ESP
Kouba
Worksheet 11 Solutions

1.) a.) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$ so

$P_3(x;0) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ and

$e^{0.1} \approx 1 + (0.1) + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 = 1.105166667$;

<u>calculator</u> : $e^{0.1} = 1.105170918$.

b.) $e^4 \approx 1 + (4) + \frac{1}{2}(4)^2 + \frac{1}{6}(4)^3 = 23.66666667$;

<u>calculator</u> : $e^4 = 54.59815003$.

c.) $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$ so $P_3(x;0) = x - \frac{1}{6}x^3$ and

$\sin(0.2) \approx (0.2) - \frac{1}{6}(0.2)^3 = 0.198666666$;

<u>calculator</u> : $\sin(0.2) = 0.19866933$.

d.) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$ so

$P_3(x;0) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ and

$\ln(1.5) = \ln(1+0.5) \approx (0.5) - \frac{1}{2}(0.5)^2 + \frac{1}{3}(0.5)^3 = 0.416666666$;

<u>calculator</u> : $\ln(1.5) = 0.405465108$.

2.) a.) $f(x) = f'(x) = f''(x) = f'''(x) = e^x$, $a_n = \frac{f^{(n)}(1)}{n!}$ so

$a_0 = e$, $a_1 = e$, $a_2 = \frac{e}{2!}$, $a_3 = \frac{e}{3!}$, $\ldots$ and

$e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots$

b.) $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = \frac{-1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$,

$a_n = \frac{f^{(n)}(1)}{n!}$ so $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{-1}{8}$, $a_3 = \frac{1}{16}$, ... and

$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \cdots$

3.) a.) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$ so

$e^{-1} \approx \cancel{1-1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} = 0.366666666$.

b.) i.) error $|R_5| < a_6 = \frac{1}{6!} = 0.001388888$.

ii.) error $|R_5(1;0)| = \left| \frac{f^{(6)}(c)}{6!} (1-0)^6 \right| = \frac{e^{-c}}{6!}$, where

$0 \le c \le 1$, so $\frac{e^{-c}}{6!} \le \frac{e^0}{6!} = \frac{1}{6!} = 0.001388888$.

iii.) <u>calculator</u>: $e^{-1} = 0.367879441$ so

error $= 0.001212775$

4.) $f(x) = 3x^4 - x^3 + 2x^2 - x + 5 \Rightarrow f'(x) = 12x^3 - 3x^2 + 4x - 1$,

$f''(x) = 36x^2 - 6x + 4$, $f'''(x) = 72x - 6$, $f^{(4)}(x) = 72$, $f^{(5)}(x) = 0$,

$f^{(n)}(x) = 0$ for $n = 6, 7, 8, \ldots$ and $a_n = \frac{f^{(n)}(1)}{n!}$ so

$a_0 = 8$, $a_1 = 12$, $a_2 = 17$, $a_3 = 11$, $a_4 = 3$ then

$3x^4 - x^3 + 2x^2 - x + 5 = 8 + 12(x-1) + 17(x-1)^2 + 11(x-1)^3 + 3(x-1)^4$.

5.) a.) $|R_n(x;0)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-0)^{n+1} \right| = \frac{e^c}{(n+1)!} |x|^{n+1}$,

where c is between 0 and x;

if $x \ge 0$, then $e^c \le e^x$ so that

$$\frac{e^c}{(n+1)!}|x|^{n+1} \le \frac{e^x}{(n+1)!}|x|^{n+1} = e^x \cdot \frac{|x|^{n+1}}{(n+1)!} \to e^x \cdot 0 = 0$$

as $n \to \infty$; if $x < 0$ then $e^c < 1$ so that

$$\frac{e^c}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!} \to 0$$ as $n \to \infty$; in both cases, it follows that $|R_n(x;0)| \to 0$ as $n \to \infty$ for all values of x.

b.) $f(x) = \ln(1+x)$, $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$,
$f'''(x) = 2(1+x)^{-3}$, $f^{(4)}(x) = -3 \cdot 2\,(1+x)^{-4}$,
$f^{(5)}(x) = 4 \cdot 3 \cdot 2\,(1+x)^{-5}$, $f^{(6)}(x) = -5 \cdot 4 \cdot 3 \cdot 2\,(1+x)^{-6}$, ...,
$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}$ then

$$|R_n(x;0)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1} \right| = \left| \frac{(-1)^{n+2}\, n!\,(1+c)^{-(n+1)}}{(n+1)!} x^{n+1} \right|$$

$= \frac{1}{n+1}\left|\frac{x}{1+c}\right|^{n+1}$, where c is between 0 and x;
if $0 \le c \le x < 1$ then $\left|\frac{x}{1+c}\right| \le \left|\frac{x}{1+0}\right| = |x| < 1$ so

$$\frac{1}{n+1}\left|\frac{x}{1+c}\right|^{n+1} \le \frac{1}{n+1}|x|^{n+1} \to 0 \quad \text{as } n \to \infty ;$$

if $-\frac{1}{2} < x \le c \le 0$ then $\left|\frac{x}{1+c}\right| \le \left|\frac{x}{1+x}\right| < \frac{\frac{1}{2}}{\frac{1}{2}} = 1$ so

$$\frac{1}{n+1}\left|\frac{x}{1+c}\right|^{n+1} \le \frac{1}{n+1}\left|\frac{x}{1+x}\right|^{n+1} \to 0 \quad \text{as } n \to \infty ;$$

in both cases, it follows that
$|R_n(x;0)| \to 0$ as $n \to \infty$.

6.) a.) $\int_0^1 \cos\sqrt{x}\,dx = \int_0^1 \left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right] dx$

$= \left(x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 4!} - \frac{x^4}{4\cdot 6!} + \frac{x^5}{5\cdot 8!} - \cdots\right)\Big|_0^1$

$= 1 - \frac{1}{2\cdot 2!} + \frac{1}{3\cdot 4!} - \frac{1}{4\cdot 6!} + \frac{1}{5\cdot 8!} - \cdots$; since this is an alternating series with $\frac{1}{5\cdot 8!} = 0.00000496 < 0.00001$,

$\int_0^1 \cos\sqrt{x}\,dx \approx 1 - \frac{1}{2\cdot 2!} + \frac{1}{3\cdot 4!} - \frac{1}{4\cdot 6!} = 0.76354$ estimates the integral with error at most 0.00001 .

b.) If $u^2 = x$, then $dx = 2u\,du$ so that

$\int \cos\sqrt{x}\,dx = 2\int u\cos u\,du$

(Let $w = u$, $dv = \cos u\,du$
$dw = du$, $v = \sin u$)

$= 2\left[u\sin u - \int \sin u\,du\right]$

$= 2u\sin u - 2(-\cos u) + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$. Thus

$\int_0^1 \cos\sqrt{x}\,dx = \left(2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}\right)\Big|_0^1$

$= 2\sin 1 + 2\cos 1 - 2\cos 0$

≈ 0.763546581352 !