Math 21C **Spring 1993** **Final Exam**

1) (10 points) Let $f(x, y) = \frac{\cos x}{y} + \ln(x^2y^3 + e^x)$. Compute f_x. Do not simplify your answer.

2. (12 points) Show that the following limit does not exist as $(x, y) \to (0, 0)$ by considering different paths of approach.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + y^2}{x^4 + y^2}.$$

3. (12 points) Evaluate the following triple integral using spherical coordinates:

$$\iiint_R e^{(x^2+y^2+z^2)^{3/2}} \, dV$$

where R is the solid ball of radius 2 centered at (0, 0, 0).

4. (12 points) Evaluate the double integral or an equivalent one.

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{9y^2}{\sqrt{1 + x^7}} \, dx \, dy.$$

5. a) (10 points) Find the critical points of the function

$$f(x, y) = x^4 - x^2y + \tfrac{3}{4}y^2 - 2y + 5.$$

b) (8 points) Use part a) to determine all relative maximum, relative minimum, and saddle points.

6. Let S be the solid region bounded below by the xy–plane, on the side by the cylinder $x^2 + y^2 = 8y$, and above by the paraboloid $z = 100 - x^2 - y^2$.

a) (12 points) Set up BUT DO NOT EVALUATE a triple integral in *rectangular coordinates* which represents the volume of the solid S.

b) (12 points) Set up BUT DO NOT EVALUATE a triple integral in *cylindrical coordinates* which represents the volume of the solid S.

7. **(9 points) Express the complex number $z = (1 + i)^{50}$ in form $z = a + bi$, where a and b are appropriately chosen real numbers.**

8. (12 points) Determine if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin(e^{\sqrt{n}})}{\sqrt{n^3 + 10}}$$

converges absolutely, conditionally or diverges. State clearly the reasons for your conclusions.

9. Determine if the following series converge or diverge. State clearly what tests you are using, justifying the use of any particular test you wish to apply.

a) (10 points) $\displaystyle\sum_{n=5}^{\infty} \frac{2}{n(\ln 4n)^4}$ b) (10 points) $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^5}{n^{5/4}}$

10. (12 points) Find the interval and radius of convergence of the following power series. (*Do not check convergence at the endpoints of the interval*):

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (x + 1)^{n-1}$$

11. (15 points) Compute the *entire* Taylor series for $f(x) = \frac{1}{2 - x}$ centered at $c = 1$. Simplify your answer as much as possible. (To receive full credit you must indicate the form of the nth term.)

12. (12 points) Use a Taylor polynomial of degree 4 for $f(x) = \dfrac{e^{-4x^2} - 1}{x^2}$ to approximate the value of the definite integral

$$\int_0^{1/2} \frac{e^{-4x^2} - 1}{x^2} \, dx .$$

13. **(12 points) Let $w = f(u, v)$ be a function whose derivatives of all orders exist.**

Suppose that

$$\frac{\partial^2 f}{\partial u^2}(3, 0) = -3$$
$$\frac{\partial^2 f}{\partial u \partial v}(3, 0) = 3$$
$$\frac{\partial^2 f}{\partial v^2}(3, 0) = -1$$

If $u = y + e^{2x}$ and $v = xy$, what is the value of $\frac{\partial^2 w}{\partial y^2}$ evaluated at the point $(x, y) = (0, 2)$?

14. Let $\{a_n\}$ be a sequence of non–zero real numbers such that $\lim_{n\to\infty} a_n = 3$.

a) (10 points) What can you conclude (if anything) about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{a_n}$? EXPLAIN.

b) (10 points) What can you conclude (if anything) about the convergence or divergence of the series $\sum_{n=1}^{\infty} (a_n - 3)$? EXPLAIN.

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1. (40 points) For each of the following series, determine convergence or divergence, stating the tests you use, and carefully justifying your answers. (Do NOT test for absolute convergence.)

a) $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

b) $\sum_{n=1}^{\infty} \frac{1/n - 3}{n!}$

c) $\sum_{n=1}^{\infty} \frac{\sin \sqrt{n}}{\sqrt{n^3 + 1}}$

d) $\sum_{n=2}^{\infty} \left[\frac{-1}{\ln n}\right]^n$.

2. (15 points) Find the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(2x + 1)^n}{n^2}.$$

3. (10 points) Determine the convergence or divergence of the following sequence. If the sequence converges, give its limit:

$$\left\{\frac{n^2}{2n - 1} - \frac{n^2}{2n + 1}\right\}.$$

4. (15 points) Find the five fifth roots of $-4 - 4i$, leaving your answers in $re^{i\theta}$ form.

5. (10 points) Let

$$u(x, y) = \ln(x^2 + y^2)$$

and

$$v(x, y) = 2 \arctan\left(\frac{y}{x}\right).$$

Show that u and v satisfy the equation:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

6. (20 points)

a) Use the Maclaurin series for $\cos x$ to state the first three nonzero terms of the Maclaurin series for $\frac{1 - \cos x}{x^2}$.

b) Use your answer in part (a) to estimate

$$\int_0^1 \frac{1 - \cos x}{x^2}\, dx .$$

c) Give a numerical bound on the error in the answer in part (b), explaining your reason for doing so.

7. (20 points) A jeweler wishes to make a rectangular pill box such that the top and bottom will be copper plated and the sides silver plated. If silver plating costs 6 dollars per cm^2 and copper plating costs 2 dollars per cm^2, what are the dimensions of the least costly pill box having $\frac{8}{3}$ cm^3 as its volume? Justify your answer.

8. (20 points) Use polar coordinates to evaluate the following integral:

$$\int_{-3}^{3} \int_{0}^{\sqrt{9 - y^2}} \sin(x^2 + y^2)\, dx\, dy .$$

9. (20 points) Let R be the three dimensional region which is bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$ (see graph). Find the volume of R.

10. (15 points) Rewrite the integral

$$\int_0^{2\pi} \int_0^{2} \int_0^{r} r^2\, dz\, dr\, d\theta$$

as an integral in spherical coordinates. (Do not evaluate.)

11. (15 points) Let $\{a_n\}$ be the sequence for which $a_1 = \sqrt{12}$ and $a_{n+1} = \sqrt{12 + a_n}$ for $n \geq 1$.

a) Find a_2, a_3, and a_4.

b) It can be shown that the sequence $\{a_n\}$ defined above converges. Assuming this to be true, find its limit L. (HINT: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$.)

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1. (15 points) Let $f(x, y) = 1 + \sqrt{x^2 + y^2}$.

   a) Sketch the graph of f, and show at least three different curves obtained by intersecting the graph with planes.

   b) Explain how the graph of f can be generated by a curve via rotation. Give an equation for such a curve.

2. (15 points) Let R be the annular region in the plane enclosed between the circles of radius 1 and 2 centered at the origin. Suppose the density per unit area of the annulus is $|xy|$. Find the mass of the annulus.

3. (25 points) Let S be the solid right circular cone of radius a and height h, with axis along the z–axis, as in the picture at the right. For each point P in S let f(P) be the square of the height of P above the (x, y)–plane.

   a) <u>Using cylindrical coordinates,</u> calculate $\int_S f(P)\, dV$.

   b) Set up BUT DO NOT EVALUATE a triple integral <u>in spherical coordinates</u> for $\int_S f(P)\, dV$.

4. (20 points) Define a function f(x, y) by

$$f(x, y) = \begin{cases} \dfrac{\sin(x^2 + xy^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

   a) Compute $f_x(x, y)$ when $(x, y) \neq (0, 0)$. (Do not simplify your answer.)

   b) Does $f_x(0, 0)$ exist? If it doesn't exist, explain why. If it does exist, compute it.

5. (15 points) Let $z = g(x^2 - y^2, y^2 - x^2 + 3)$, where g is a differentiable function of two variables. Show that

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0 .$$

6. (40 points) In each of the following four parts, determine if the given series converges absolutely, converges conditionally, or diverges, and justify your answers.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1\cdot 5\cdot 9 \cdots (4n+1)}{5\cdot 8\cdot 11 \cdots (3n+5)}$.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{5+\sin(\frac{1}{n})}{n^2}$

c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$

d) $\displaystyle\sum_{n=1}^{\infty} 2^{(-2/n^3)}$

7. (15 points) Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n4^n} .$$

8. (15 points) Let $f(t) = \frac{1}{2}(e^t + e^{-t})$. Use a Maclaurin series representation for $f(t)$ to obtain a power series for $\int_0^x f(t)\,dt$. Express your answer in summation notation.

9. (20 points) Find the minimum cost of a closed rectangular box with volume 48 cubic feet, where the front and back of the box cost one dollar per square foot, the top and bottom of the box cost two dollars per square foot, and the two ends of the box cost three dollars per square foot.

10. (20 points) The following is an open ended question. The grade will be based on the quality of writing and clarity of discussion as well as mathematical content and thoroughness.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and suppose there are real constants m and M with $m > 0, M > 0$, and $m < a_n < M$ for each n.

a) What can be said about the convergence or divergence of $\{a_n\}_{n=1}^{\infty}$?

b) What can be said about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$?