ESP
Kouba
Worksheet 9 Solutions

1.) a.) $\int_1^3 \frac{x}{\sqrt{x^2+5}}\,dx$ (Let $u = x^2+5$, $du = 2x\,dx \Rightarrow$ $\frac{1}{2}\,du = x\,dx$, $x: 1 \to 3$ so $u: 6 \to 14$)

$$= \int_6^{14} \frac{\frac{1}{2}}{\sqrt{u}}\,du = \sqrt{u}\Big|_6^{14} = \sqrt{14} - \sqrt{6}\,.$$

b.) $\int_1^2 \frac{2x+1}{(x^2+x-1)^2}\,dx = \frac{-1}{x^2+x-1}\Big|_1^2 = \frac{4}{5}$

c.) $\int_1^3 \left(x - \frac{3}{x}\right)^5\left(1+\frac{3}{x^2}\right)dx$ (Let $u = x - \frac{3}{x}$, $du = 1 + \frac{3}{x^2} \Rightarrow$ $x: 1 \to 3$ so $u: -2 \to 2$)

$$= \int_{-2}^{2} u^5\,du = \frac{u^6}{6}\Big|_{-2}^{2} = 0$$

d.) $\int_0^{\frac{\pi}{6}} \sec^2 x\,dx = \tan x\Big|_0^{\pi/6} = \frac{1}{\sqrt{3}}$

e.) $\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = (-\cos 2\pi) - (-\cos 0) = 0$

f.) $\int_0^{2\pi} \sqrt{1-\cos^2 x}\,dx = \int_0^{2\pi} \sqrt{\sin^2 x}\,dx$

$$= \int_0^{2\pi} |\sin x|\,dx = \int_0^{\pi} (\sin x)\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx$$

$$= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi} = (-\cos\pi) - (-\cos 0)$$

$$+ (\cos 2\pi) - (\cos \pi) = 1 + 1 + 1 - (-1) = 4$$

g.) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$= \int \left[\frac{1}{2} - \frac{1}{2}\cos 2x\right] dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

h.) $\int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3 x + c$

i.) $\int \sin^3 x \, dx = \int \sin x \cdot \sin^2 x \, dx = \int \sin x \, (1 - \cos^2 x) \, dx$

$= \int [\sin x - \cos^2 x \sin x] \, dx = -\cos x + \frac{1}{3}\cos^3 x + c$

j.) $\int (\sin^2 x + 2\sin x \cos x + \cos^2 x) \, dx = \int [1 + 2\sin x \cos x] \, dx$

$= x + \sin^2 x + c$

k.) $\int \frac{4}{1+x^2} dx = 4 \arctan x + c$

l.) $\int \frac{1}{4+x^2} dx = \frac{1}{4} \int \frac{1}{1+\left(\frac{x}{2}\right)^2} dx$ (Let $u = \frac{x}{2}$,

$du = \frac{1}{2} dx \Rightarrow 2\,du = dx$)

$= \frac{1}{4} \int \frac{2}{1+u^2} du = \frac{1}{2} \arctan u + c = \frac{1}{2} \arctan\left(\frac{x}{2}\right) + c$

m.) $\int \frac{1}{(x+3)^2} dx = \frac{-1}{x+3} + c$

n.) $\int \frac{1}{(x+4)(x+2)} dx = \int \left[\frac{A}{x+4} + \frac{B}{x+2}\right] dx$

$= \int \left[\frac{-\frac{1}{2}}{x+4} + \frac{\frac{1}{2}}{x+2}\right] dx = \frac{-1}{2} \ln|x+4| + \frac{1}{2} \ln|x+2| + c$

o.) $\int \frac{1}{(x+3)^2+1} dx = \arctan(x+3) + c$

p.) $\int \frac{x+3}{x^2+6x+10} dx = \frac{1}{2} \ln|x^2+6x+10| + c$

q.) $\int \frac{x}{x^2+6x+10}\,dx = \int \frac{x+3-3}{x^2+6x+10}\,dx$

$= \int \left[\frac{x+3}{x^2+6x+10} - \frac{3}{(x+3)^2+1} \right] dx$

$= \frac{1}{2} \ln|x^2+6x+10| - 3 \arctan(x+3) + c$

r.) $\int (1+x)^{1/2}\,dx = \frac{2}{3}(1+x)^{3/2} + c$

s.) $\int \sqrt{1+\sqrt{x}}\,dx$ (Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow$
$2u\,du = dx$)

$= \int \sqrt{1+u} \cdot 2u\,du$ (Let $w = 1+u$, $u = w-1$,
$dw = du$)

$= \int \sqrt{w} \cdot 2(w-1)\,dw = 2\int (w^{3/2} - w^{1/2})\,dw$

$= 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + c = \frac{4}{5}(1+u)^{5/2} - \frac{4}{3}(1+u)^{3/2} + c$

$= \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + c$

t.) $\int \frac{1}{x+5}\,dx = \ln|x+5| + c$

u.) $\int \frac{x}{x+5}\,dx = \int \frac{x+5-5}{x+5}\,dx = \int \left[1 - \frac{5}{x+5}\right] dx$

$= x - 5\ln|x+5| + c$

v.)
$$\begin{array}{r|l} & x-5 \\ x+5 & x^2 \\ & \underline{x^2+5x} \\ & -5x \\ & \underline{-5x-25} \\ & 25 \end{array}$$

so $\int \frac{x^2}{x+5}\,dx = \int \left[x - 5 + \frac{25}{x+5}\right] dx$

$= \frac{x^2}{2} - 5x + 25\ln|x+5| + c$

w.)

$$\begin{array}{r|l} & x - 3 \\ \hline x+5 & x^2 + 2x - 3 \\ & \underline{x^2 + 5x} \\ & \quad -3x - 3 \\ & \quad \underline{-3x - 15} \\ & \qquad 12 \end{array}$$

so $\displaystyle\int \frac{x^2+2x-3}{x+5}\,dx$

$$= \int \left[x - 3 + \frac{12}{x+5} \right] dx$$

$$= \frac{x^2}{2} - 3x + 12 \ln|x+5| + c$$

x.) $$\int \frac{1}{x(x-1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x-1} \right] dx = \int \left[\frac{-1}{x} + \frac{1}{x-1} \right] dx$$

$$= -\ln|x| + \ln|x-1| + c$$

y.) $$\int \frac{1}{x^2(x-1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \right] dx$$

$$= \int \left[\frac{-1}{x} + \frac{-1}{x^2} + \frac{1}{x-1} \right] dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + c$$

z.) $$\int \frac{1}{x^3(x-1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} \right] dx$$

$$= \int \left[\frac{-1}{x} + \frac{-1}{x^2} + \frac{-1}{x^3} + \frac{1}{x-1} \right] dx$$

$$= -\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + \ln|x-1| + c$$

A.) $$\int \frac{1}{x^2(x-1)^2}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \right] dx$$

$$= \int \left[\frac{2}{x} + \frac{1}{x^2} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} \right] dx$$

$$= 2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{x-1} + c$$

B.) $$\int \frac{x+2}{x^2(x^2+1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \right] dx$$

$$= \int \left[\frac{1}{x} + \frac{2}{x^2} + \frac{-x-2}{x^2+1} \right] dx = \int \left[\frac{1}{x} + \frac{2}{x^2} + \frac{-x}{x^2+1} + \frac{-2}{x^2+1} \right] dx$$

$$= \ln|x| - \frac{2}{x} - \frac{1}{2}\ln(x^2+1) - 2\arctan x + C$$

c.) $$\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\ln|x^3-1| + C$$

D.) $$\int \frac{x}{x^3-1}\,dx = \int \frac{x}{(x-1)(x^2+x+1)}\,dx = \int \left[\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \right] dx$$

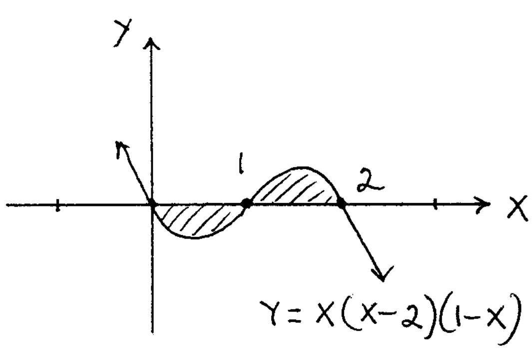
$$= \int \left[\frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} \right] dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \left[\frac{x+\frac{1}{2}}{x^2+x+1} + \frac{-\frac{3}{2}}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right] dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\left\{ \frac{1}{2}\ln|x^2+x+1| - \frac{3}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \arctan \frac{x+\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)} \right\} + C$$

Fact: $$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

2.)



$$\text{Area} = -\int_0^1 (-x^3 + 3x^2 - 2x)\,dx + \int_1^2 (-x^3 + 3x^2 - 2x)\,dx$$

$$= -\left(-\frac{x^4}{4} + x^3 - x^2\right)\Big|_0^1 + \left(-\frac{x^4}{4} + x^3 - x^2\right)\Big|_1^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

3.) a.) $$\int_0^{\frac{\pi}{2}} f'(x)\,dx = f(x)\Big|_0^{\frac{\pi}{2}} = f\left(\frac{\pi}{2}\right) - f(0) = 1 - 0 = 1.$$

b.) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\,dx = -\ln|1+\cos x|\ \Big|_0^{\frac{\pi}{2}}$

$= -\ln 1 - {}^{-}\ln 2 = \ln 2$

c.) $\displaystyle\int_0^{\frac{\pi}{2}} f(x)\,f'(x)\,dx = \frac{1}{2}[f(x)]^2\ \Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[f\left(\frac{\pi}{2}\right)\right]^2 - \frac{1}{2}[f(0)]^2 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$

d.) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{f'(x)}{1+f(x)}\,dx = \ln|1+f(x)|\ \Big|_0^{\frac{\pi}{2}}$

$= \ln\left|1+f\left(\frac{\pi}{2}\right)\right| - \ln|1+f(0)| = \ln 2 - \ln 1 = \ln 2$

e.) $\displaystyle g(x) = \int_0^{f(x)} f(t)\,dt \Rightarrow$

$g'(x) = f(f(x))\cdot f'(x) = f\left(\frac{\sin x}{1+\cos x}\right)\cdot \frac{2}{(1+\cos x)^2}$

$$= \frac{\sin\left(\frac{\sin x}{1+\cos x}\right)}{1+\cos\left(\frac{\sin x}{1+\cos x}\right)} \cdot \frac{2}{(1+\cos x)^2}$$

4.) True, since $D\,\dfrac{x}{(x^2+1)^2} = \dfrac{1-3x^2}{(x^2+1)^3}$.