ESP
Kouba
Worksheet 5 Solutions

1.) a.) $z_x = e^{\tan(y-x)} \cdot \frac{2x}{x^2+y^3} + \ln(x^2+y^3) \cdot e^{\tan(y-x)} \cdot \sec^2(y-x) \cdot (-1)$

$$z_{xx} = \frac{(x^2+y^3)\left[e^{\tan(y-x)} \cdot 2 + 2x e^{\tan(y-x)} \cdot \sec^2(y-x) \cdot (-1)\right] - 2x\, e^{\tan(y-x)} \cdot 2x}{(x^2+y^3)^2}$$

$$- \frac{2x}{x^2+y^3} \cdot e^{\tan(y-x)} \cdot \sec^2(y-x) + \ln(x^2+y^3) \cdot e^{\tan(y-x)} \cdot \sec^4(y-x)$$

$$+ \ln(x^2+y^3) \cdot e^{\tan(y-x)} \cdot 2\sec^2(y-x) \cdot \tan(y-x) .$$

b.) $z_x = g_u \cdot u_x + g_v \cdot v_x$ so

$$\begin{aligned} z_{xx} &= g_u \cdot u_{xx} + \frac{\partial}{\partial x}(g_u) \cdot u_x + g_v \cdot v_{xx} + \frac{\partial}{\partial x}(g_v) \cdot v_x \\ &= g_u \cdot u_{xx} + (g_{uu} \cdot u_x + g_{uv} \cdot v_x) \cdot u_x \\ &\qquad + g_v \cdot v_{xx} + (g_{vu} \cdot u_x + g_{vv} \cdot v_x) \cdot v_x \\ &= g_u \cdot (0) + (g_{uu} \cdot (2) + g_{uv} \cdot (2x)) \cdot (2) \\ &\qquad + g_v \cdot (2) + (g_{uv} \cdot (2) + g_{vv} \cdot (2x)) \cdot (2x) \\ &= (2) \cdot g_v + (4) \cdot g_{uu} + (4x^2) g_{vv} + (8x) \cdot g_{uv} \end{aligned}$$

2.) $z_x = f'\left(\frac{x}{y}\right) \cdot \frac{1}{y}$ and $z_y = f'\left(\frac{x}{y}\right) \cdot \frac{-x}{y^2}$ so

$$x \cdot z_x + y \cdot z_y = \frac{x}{y} \cdot f'\left(\frac{x}{y}\right) - \frac{x}{y} \cdot f'\left(\frac{x}{y}\right) = 0 .$$

3.) $z = xy^2 - x^2y + x - y \rightarrow$

$z_x = y^2 - 2xy + 1 = 0$ and $z_y = 2xy - x^2 - 1 = 0 \rightarrow$

$x = \frac{y^2+1}{2y}$ and $2y\left(\frac{y^2+1}{2y}\right) - \left(\frac{y^2+1}{2y}\right)^2 - 1 = 0 \rightarrow$

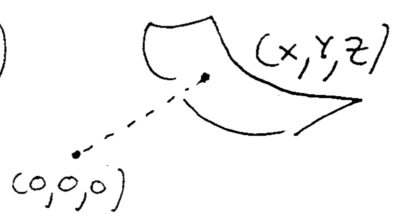
$y^2 + 1 - \frac{(y^2+1)^2}{4y^2} - 1 = 0 \rightarrow 4y^4 = (y^2+1)^2 \rightarrow$

$2y^2 = y^2 + 1 \rightarrow y^2 = 1 \rightarrow y = \pm 1$ so critical points are $(1,1)$ and $(-1,-1)$;

$z_{xx} = -2y$, $z_{yy} = 2x$, $z_{xy} = 2y - 2x$

$\underline{(1,1)}$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (-2)(2) - (0)^2 = -4 < 0$ so $(1,1)$ determines a saddle point at $z = 0$;

$\underline{(-1,-1)}$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (2)(-2) - (0)^2 = -4 < 0$ so $(-1,-1)$ determines a saddle point at $z = 0$.

4.) 

$xyz = 8$ or $z = \frac{8}{xy}$, minimize distance

$L = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2 + \frac{64}{x^2y^2}} \rightarrow$

$L_x = \frac{1}{2}(\sim)^{\frac{-1}{2}}\left[2x - \frac{128}{x^3y^2}\right] = 0 \rightarrow x = \frac{64}{x^3y^2}$ and

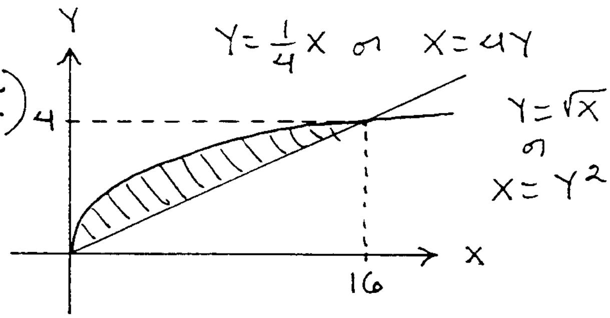
$L_y = \frac{1}{2}(\sim)^{\frac{-1}{2}}\left[2y - \frac{128}{x^2y^3}\right] = 0 \rightarrow y = \frac{64}{x^2y^3}$ so

$$\left.\begin{aligned} X^4 Y^2 &= 64 \\ X^2 Y^4 &= 64 \end{aligned}\right\} \qquad Y^2 = \frac{64}{X^4} \longrightarrow X^2 \cdot \frac{(64)^2}{X^8} = 64 \rightarrow 64 = X^6$$

$\rightarrow X = \pm 2$ and $Y = \pm 2$ so critical points $(2,2)$, $(2,-2)$, $(-2,2)$, and $(-2,-2)$ all determine a minimum distance of

$$L = \sqrt{12} = 2\sqrt{3} .$$

5.)



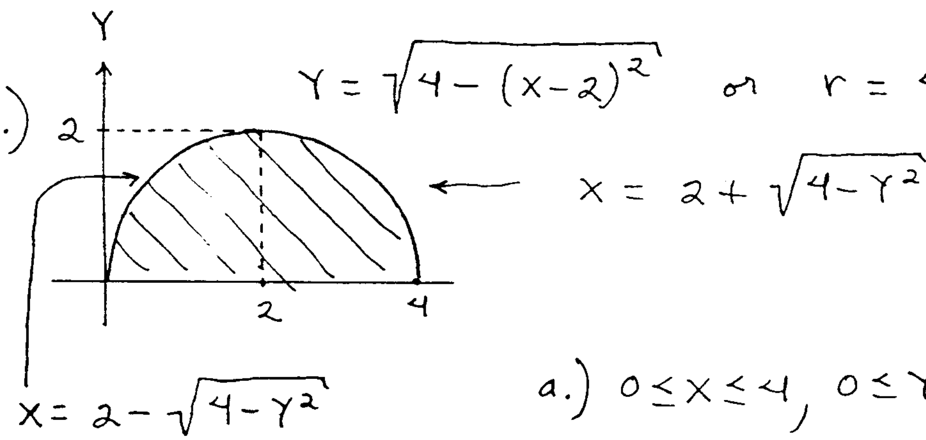
a.) $0 \le X \le 16,\ \frac{1}{4}X \le Y \le \sqrt{X}$

b.) $0 \le Y \le 4,\ Y^2 \le X \le 4Y$

c.) i.) $\int_0^4 \int_{Y^2}^{4Y} f(x,Y)\,dx\,dy$

ii.) $\int_0^{16} \int_{\frac{1}{4}X}^{\sqrt{X}} f(x,Y)\,dy\,dx$

6.)



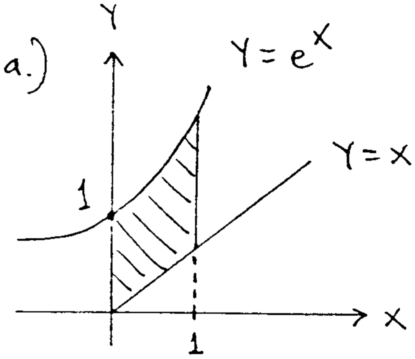
a.) $0 \le X \le 4,\ 0 \le Y \le \sqrt{4-(X-2)^2}$

b.) $0 \le \quad \le 2,\ 2-\sqrt{4-Y^2} \le X \le 2+\sqrt{4-Y^2}$

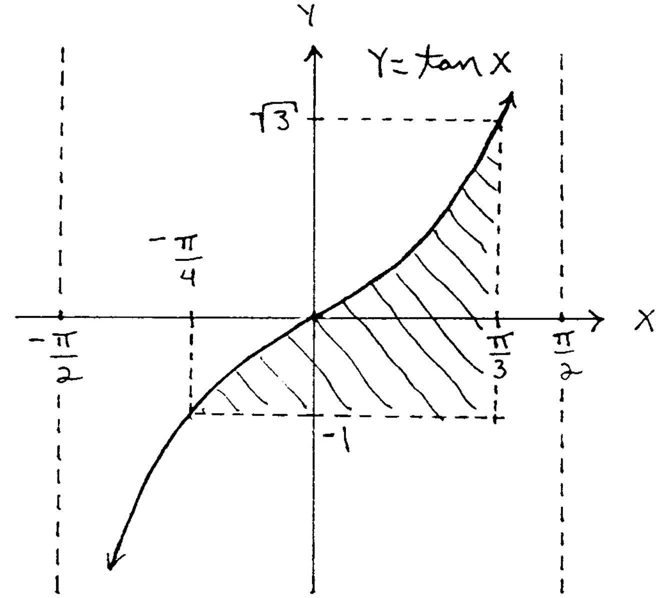
c.) $0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 4\cos\theta$

d.) $0 \le r \le 4$, $0 \le \theta \le \arccos\left(\frac{r}{4}\right)$

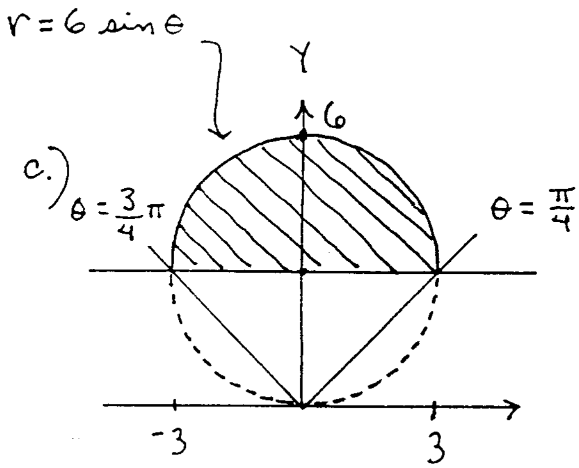
7.) a.)



b.)



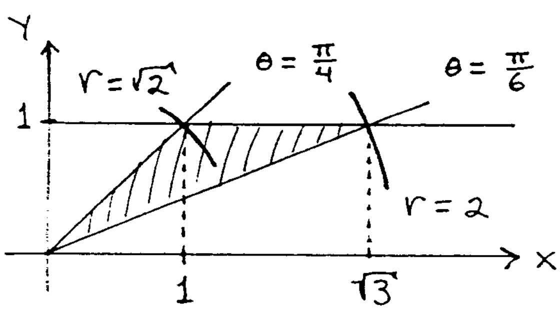
c.)



$\leftarrow \; Y = 3 \rightarrow r\sin\theta = 3 \rightarrow$

$r = 3\csc\theta$

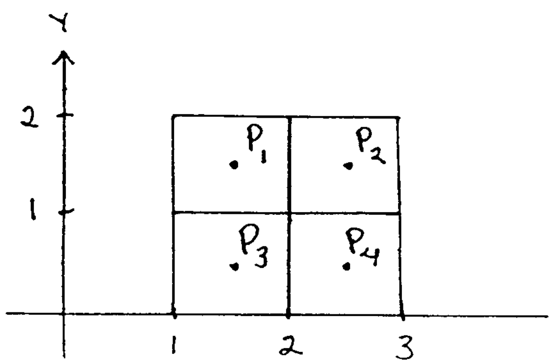
d.)



$Y = 1 \rightarrow r\sin\theta = 1 \rightarrow$

$\theta = \arcsin\left(\frac{1}{r}\right)$

8.)



$P_1 = \left(3/2, 3/2\right)$

$P_2 = \left(5/2, 3/2\right)$

$P_3 = \left(3/2, 1/2\right)$

$P_4 = \left(5/2, 1/2\right)$

$A_1 = A_2 = A_3 = A_4 = 1$ so

$$\sum_{i=1}^{4} f(P_i)\,A_i = \sum_{i=1}^{4} f(P_i)\cdot 1 = f(P_1)+f(P_2)+f(P_3)+f(P_4)$$

$$= \frac{9}{2} + \frac{17}{2} + \frac{5}{2} + \frac{13}{2} = \frac{44}{2} = 22$$

9.) a.) $\int_0^1 \int_2^3 2x^2 y \, dy \, dx = \int_0^1 \left(x^2 y^2 \Big|_{y=2}^{y=3} \right) dx$

$$= \int_0^1 5x^2 \, dx = \frac{5}{3} x^3 \Big|_0^1 = \frac{5}{3}$$

b.) $\int_1^2 \int_1^x \frac{x^2}{y^2} \, dy \, dx = \int_1^2 \left(-\frac{x^2}{y} \Big|_{y=1}^{y=x} \right) dx$

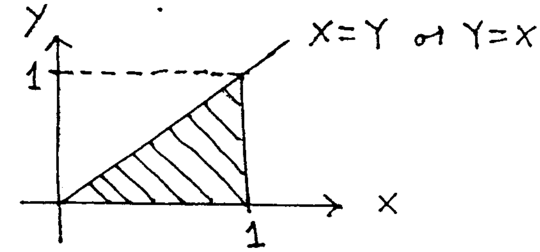
$$= \int_1^2 (-x + x^2) \, dx = \left(-\frac{x^2}{2} + \frac{x^3}{3} \right) \Big|_1^2 = \frac{5}{6}$$

c.) $\int_0^{\sqrt{\pi}} \int_0^y \sin(y^2) \, dx \, dy = \int_0^{\sqrt{\pi}} \left(x \cdot \sin(y^2) \Big|_{x=0}^{x=y} \right) dy$

$$= \int_0^{\sqrt{\pi}} y \sin(y^2) \, dy = \frac{-1}{2} \cos(y^2) \Big|_0^{\sqrt{\pi}}$$

$$= \frac{-1}{2}\cos(\pi) - \frac{-1}{2}\cos(0) = \frac{1}{2} + \frac{1}{2} = 1$$

d.) $\int_0^1 \int_y^1 \sqrt{1+x^2} \, dx \, dy$



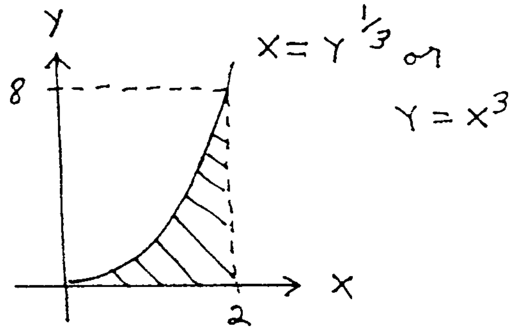
(switch order) $= \int_0^1 \int_0^x \sqrt{1+x^2} \, dy \, dx$

$$= \int_0^1 \left(y\sqrt{1+x^2} \Big|_{y=0}^{y=x} \right) dx = \int_0^1 x\sqrt{1+x^2} \, dx$$

$$= \frac{1}{3}\cdot(1+x^2)^{3/2}\Big|_0^1 = \boxed{\frac{1}{3}(2)^{3/2} - \frac{1}{3}(1)}$$

e.) $\int_0^8 \int_{y^{1/3}}^2 e^{x^4}\,dx\,dy$

(switch order)



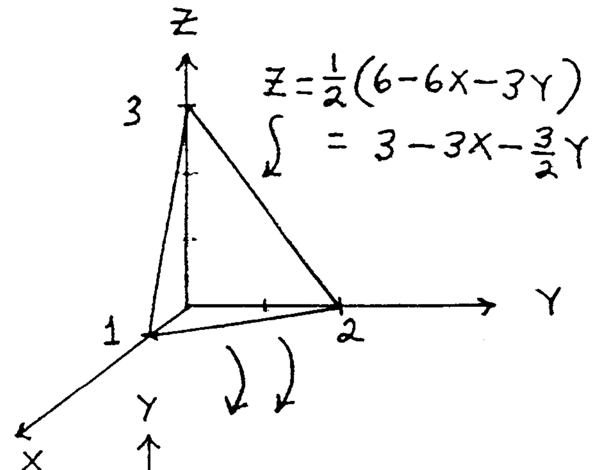
$$= \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

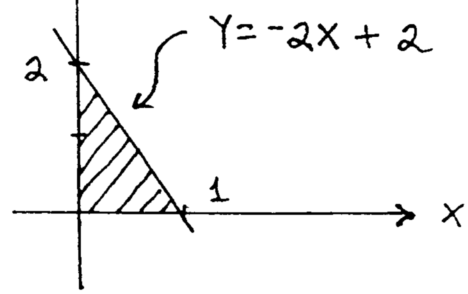
$$= \int_0^2 \left(y\cdot e^{x^4}\Big|_{y=0}^{y=x^3}\right)dx = \int_0^2 x^3\cdot e^{x^4}\,dx$$

$$= \frac{1}{4}e^{x^4}\Big|_0^2 = \boxed{\frac{1}{4}e^{16} - \frac{1}{4}}$$

10.) a.) $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$

or $6x + 3y + 2z = 6$





b.) $\text{Volume} = \int_0^1 \int_0^{2-2x} (3 - 3x - \tfrac{3}{2}y)\,dy\,dx$

$$= \int_0^1 \left(3y - 3xy - \frac{3}{4}y^2\right)\Big|_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 (3x^2 - 6x + 3)\,dx$$

$$= (x^3 - 3x^2 + 3x)\Big|_0^1 = 1$$