ESP
Kouba
Worksheet 9 Solutions

1.) a.) $\sum_{n=3}^{\infty} 2^{-n} = \sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series since $r = \frac{1}{2}$ satisfies $-1 < r < 1$.

b.) $\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n$ is a divergent geometric series since $r = \frac{\pi}{e} > 1$.

c.) $\lim_{n\to\infty} n \cdot \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \overset{\frac{0}{0}}{=} \lim_{n\to\infty} \frac{\cos\frac{1}{n} \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}}$

$= \cos 0 = 1 \neq 0$ so $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$ diverges by the nth-term test.

d.) limit comparison : $\lim_{n\to\infty} \frac{\arcsin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{\frac{0}{0}}{=} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}} \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = 1$

so $\sum_{n=2}^{\infty} \arcsin\left(\frac{1}{n}\right)$ diverges since $\sum_{n=2}^{\infty} \frac{1}{n}$ is a divergent p-series $(p=1)$.

e.) $\lim_{n\to\infty} \frac{3^n}{2^n+3^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n + 1} = \frac{1}{0+1} = 1 \neq 0$ so $\sum_{n=0}^{\infty} \frac{3^n}{2^n+3^n}$ diverges by nth-term test.

f.) ratio test : $\lim_{n\to\infty} \frac{2^{(n+1)+1}}{(n+1)!} \cdot \frac{n!}{2^{n+1}} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

so $\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!}$ converges.

g.) ratio test: $\lim_{n\to\infty} \frac{e^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{e^n} = \lim_{n\to\infty} \frac{e}{(2n+2)(2n+1)} = 0 < 1$

so $\sum_{n=1}^{\infty} \frac{e^n}{(2n)!}$ converges.

h.) ratio test: $\lim_{n\to\infty} \frac{(n+1)^6}{8^{(n+1)+1}} \cdot \frac{8^{n+1}}{n^6} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^6 \cdot \frac{1}{8} = \frac{1}{8} < 1$

so $\sum_{n=1}^{\infty} \frac{n^6}{8^{n+1}}$ converges.

i.) ratio test: $\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$

so $\sum_{n=0}^{\infty} \frac{n^n}{n!}$ diverges.

j.) root test: $\lim_{n\to\infty} \left(\frac{3^{n+1}}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{3^{\frac{n+1}{n}}}{n} = \text{"}\frac{3}{\infty}\text{"} = 0 < 1$

so $\sum_{n=1}^{\infty} \frac{3^{n+1}}{n^n}$ converges.

k.) root test: $\lim_{n\to\infty} \left(\frac{n}{2n+3}\right)^{n\cdot\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{2+\frac{3}{n}}\right) = \frac{1}{2} < 1$

so $\sum_{n=0}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converges.

l.) limit comparison: $\lim_{n\to\infty} \frac{\frac{7}{n^2+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{7n^2}{n^2+3}$

$= \lim_{n\to\infty} \left(\frac{7}{1+\frac{3}{n^2}}\right) = 7$ so $\sum_{n=7}^{\infty} \frac{7}{n^2+3}$ converges

since $\sum_{n=7}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2$).

m.) limit comparison : $\lim\limits_{n\to\infty} \dfrac{\sqrt{\frac{n}{n^2-1}}}{\frac{1}{\sqrt{n}}} = \lim\limits_{n\to\infty} \sqrt{\dfrac{n}{n^2-1}} \cdot \sqrt{n}$

$= \lim\limits_{n\to\infty} \sqrt{\dfrac{n^2}{n^2-1}} = \lim\limits_{n\to\infty} \sqrt{\dfrac{1}{1-\frac{1}{n^2}}} = 1$ so

$\sum\limits_{n=2}^{\infty} \sqrt{\dfrac{n}{n^2-1}}$ diverges since $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$ is a

divergent p-series $(p=\frac{1}{2})$.

n.) $\lim\limits_{n\to\infty} \ln n = \infty \neq 0$ so $\sum\limits_{n=1}^{\infty} \ln n$ diverges

by the nth-term test.

o.) limit comparison : $\lim\limits_{n\to\infty} \dfrac{\frac{1}{\ln n}}{\frac{1}{\sqrt{n}}} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{2} = \infty$ so $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges since

$\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$ is a divergent p-series $(p=\frac{1}{2})$.

p.) integral test : $f(x) = \dfrac{1}{x \ln x}$ is +, continuous,

and decreasing and $\int_2^{\infty} \dfrac{1}{x \ln x}\,dx = \lim\limits_{b\to\infty} \ln(\ln x)\Big|_2^b$

$= \lim\limits_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$ so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges.

q.) sequence of partial sums : $\ln\left(\dfrac{n+1}{n}\right) = \ln(n+1) - \ln n$

$S_1 = \ln 2 - \ln 1$

$S_2 = (\ln 3 - \cancel{\ln 2}) - (\cancel{\ln 2} - \ln 1)$

$$s_3 = (\ln 4 - \cancel{\ln 3}) + (\cancel{\ln 3} - \cancel{\ln 2}) + (\cancel{\ln 2} - \ln 1)$$

$\vdots$

$$s_n = \ln(n+1) - \cancel{\ln 1}^{\,0} = \ln(n+1) \quad \text{and}$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln n] = \lim_{n\to\infty} s_n$$

$$= \lim_{n\to\infty} \ln(n+1) = \infty \quad \text{so series diverges.}$$

r.) nth term test: $\lim\limits_{n\to\infty} \ln(\ln n)^{1/n} = \lim\limits_{n\to\infty} \dfrac{\ln(\ln n)}{n}$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1} = 0 \quad \text{so} \quad \lim_{n\to\infty} (\ln n)^{1/n} = 1 \quad \text{and}$$

$$\lim_{n\to\infty} \frac{1}{(\ln n)^{1/n}} = \frac{1}{1} = 1 \neq 0 \quad \text{so} \quad \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{1/n}} \text{ diverges.}$$

s.) root test: $\lim\limits_{n\to\infty} \left(\dfrac{1}{(\ln n)^n}\right)^{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0 < 1$ so

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n} \text{ converges.}$$

t.) limit comparison: $\lim\limits_{n\to\infty} \dfrac{\frac{2}{1+3^n}}{\left(\frac{1}{3}\right)^n} = \lim\limits_{n\to\infty} \dfrac{2\cdot 3^n}{1+3^n} = \lim\limits_{n\to\infty} \dfrac{2}{\frac{1}{3^n}+1}$

$$= 2 \quad \text{so} \quad \sum_{n=0}^{\infty} \frac{2}{1+3^n} \text{ converges since } \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

is a convergent geometric series $(r = \frac{1}{3})$.

u.) limit comparison: $\lim\limits_{n\to\infty} \dfrac{\frac{100n}{n^2+2}\left(\frac{1}{4}\right)^{2n}}{\left(\frac{1}{16}\right)^n} = \lim\limits_{n\to\infty} \dfrac{100n}{n^2+2}$

$= \lim_{n\to\infty} \left(\frac{\frac{100}{n}}{1+\frac{2}{n^2}} \right) = \frac{0}{1+0} = 0$ so $\sum_{n=0}^{\infty} \frac{100n}{n^2+2} \left(\frac{1}{4}\right)^{2n}$

converges since $\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n$ is a convergent geometric series $\left(r = \frac{1}{16}\right)$.

v.) alternating series test: $a_n = \frac{1}{n+7}$ is $+, \downarrow$, and $\to 0$ so $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+7}$ converges

w.) nth term test: $\lim_{n\to\infty} (-1)^n \frac{3n^2+1}{5n^2+9} = \lim_{n\to\infty} (-1)^n \cdot \frac{3+\frac{1}{n^2}}{5+\frac{9}{n^2}}$

$\neq 0$ so $\sum_{n=1}^{\infty} (-1)^n \frac{3n^2+1}{5n^2+9}$ diverges.

x.) alternating series test: $a_n = \frac{n}{(n+5)^2}$ is $+, \downarrow$, and $\to 0$ for $n \geq 6$ so $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n}{(n+5)^2}$ converges.

y.) $\sum_{n=3}^{\infty} \frac{1+2(-1)^n}{n^2} = \frac{-1}{3^2} + \frac{3}{4^2} + \frac{-1}{5^2} + \frac{3}{6^2} + \frac{-1}{7^2} + \frac{3}{8^2} + \cdots$;

since $\frac{1}{3^2} + \frac{3}{4^2} + \frac{1}{5^2} + \frac{3}{6^2} + \cdots < \frac{3}{3^2} + \frac{3}{4^2} + \frac{3}{5^2} + \frac{3}{6^2} + \cdots$

$= 3 \sum_{n=3}^{\infty} \frac{1}{n^2} < \infty$, $\sum_{n=3}^{\infty} \frac{1+2(-1)^n}{n^2}$ converges by the absolute convergence test.

z.) $\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n = \frac{1}{1-\left(\frac{-1}{3}\right)} = \frac{3}{4}$ is a convergent geometric series.

A.) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ so

$1-\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}-\frac{1}{243}+\cdots$ converges by the absolute convergence test.

B.) limit comparison: $\lim_{n\to\infty}\frac{\frac{1}{n+\ln n^2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+2\ln n}$

$\overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}} = 1$ so $\sum_{n=1}^{\infty}\frac{1}{n+\ln n^2}$ diverges

since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. (harmonic series)

C.) root test: $\lim_{n\to\infty}\left[\frac{\ln n}{\ln(n^2+3)}\right]^{n\cdot\frac{1}{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{2n}{n^2+3}}$

$= \lim_{n\to\infty}\frac{n^2+3}{2n^2} = \frac{1}{2}<1$ so $\sum_{n=1}^{\infty}\left[\frac{\ln n}{\ln(n^2+3)}\right]^n$ converges

D.) $\frac{\sqrt{n+1}-\sqrt{n}}{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$ so limit comparison:

$\lim_{n\to\infty}\frac{\frac{1}{n(\sqrt{n+1}+\sqrt{n})}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+n^2}+n^{3/2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

so $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges since $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges

(p-series, $p=\frac{3}{2}$)

E.) root test: $\lim_{n\to\infty}\left(\frac{n+1}{n+3}\right)^{n^2\cdot\frac{1}{n}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}\cdot 3}} = \frac{e}{e^3} = \frac{1}{e^2}<$

so $\sum_{n=0}^{\infty}\left(\frac{n+1}{n+3}\right)^{n^2}$ converges.

F.) Let $f(x)=\frac{1}{4+x^2}$ on $[0,1]$ and $x_i=\frac{i}{n}$ so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{4+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$$

$$= \int_0^1 f(x)\,dx = \frac{1}{2}\arctan\frac{x}{2}\Big|_0^1 = \frac{1}{2}\arctan\frac{1}{2}$$

G.) Let $f(x)=\sqrt{2+x}$ on $[0,3]$ and $x_i = \frac{3i}{n}$ so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2+\frac{3i}{n}} \cdot \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$$

$$= \int_0^3 f(x)\,dx = \frac{2}{3}(2+x)^{3/2}\Big|_0^3 = \frac{2}{3}(5)^{3/2} - \frac{2}{3}(2)^{3/2}$$

2.) a.) $a_n = \frac{1}{n^5}$ so $|R_n| < a_{n+1} = \frac{1}{(n+1)^5} < 0.00001 \rightarrow$

$100{,}000 < (n+1)^5 \rightarrow 10 < n+1 \rightarrow 9 < n \rightarrow \boxed{n=10}$

b.) $a_n = \frac{1}{n}$ so $|R_n| < a_{n+1} = \frac{1}{n+1} < 0.00001 \rightarrow$

$100{,}000 < n+1 \rightarrow 99{,}999 < n \rightarrow \boxed{n = 100{,}000}$

c.) $f(x) = \frac{1}{1+x^2}$ is $+$, $\downarrow$, continuous so

$$R_n < \int_n^\infty \frac{1}{1+x^2}\,dx = \lim_{B\to\infty} \arctan x \Big|_n^B$$

$$= \lim_{B\to\infty}(\arctan B - \arctan n) = \frac{\pi}{2} - \arctan n < 0.00001$$

$$\rightarrow \frac{\pi}{2} - 0.00001 < \arctan n \rightarrow \tan\left(\frac{\pi}{2} - 0.00001\right) < n$$

$$\rightarrow 99999.9 < n \rightarrow \boxed{n = 100{,}000}$$