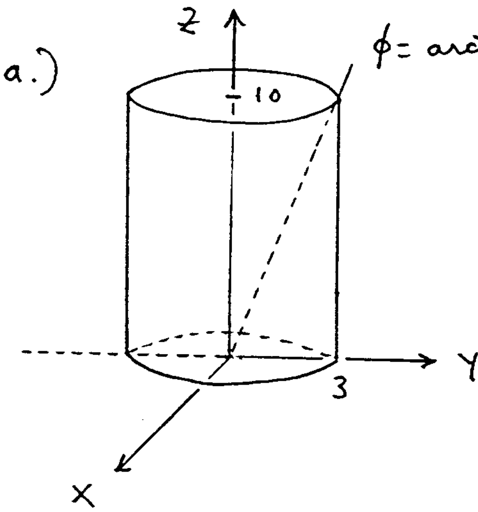
ESP
Kouba
Worksheet 7 Solutions

1.) a.)



plane: $z = 10 \rightarrow \rho \cos\phi = 10$

$\rightarrow \rho = 10 \sec\phi$

cylinder: $x^2 + y^2 = 9 \rightarrow$

$(\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 = 9 \rightarrow$

$\rho^2 \sin^2\phi \, \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = 9 \rightarrow$

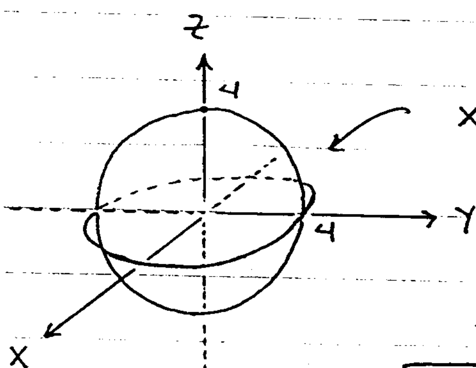
$\rho = 3 \csc\phi$

rect.: $-3 \le x \le 3$, $-\sqrt{9-x^2} \le y \le +\sqrt{9-x^2}$, $0 \le z \le 10$

cyl.: $0 \le \theta < 2\pi$, $0 \le r \le 3$, $0 \le z \le 10$

spher.: $0 \le \theta < 2\pi$, $0 \le \phi \le \arctan\frac{3}{10}$, $0 \le \rho \le 10 \sec\phi$ and
$0 \le \theta < 2\pi$, $\arctan\frac{3}{10} \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le 3\csc\phi$

b.)



$x^2 + y^2 + z^2 = 16$

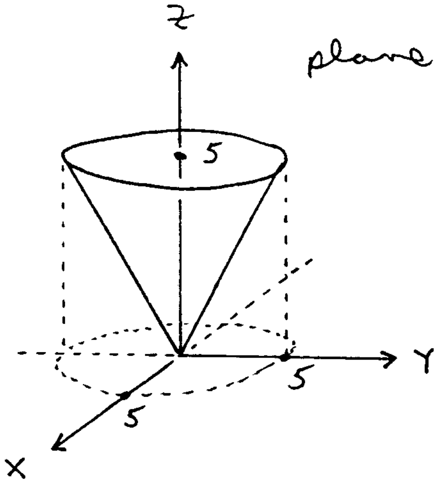
rect.: $-4 \le x \le 4$,
$-\sqrt{16-x^2} \le y \le +\sqrt{16-x^2}$,
$-\sqrt{16-x^2-y^2} \le z \le +\sqrt{16-x^2-y^2}$

cyl.: $0 \le \theta < 2\pi$, $0 \le r \le 4$, $-\sqrt{16-r^2} \le z \le +\sqrt{16-r^2}$

spher.: $0 \le \theta < 2\pi$, $0 \le \phi \le \pi$, $0 \le \rho \le 4$

$\text{plane: } z=5 \rightarrow \rho\cos\phi = 5 \rightarrow \rho = 5\sec\phi$

c.)

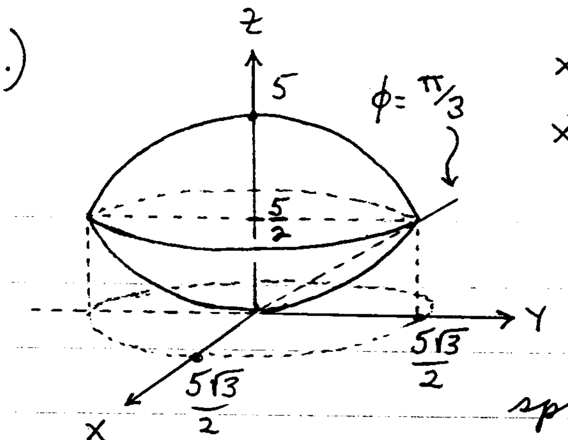


rect.: $-5 \le x \le 5$,
$-\sqrt{25-x^2} \le y \le +\sqrt{25-x^2}$,
$\sqrt{x^2+y^2} \le z \le 5$

cyl.: $0 \le \theta < 2\pi$, $0 \le r \le 5$, $r \le z \le 5$

spher.: $0 \le \theta < 2\pi$, $0 \le \phi \le \frac{\pi}{4}$, $0 \le \rho \le 5\sec\phi$

d.)



$x^2+y^2+z^2 = 25$ and
$x^2+y^2+(z-5)^2 = 25 \rightarrow$
$z^2-(z-5)^2 = 0 \rightarrow$
$z = 5/2 \rightarrow$
spheres intersect in
circle $x^2+y^2 = 75/4$,

$x^2+y^2+(z-5)^2 = 25 \rightarrow x^2+y^2+z^2-10z+25 = 25 \rightarrow$
$\rho^2 = 10\rho\cos\phi \rightarrow \rho = 10\cos\phi$;

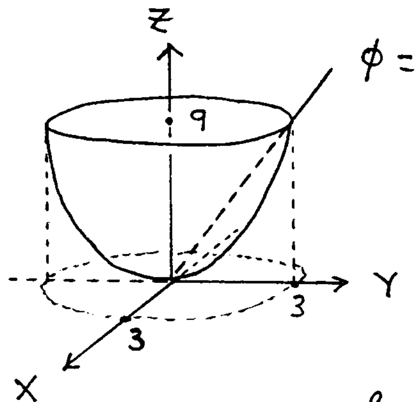
rect.: $-\frac{5\sqrt{3}}{2} \le x \le \frac{5\sqrt{3}}{2}$, $-\sqrt{\frac{75}{4}-x^2} \le y \le +\sqrt{\frac{75}{4}-x^2}$,
$5-\sqrt{25-x^2-y^2} \le z \le +\sqrt{25-x^2-y^2}$

cyl.: $0 \le \theta < 2\pi$, $0 \le r \le \frac{5\sqrt{3}}{2}$, $5-\sqrt{25-r^2} \le z \le \sqrt{25-r^2}$

spher.: $0 \le \theta < 2\pi$, $0 \le \phi \le \frac{\pi}{3}$, $0 \le \rho \le 5$ and

$0 \le \theta < 2\pi$, $\frac{\pi}{3} \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le 10 \cos\phi$

e.)



$\phi = \arctan \frac{1}{3}$

paraboloid: $z = x^2 + y^2 \rightarrow$

$\rho \cos\phi = (\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 \rightarrow$

$\rho \cos\phi = \rho^2 \sin^2\phi \, \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} \rightarrow$

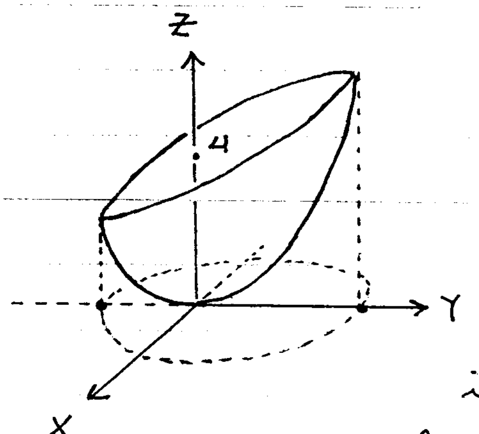
$\rho = \csc\phi \cot\phi$,

rect.: $-3 \le x \le 3$, $-\sqrt{9-x^2} \le y \le +\sqrt{9-x^2}$, $x^2 + y^2 \le z \le 9$

cyl.: $0 \le \theta < 2\pi$, $0 \le r \le 3$, $r^2 \le z \le 9$

spher.: $0 \le \theta < 2\pi$, $0 \le \phi \le \arctan\frac{1}{3}$, $0 \le \rho \le 9 \sec\phi$ and
$0 \le \theta < 2\pi$, $\arctan\frac{1}{3} \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le \csc\phi \cot\phi$

f.)



$z = x^2 + y^2$ and $z = 2y + 4 \rightarrow$

$x^2 + y^2 = 2y + 4 \rightarrow$

$x^2 + y^2 - 2y + 1 = 5 \rightarrow$ circle

$x^2 + (y-1)^2 = 5$

is intersection of plane and paraboloid;

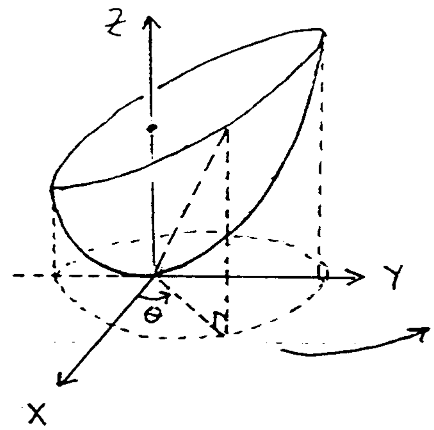
paraboloid:

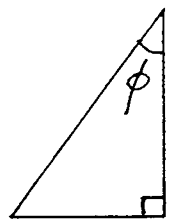
$x^2 + y^2 = 2y + 4 \rightarrow \quad r^2 = 2r \sin\theta + 4 \rightarrow$

$r^2 - 2\sin\theta \cdot r - 4 = 0 \rightarrow$

$$r = \frac{2\sin\theta \pm \sqrt{4\sin^2\theta + 16}}{2} = \frac{2\sin\theta + \sqrt{4\sin^2\theta+16}}{2}$$

$$= F(\theta) \;;$$





$F(\theta) = r$

$$[F(\theta)]^2 = r^2$$

$$\phi = \arctan\left({}^{1}\!/_{F(\theta)}\right) \;;$$

paraboloid : $z = x^2 + y^2 \rightarrow$

$\rho\cos\phi = \rho^2\cos^2\theta\,\sin^2\phi + \rho^2\sin^2\theta\,\sin^2\phi = \rho^2\sin^2\phi \rightarrow$

$\rho = \dfrac{\cos\phi}{\sin^2\phi} = \cot\phi\,\csc\phi = G(\phi) \;;$

plane : $z = 2y + 4 \rightarrow \rho\cos\phi = 2\rho\sin\theta\sin\phi + 4 \rightarrow$

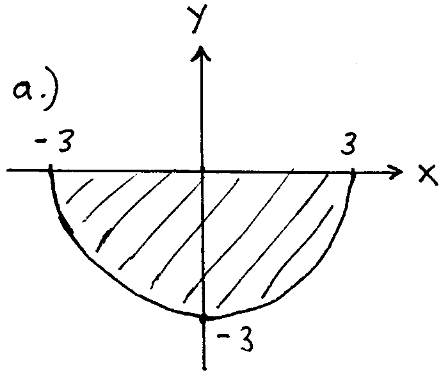
$$\rho = \frac{4}{\cos\phi - 2\sin\theta\sin\phi} = H(\theta,\phi)$$

rect. : $1-\sqrt{5} \le y \le 1+\sqrt{5}$, $-\sqrt{5-(y-1)^2} \le x \le +\sqrt{5-(y-1)^2}$,
$x^2+y^2 \le z \le 2y+4$

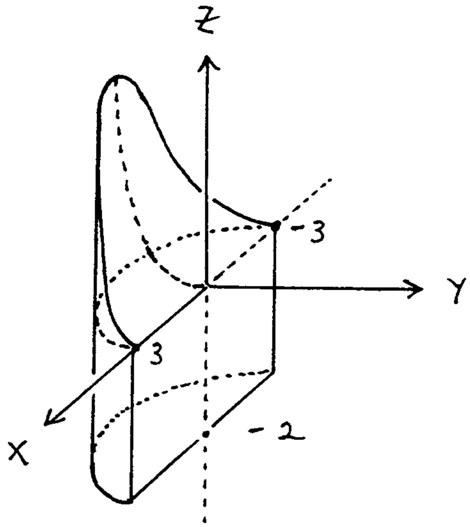
cyl. : $0 \le \theta < 2\pi$, $0 \le r \le F(\theta)$, $r^2 \le z \le 2r\sin\theta + 4$

spher. : $0 \le \theta < 2\pi$, $0 \le \phi \le \arctan\left({}^{1}\!/_{F(\theta)}\right)$, $0 \le \rho \le H(\theta,\phi)$ and
$0 \le \theta < 2\pi$, $\arctan\left({}^{1}\!/_{F(\theta)}\right) \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le G(\phi)$.

2.) a.)

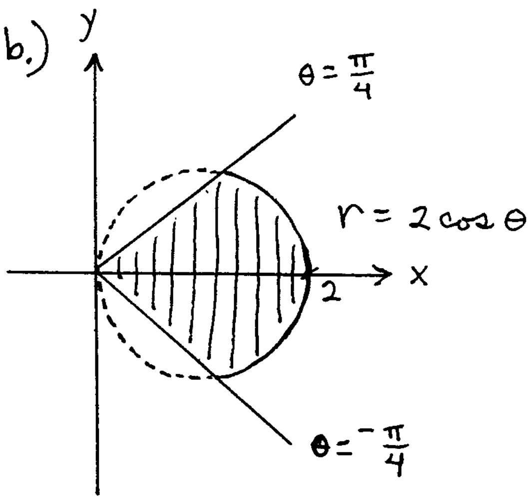


Y
-3
3
X
-3

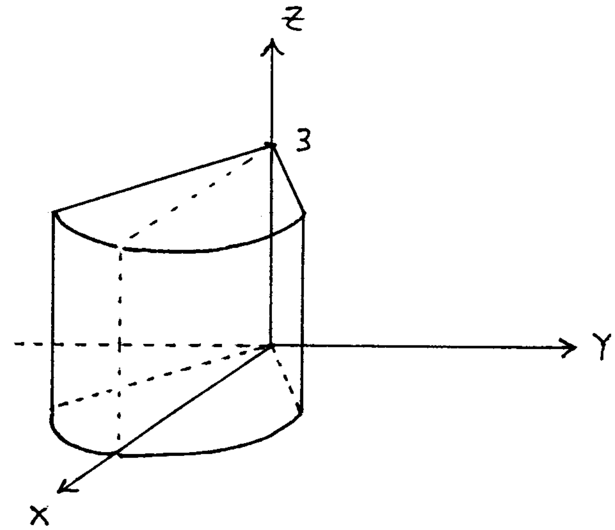


Z
-3
Y
3
X
-2

b.)

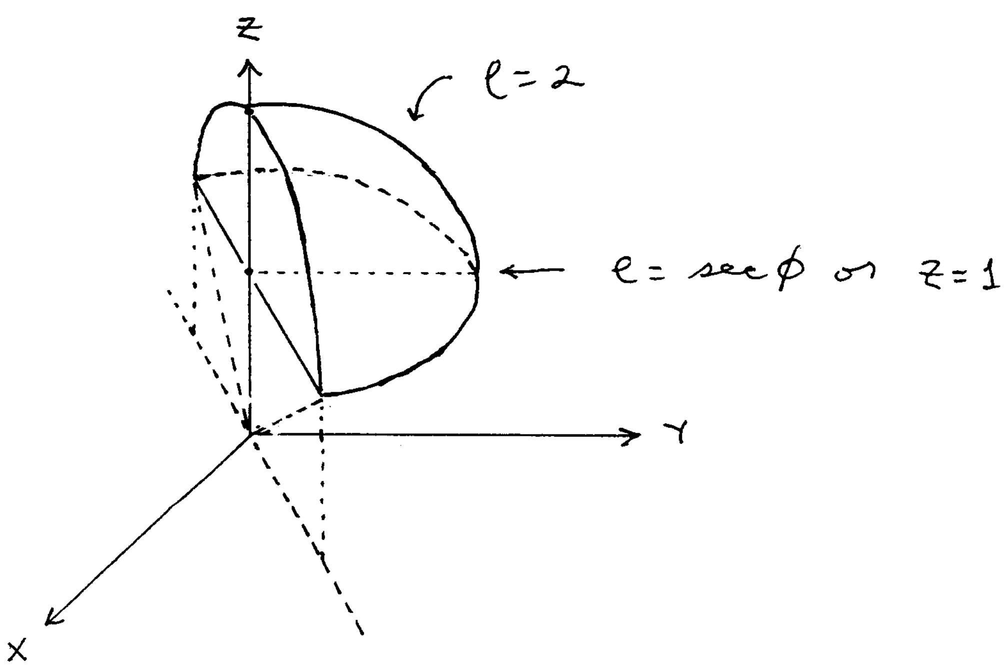


y
$\theta = \frac{\pi}{4}$
$r = 2\cos\theta$
X
2
$\theta = -\frac{\pi}{4}$

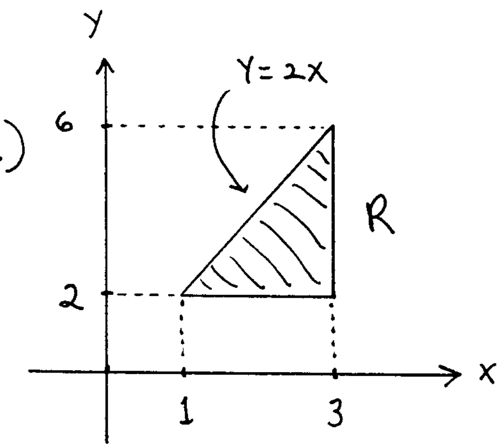


Z
3
Y
X

c.)



Z
$\rho = 2$
$\rho = \sec\phi$ or $z = 1$
Y
X

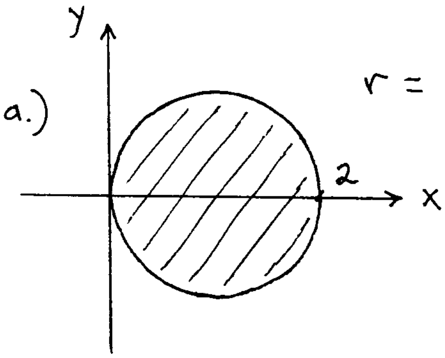
3.) 

$$\text{AVE} = \frac{1}{\text{area } R} \int_1^3 \int_2^{2x} (x+y)\,dy\,dx$$

$$= \frac{1}{4} \int_1^3 \left(xy + \frac{1}{2}y^2\right)\Big|_{y=2}^{y=2x} dx$$

$$= \frac{1}{4} \int_1^3 (4x^2 - 2x - 2)\,dx$$

$$= \frac{1}{4}\left(\frac{4}{3}x^3 - x^2 - 2x\right)\Big|_1^3 = \frac{17}{3}$$

4.) a.) 

$r = 2\cos\theta \rightarrow \theta = \arccos\frac{r}{2}$

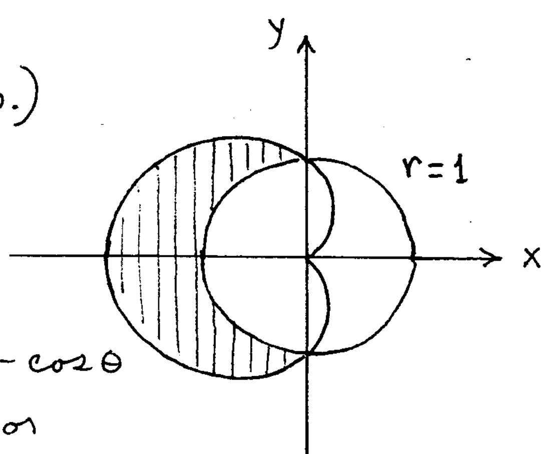
for $0 \le \theta \le \frac{\pi}{2}$ and

$\theta = -\arccos\frac{r}{2}$

for $\quad \le \theta \le \quad$;

i.) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2\cos\theta$

ii.) $0 \le r \le 2, \quad -\arccos\frac{r}{2} \le \theta \le +\arccos\frac{r}{2}$

b.) 

i.) $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, \quad 1 \le r \le 1 - \cos\theta$

ii.) $1 \le r \le 2$,

$\arccos(1-r) \le \theta \le 2\pi - \arccos(1-r)$

$r = 1 - \cos\theta$

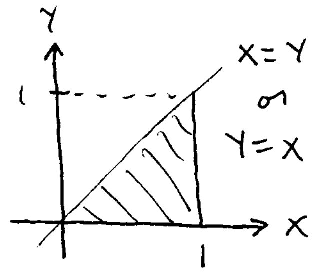
or

$\theta = \arccos(1-r)$ for $\frac{\pi}{2} \le \theta \le \pi$ and

$\theta = 2\pi - \arccos(1-r)$ for $\pi \le \theta \le \frac{3\pi}{2}$

5.) a.) $\int_1^2 \int_1^x \frac{x^2}{y^2}\, dy\, dx = \int_1^2 \left. \frac{-x^2}{y} \right|_{y=1}^{y=x} dx$

$= \int_1^2 (-x + x^2)\, dx = \left(\frac{-1}{2}x^2 + \frac{1}{3}x^3 \right)\Big|_1^2 = 5/6$

b.) 

$\int_0^1 \int_y^1 \sqrt{1+x^2}\, dx\, dy = \int_0^1 \int_0^x \sqrt{1+x^2}\, dy\, dx$

$= \int_0^1 x\sqrt{1+x^2}\, dx$

$= \frac{1}{3}\left(1+x^2\right)^{3/2}\Big|_0^1 = \frac{1}{3}(2)^{3/2} - \frac{1}{3}$

c.) $\int_2^3 \left. x^2 \right|_{-\sqrt{18-2y^2}}^{+\sqrt{18-2y^2}} dy = \int_2^3 0\, dy = 0$

d.) $\int_1^2 \int_0^{\sqrt{\ln\theta}} \int_0^{2r\theta} \frac{e^{r^2}}{\theta^2+4}\, dz\, dr\, d\theta = \int_1^2 \int_0^{\sqrt{\ln\theta}} \frac{2r\theta e^{r^2}}{\theta^2+4}\, dr\, d\theta$

$= \int_1^2 \frac{\theta}{\theta^2+4} \left. e^{r^2} \right|_{r=0}^{r=\sqrt{\ln\theta}} d\theta = \int_1^2 \frac{\theta\cdot(\theta-1)}{\theta^2+4}\, d\theta$

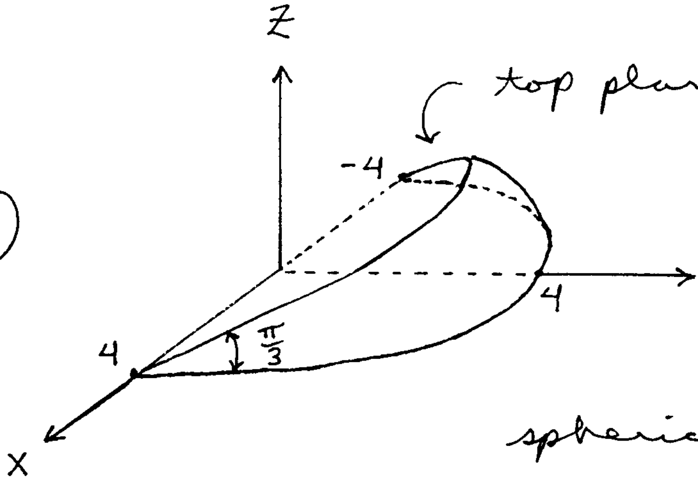
$= \int_1^2 \left(\frac{\theta^2}{\theta^2+4} - \frac{\theta}{\theta^2+4} \right) d\theta = \int_1^2 \left[\frac{\theta^2+4}{\theta^2+4} + \frac{-4}{\theta^2+4} - \frac{\theta}{\theta^2+4} \right] d\theta$

$= \left(\theta - 4\cdot\frac{1}{2}\arctan\frac{\theta}{2} - \frac{1}{2}\ln(\theta^2+4) \right)\Big|_1^2$

$= \left(2 - 2\cdot\frac{\pi}{4} - \frac{1}{2}\ln 8\right) - \left(1 - 2\cdot\arctan\frac{1}{2} - \frac{1}{2}\ln 5\right)$

$= 1 - \frac{\pi}{2} + 2\arctan\frac{1}{2} - \frac{1}{2}\ln 8 + \frac{1}{2}\ln 5$

6.)



top plane : $z = \sqrt{3}\, y$ or

$\rho \cos\phi = \sqrt{3}\, \rho \sin\theta \sin\phi \rightarrow$

$\phi = \operatorname{arccot}(\sqrt{3} \sin\theta)$

so solid R in spherical coordinates is

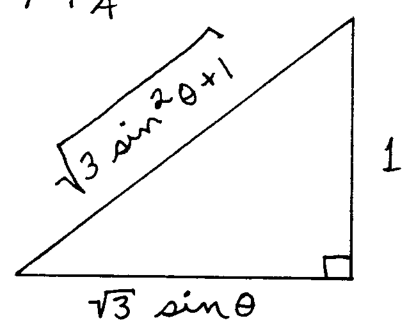
$0 \le \theta \le \pi$, $\underbrace{\operatorname{arccot}(\sqrt{3}\sin\theta)}_{A} \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le 4$ so

$$\text{Volume} = \int_R 1\, dV = \int_0^{\pi} \int_A^{\frac{\pi}{2}} \int_0^4 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi} \int_A^{\frac{\pi}{2}} \frac{64}{3} \sin\phi \, d\phi\, d\theta = -\frac{64}{3} \int_0^{\pi} \cos\phi \Big|_A^{\frac{\pi}{2}} d\theta$$

$$= \frac{64}{3} \int_0^{\pi} \cos A \, d\theta$$

$$= \frac{64}{3} \int_0^{\pi} \cos(\operatorname{arccot}(\sqrt{3}\sin\theta))\, d\theta \longrightarrow$$



$$= \frac{64}{3} \int_0^{\pi} \frac{\sqrt{3}\sin\theta}{\sqrt{3\sin^2\theta + 1}}\, d\theta$$

by symmetry

$$= \frac{64}{\sqrt{3}} \int_0^{\pi} \frac{\sin\theta}{\sqrt{3\sin^2\theta + 1}}\, d\theta = \frac{128}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{3\sin^2\theta + 1}}\, d\theta$$

(let $u = \sin\theta$, $\cos\theta = \sqrt{1-u^2}$, $du = \cos\theta\, d\theta$)

$$= \frac{128}{\sqrt{3}} \int_0^1 \frac{u}{\sqrt{3u^2+1}} \cdot \frac{1}{\sqrt{1-u^2}}\, du$$

$$= \frac{128}{\sqrt{3}} \int_0^1 \frac{u}{\sqrt{-3(u^2-\frac{1}{3})^2 + \frac{4}{3}}}\, du$$

$$= \frac{128}{\sqrt{3}} \int_0^1 \frac{u}{\sqrt{\frac{4}{3} - (\sqrt{3}u^2 - \frac{\sqrt{3}}{3})^2}}\, du$$

$$= \frac{\frac{128}{\sqrt3}}{\frac{2}{\sqrt3}} \int_0^1 \frac{u}{\sqrt{1-\left(\frac{\sqrt3 u^2 - \frac{\sqrt3}{3}}{2/\sqrt3}\right)^2}}\, du$$

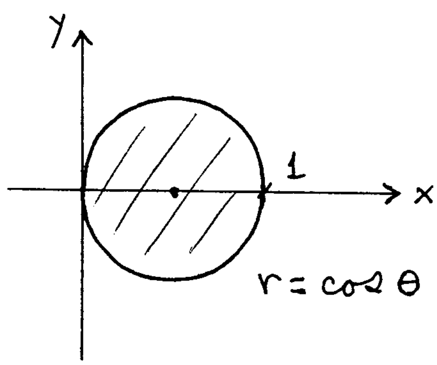
$$= 64 \int_0^1 \frac{u}{\sqrt{1-\left(\frac32 u^2 - \frac12\right)^2}}\, du$$

$$= 64 \cdot \frac13 \arcsin\left(\frac32 u^2 - \frac12\right)\Big|_0^1$$

$$= \frac{64}{3}\left(\arcsin 1 - \arcsin\left(-\tfrac12\right)\right)$$

$$= \frac{64}{3}\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \frac{128}{9}\pi .$$

7.) $\left.\begin{array}{l} z = x^2 + y^2 \\ z = x \end{array}\right\}$ $x^2 + y^2 = x \to x^2 - x + \frac14 + y^2 = \frac14 \to$
$(x - \frac12)^2 + y^2 = (\frac12)^2$ so



solid R in cylindrical coordinates is

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$,

$0 \le r \le \cos\theta$,

$r^2 \le z \le r\cos\theta$ so

$$\text{Volume} = \int_R 1\, dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_{r^2}^{r\cos\theta} r\, dz\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} rz \Big|_{z=r^2}^{z=r\cos\theta} dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} (r^2\cos\theta - r^3)\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \cos\theta - \frac{1}{4} r^4 \right) \Big|_{r=0}^{r=\cos\theta} \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{12} \cos^4\theta \, d\theta = \frac{1}{12} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}(1+\cos 2\theta) \right)^2 d\theta$$

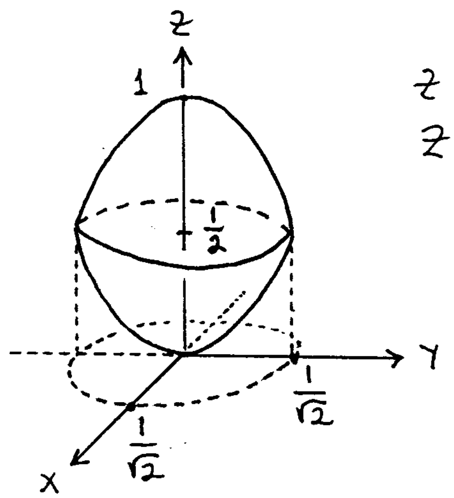
$$= \frac{1}{48} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta) \right) d\theta$$

$$= \frac{1}{48} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta$$

$$= \frac{1}{48} \left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{48} \cdot \frac{3}{2} \cdot \left(\frac{\pi}{2} - -\frac{\pi}{2} \right) = \frac{1}{32}\pi \, .$$

8.)



$$\left. \begin{aligned} z &= 1 - x^2 - y^2 \\ z &= x^2 + y^2 \end{aligned} \right\} \quad 1 - x^2 - y^2 = x^2 + y^2 \rightarrow$$

$$x^2 + y^2 = \frac{1}{2}$$

a.) $\frac{-1}{\sqrt{2}} \le x \le \frac{1}{2}$,

$-\sqrt{\frac{1}{2} - x^2} \le y \le +\sqrt{\frac{1}{2} - x^2}$,

$x^2 + y^2 \le z \le 1 - x^2 - y^2$

b.) $0 \le \theta < 2\pi$, $0 \le r \le \frac{1}{\sqrt{2}}$, $r^2 \le z \le 1 - r^2$

c.) $z = 1 - x^2 - y^2 \rightarrow \rho \cos\phi = 1 - \rho^2 \sin^2\phi \rightarrow$

$\rho^2 \sin^2\phi + \rho \cos\phi - 1 = 0 \rightarrow$

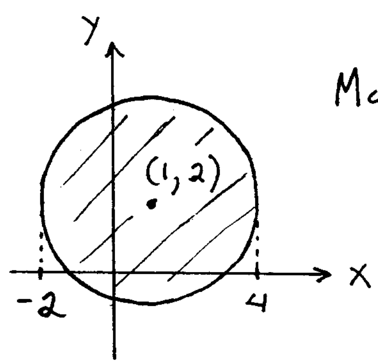
$$\rho = \frac{-\cos\phi \pm \sqrt{\cos^2\phi + 4\sin^2\phi}}{2\sin^2\phi}$$

$$= \frac{-\cos\phi + \sqrt{1 + 3\sin^2\phi}}{2\sin^2\phi} = F(\phi) \ ;$$

$$z = x^2 + y^2 \rightarrow \rho\cos\phi = \rho^2\sin^2\phi \rightarrow \rho = \csc\phi\cot\phi \ ;$$

$0 \le \theta < 2\pi$, $0 \le \phi \le \arctan\sqrt{2}$, $0 \le \rho \le F(\phi)$ and
$0 \le \theta < 2\pi$, $\arctan\sqrt{2} \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le \csc\phi\cot\phi$.

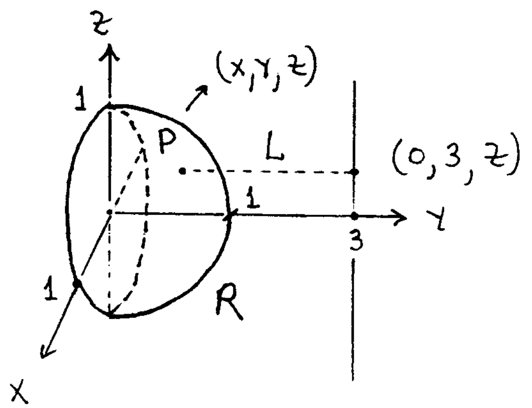
9.) $$\left.\begin{array}{l} z = x^2 + y^2 \\ z = 4 + 2x + 4y \end{array}\right\} \quad \begin{array}{l} x^2 + y^2 = 4 + 2x + 4y \rightarrow \\ x^2 - 2x + 1 + y^2 - 4y + 4 = 9 \rightarrow \end{array}$$
$(x-1)^2 + (y-2)^2 = 3^2$



$$\text{Mass} = \int_R \delta(P)\, dV$$

$$= \int_{-2}^{4} \int_{2-\sqrt{9-(x-1)^2}}^{2+\sqrt{9-(x-1)^2}} \int_{x^2+y^2}^{4+2x+4y} \sqrt{x^2+y^2}\; dz\, dy\, dx$$

10.)



distance L squared is

$$f(P) = x^2 + (y-3)^2$$

so average value of f is

$$AVE = \frac{1}{\text{vol } R} \int_R f(P)\, dV$$

$$= \frac{1}{\frac{1}{2}\cdot\frac{4}{3}\pi(1)^3} \int_0^{\pi}\int_0^{\pi}\int_0^{1} f(P)\cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{1} (\rho^2\sin^2\phi - 6\rho\sin\theta\sin\phi + 9)\,\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{1} \left(\rho^4\sin^3\phi - 6\rho^3\sin\theta\sin^2\phi + 9\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta$$

$$= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \left(\frac{1}{5}\sin^3\phi - \frac{3}{2}\sin\theta\cdot\sin^2\phi + 3\sin\phi\right) d\phi\, d\theta$$

$$= \frac{3}{2\pi}\int_0^{\pi}\int_0^{\pi} \left(\frac{1}{5}(\sin\phi - \sin\phi\cos^2\phi) - \frac{3}{2}\sin\theta\cdot\frac{1}{2}(1-\cos 2\phi) + 3\sin\phi\right) d\phi\, d\theta$$

$$= \frac{3}{2\pi}\int_0^{\pi}\left[\frac{1}{5}\left(-\cos\phi + \frac{1}{3}\cos^3\phi\right) - \frac{3}{4}\sin\theta\left(\phi - \frac{1}{2}\sin 2\phi\right) - 3\cos\phi\right]_{\phi=0}^{\phi=\pi} d\theta$$

$$= \frac{3}{2\pi}\int_0^{\pi}\left[\frac{94}{15} - \frac{3}{4}\sin\theta\cdot\pi\right] d\theta$$

$$= \frac{3}{2\pi}\left[\frac{94}{15}\theta + \frac{3}{4}\pi\cos\theta\right]\Big|_0^{\pi}$$

$$= \frac{143}{20}$$