ESP
Kouba
Worksheet 12 Solutions

1.) a.) $\lim_{n\to\infty} \frac{e^n}{n+e^n} = \lim_{n\to\infty} \frac{1}{\frac{n}{e^n}+1} = \lim_{n\to\infty} \frac{1}{\frac{1}{e^n}+1} = 1$

b.) $\lim_{n\to\infty} \frac{\ln(3n)}{\ln(4n)} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1$

c.) $\lim_{n\to\infty} \left[1+\left(\frac{3}{2}\right)^n\right]^{\frac{1}{n}} = \infty^0$ (indeterminant) so

$$\lim_{n\to\infty} \ln\left[1+\left(\tfrac{3}{2}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln\left[1+\left(\frac{3}{2}\right)^n\right]}{n}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\left(\frac{3}{2}\right)^n \cdot \ln\left(\frac{3}{2}\right)}{1+\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty} \frac{\ln\left(\frac{3}{2}\right)}{\left(\frac{2}{3}\right)^n + 1} = \ln\left(\frac{3}{2}\right) \text{ so}$$

$$\lim_{n\to\infty} \left[1+\left(\tfrac{3}{2}\right)^n\right]^{\frac{1}{n}} = \frac{3}{2}$$

d.) $\lim_{n\to\infty} (2^n+3^n)^{1/n} = \lim_{n\to\infty} \left[3^n\left(\left(\frac{2}{3}\right)^n+1\right)\right]^{1/n}$

$$= \lim_{n\to\infty} 3 \cdot \left[\left(\tfrac{2}{3}\right)^n+1\right]^{1/n} = 3(0+1)^0 = 3\cdot 1 = 3$$

2.) a.) limit comparison: $\int_1^\infty \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2\Big|_1^\infty = \infty$
so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges, and

$$\lim_{n\to\infty} \frac{\frac{\ln(n+3)}{n}}{\frac{\ln n}{n}} = \lim_{n\to\infty} \frac{\ln(n+3)}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+3}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{n+3} = 1 \text{ so series diverges.}$$

b.) absolute convergence: $0 \le \frac{|\cos n|^4}{n^3} \le \frac{1}{n^3}$
and since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p = 3 > 1$)
so does $\sum_{n=1}^{\infty} \frac{|\cos n|^4}{n^3}$; thus the series converges.

c.) integral test: $f(x) = xe^{-x^2}$ is $+$, $\downarrow$, and
cont. with $\int_1^{\infty} xe^{-x^2}dx = \frac{-1}{2}e^{-x^2}\Big|_1^{\infty} = 0 - \frac{-1}{2}e^{-1} = \frac{1}{2e}$
so series converges.

d.) integral test: $f(x) = \frac{1}{x(\ln x)^{3/4}}$ is $+$, $\downarrow$, and
cont. with $\int_3^{\infty} \frac{1}{x(\ln x)^{3/4}}dx = 4(\ln x)^{1/4}\Big|_3^{\infty} = \infty$,
so series diverges.

e.) limit comparison: $\lim_{n\to\infty} \frac{\frac{n}{(n^4+10)^{1/3}}}{\frac{1}{n^{1/3}}} = \lim_{n\to\infty} \frac{n^{4/3}}{(n^4+10)^{1/3}}$

$= \lim_{n\to\infty} \left(\frac{n^4}{n^4+10}\right)^{1/3} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{10}{n^4}}\right)^{1/3} = 1$, so series
diverges since $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges (p-series, $p = \frac{1}{3} < 1$)

f.) limit comparison: $\lim_{n\to\infty} \frac{\frac{2^n}{4^n - 100}}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \frac{4^n}{4^n - 100}$

$= \lim_{n\to\infty} \frac{1}{1 - \frac{100}{4^n}} = 1$, so series converges
since $\sum_{n=10}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent geometric
series.

g.) integral test: $f(x) = \frac{1}{x \ln x^2} = \frac{1}{2x \ln x}$ is +, ↓, and cont. with $\int_2^\infty \frac{1}{2x\ln x}\,dx = \frac{1}{2}\ln(\ln x)\Big|_2^\infty = \infty$, so series diverges.

h.) comparison test: $\frac{1}{n^2 \ln n^2} < \frac{1}{n^2}$ so series converges since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$)

i.) ratio test: $\lim_{n\to\infty} \frac{(n+1)!}{1\cdot 3\cdot 5\cdot \dots (2n-1)(2n+1)} \cdot \frac{1\cdot 3\cdot 5\cdot \dots (2n-1)}{n!}$

$= \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$ so series converges.

3.) a.) $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n^2}$ then $a_n b_n = \frac{1}{n^4}$

b.) $a_n = (-1)^n \cdot \frac{1}{\sqrt{n}}$, $b_n = (-1)^n \cdot \frac{1}{\sqrt{n}}$ then $a_n b_n = \frac{1}{n}$

4.) a.) $(8-3i) - (7-6i) = 1 + 3i$

b.) $(4+3i)(3-2i) = 12 - 8i + 9i + 6 = 18 + i$

c.) $\frac{2+3i}{4+3i} \cdot \frac{4-3i}{4-3i} = \frac{8+6i+9}{25} = \frac{17}{25} + \frac{6}{25}i$

5.) a.) $\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = 1\cdot\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$

b.) $i = 1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

c.) $-5 = 5\left(\cos\pi + i \sin\pi\right)$

d.) $3 - 3i = 3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 3\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

6.) a.) $\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^4$

$= \cos 4\left(\frac{3\pi}{4}\right) + i\sin 4\left(\frac{3\pi}{4}\right) = -1$

b.) $(\sqrt{3}+i)^{10} = \left[2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right]^{10} = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{10}$

$= 2^{10}\left(\cos 10\left(\frac{\pi}{6}\right) + i\sin 10\left(\frac{\pi}{6}\right)\right) = 1024\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

c.) $\dfrac{\left[2(\cos 60^\circ + i\sin 60^\circ)\right]^3}{16(\cos 135^\circ + i\sin 135^\circ)^4} = \dfrac{8(\cos 180^\circ + i\sin 180^\circ)}{16(\cos 540^\circ + i\sin 540^\circ)}$

$= \frac{1}{2}\left(\cos(-360^\circ) + i\cdot\sin(-360^\circ)\right) = \frac{1}{2}$

7.) $z^2 - 2z + 2 = 0 \rightarrow z = \dfrac{-(-2) \pm \sqrt{-4}}{2} = 1 \pm i$

8.) The solutions to

$z^4 = 16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

are

$z_1 = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$,

$z_2 = 2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$,

$z_3 = 2\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$,

$z_4 = 2\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$.