Proof by Mathematical Induction
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Proof by Mathematical Induction is a method used to formally verify the validity of a statement which is true for all numbers in the set of natural numbers $N = \{1, 2, 3, 4, 5, ...\}$. (The following method of proof can be used if the set under consideration is the set N with a finite number of integers removed from the beginning of the set. For example, the method of proof by induction could be applied to a set like $S = \{3, 4, 5, ...\}$. In such an instance we will refer to the process as the "Generalized" Proof by Mathematical Induction.)

PROOF BY MATHEMATICAL INDUCTION : Prove that a given statement is true for all natural numbers $n \in N$.

Step i.) Illustrate that the statement is true for $n = 1$.

Step ii.) ASSUME that the statement is true for some fixed natural number k. This step is called the INDUCTIVE HYPOTHESIS.

Step iii.) SHOW that the statement is true for $k + 1$, by USING the inductive hypothesis. The completion of this step is the end of the proof.

PROBLEMS :

1.) Check that each of the following statements is true for $n = 1, n = 2, n = 3$ and $n = 4$. Then use the method of Proof by Mathematical Induction to prove each of the following statements.

a.) Prove that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \in N$.

b.) Prove that $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$ for all $n \in N$.

c.) Prove that $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$ for all $n \in N$.

d.) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in N$.

e.) Prove that $1 + r + r^2 + r^3 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for all $r \neq 1$ and for all $n \in N$.

f.) Prove that $(n + 3)! > 5^n$ for all $n \in N$.

g.) Prove that $4^n - 1$ is divisible by 3 for all $n \in N$. (Definition: Integer w is divisible by 3 means $w = 3m$ for some integer m.)

h.) Prove that $10^n + 48(4^n) + 5$ is divisible by 9 for all $n \in N$. (Definition: Integer w is divisible by 9 means $w = 9m$ for some integer m.)

i.) (For those who have taken differential calculus) Prove that if $f(x) = x^n$, then its derivative is $f'(x) = n \cdot x^{n-1}$ for all $n \in N$.

j.) Prove that $\dfrac{n^4}{4} < 1^3 + 2^3 + 3^3 + \cdots + n^3$ for all $n \in N$.

2.) Use the method of "Generalized" Proof by Mathematical Induction to prove each of the following statements.

a.) Prove that $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{(n-1) \cdot n} = 1 - \dfrac{1}{n}$ for all natural numbers $n \geq 2$.

b.) Prove that $2^n > n^2$ for all natural numbers $n \geq 5$.

c.) Prove that $\sqrt{n} < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$.

d.) Prove that $2(3^n) > (n + 1)^3$ for all natural numbers $n \geq 4$.

3.) Determine a formula for each of the following and then PROVE it is valid using Proof by Mathematical Induction.

a.) $1^2 + 4^2 + 7^2 + \cdots + (3n + 1)^2$ where $n \in N$

b.) $1^3 + 2^3 + 3^3 + \cdots + n^3$ where $n \in N$

c.) $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$ where $n \in N$