})\, dx = \frac{1}{4}\left(\frac{1}{4}x^4 - 2x^{-2}\right)\Big|_2^3$

$= \frac{1}{4}\left(\frac{81}{4} - \frac{2}{9}\right) - \frac{1}{4}\left(4 - \frac{1}{2}\right) = \frac{67}{16} - \frac{1}{18} = \frac{595}{144}$

c.) $\text{Arc} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} t\, dt$

$= \frac{1}{2}t^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{2}\cdot\frac{\pi^2}{16} - \frac{1}{2}\cdot\frac{\pi^2}{36} = \frac{5}{288}\pi^2$

d.) $\text{Arc} = \int_0^{\pi} \sqrt{r^2 + (r'(\theta))^2}\, d\theta$

$= \int_0^{\pi} \sqrt{\sin^4\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= \int_0^{\pi} \sin\frac{\theta}{2}\, d\theta = -2\cdot\cos\frac{\theta}{2}\Big|_0^{\pi} = 2$

6.) a.) $\text{Speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$ or

$S(t) = \sqrt{(1-\sin t)^2 + (1-\cos t)^2} = \sqrt{3 - 2\sin t - 2\cos t}$

b.) $S(0) = 1$ ft./sec.
$S(\pi/2) = 1$ ft./sec.
$S(100) = 1.513$ ft./sec.

7.) a.) $K = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{|6x|}{(1+(3x^2)^2)^{3/2}} = \frac{|6x|}{(1+9x^4)^{3/2}}$

at $x=-1$ $\quad K = \frac{6}{10^{3/2}}$

b.) $$K = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{4x^2e^{x^2} + 2e^{x^2}}{(1+(2xe^{x^2})^2)^{3/2}}$$

$$= \frac{4x^2e^{x^2} + 2e^{x^2}}{(1+4x^2e^{2x^2})^{3/2}} \quad \text{at } x=1$$

$$K = \frac{6e}{(1+4e^2)^{3/2}}$$

c.) $$K = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{((\dot{x})^2 + (\dot{y})^2)^{3/2}}$$

$$= \frac{|(2t-1)(2) - (2t+1)(2)|}{[(2t-1)^2 + (2t+1)^2]^{3/2}} = \frac{4}{(8t^2+1)^{3/2}} \quad \text{at } t=1$$

$$K = \frac{4}{9^{3/2}} = \frac{4}{27}.$$