Permutations

Consider all of the different *ordered* arrangements of the letters, a, b, and c. They are :

abc, acb, bac, bca, cab, cba

These ordered arrangements are called permutations of the letters a, b, and c. Now consider all of the different ordered arrangements of the letters a, b, and c by *taking only 2 letters at a time* :

ab, ba, ac, ca, bc, cb

PROBLEMS :

1.) List all permutations of the digits 1, 3, 5, and 7 if we take

   a.) 1 number at a time.

   b.) 2 numbers at a time.

   c.) 3 numbers at a time.

   d.) 4 numbers at a time.

Definition : The symbol $n!$, read n factorial, is defined in the following way: $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, and for any natural number $n \geq 1$,

$$n! = n \cdot (n-1)! \, .$$

Examples :

1.) $3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1 = 6$

2.) $4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

3.) $7! = 7 \cdot 6! = 7 \cdot 6 \cdot 5! = 7 \cdot 6 \cdot 5 \cdot 4! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

Shortcut : We have $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$ and for $n = 3, 4, 5 \cdots$ we can write

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \, .$$

Example :

4.) $9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880$

PROBLEMS :

2.) Simplify each factorial expression.

a.) $4!$

b.) $10!$

c.) $\dfrac{7!}{5!}$

d.) $\dfrac{(7!)^2}{5! \cdot 3!}$

Consider all of the distinct ordered arrangements using any four letters in the word *upgrade* . Examples are *grad, rade, ugde, drag, erap,* etc. These are called permutations of 7 letters taken 4 at a time, $P(7,4)$. Using the Fundamental Principle of Counting and considering this exercise as a four-step process (Step 1– Choose the first letter. Step 2– Choose the second letter. Step 3– Choose the third letter. Step 4– Choose the fourth letter.), it follows that

$$P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840 \ .$$

Note also that

$$P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{7!}{3!} = \frac{7!}{(7-4)!} \ .$$

This example can be easily generalized using the Fundamental Principle of Counting in the following manner.

PERMUTATION RULE : The number of permutations of n distinct objects taken k at a time is

$$P(n,k) = \frac{n!}{(n-k)!} \ .$$

PROBLEMS :

3.) Simplify the following permutations.

a.) $P(6,3)$

b.) $P(7,4)$

c.) $P(5,0)$

d.) $P(5,5)$

e.) $P(0,0)$

PROBLEMS : Use permutations to solve the following problems.

4.) Thirteen different students, 8 women and 5 men, are running for student council seats at the local high school. There are 5 positions to fill– president, vice-president, secretary, treasurer, and fundraiser– and no student can hold more than 1 position.

a.) How many distinct outcomes are there to this election ?

b.) How many distinct outcomes are there to this election if all of the positions must be filled

i.) by women ?

ii.) by men ?

c.) How many distinct outcomes are there to this election if the president must be a woman and the other positions can be filled by anyone?

d.) How many distinct outcomes are there to this election if the president must be a man, the vice-president must be a woman, and the other positions can be filled by anyone ?

e.) How many distinct outcomes are there to this election if the president, treasurer, and fundraiser must be women and the other positions must be men ?

f.) How many distinct outcomes are there to this election if the vice-president must be a man, the treasurer and secretary must be women, and the other positions can be filled by anyone ?

5.) A baseball coach has 15 players from which to choose a 9-player batting order. How many different choices does the coach have ?

6.) How many distinct permutations are there of the letters in the word

a.) *brown* ?

b.) *friendly* ?

c.) *dermatoglyphics* ? (This word with 15 letters is the longest word in the English language without a repeated letter. There is one other 15-letter word in the English language with no repeated letters. See if you can find it.)

d.) *leek* ? (HINT: The answer is not $4! = 24$.)

e.) *booboo* ? (HINT: The answer is not $6! = 720$.)

7.) How many distinct k-letter permutations are there of the letters in the word *bike* if

a.) $k = 1$?

b.) $k = 2$?

c.) $k = 3$?

d.) $k = 4$?

8.) How many distinct k-letter permutations are there of the letters in the word *dermatoglyphics* if

a.) $k = 0$?

b.) $k = 3$?

c.) $k = 7$?

d.) $k = 11$?

9.) How many distinct k-letter permutations are there of the letters in the word *moon* if

a.) $k = 1$?

b.) $k = 2$?

c.) $k = 3$?

d.) $k = 4$?