Summation Notation, Some Useful Formulae
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DEFINITION : Let $f(i)$ be some expression (function value) involving the whole number i. Then

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \cdots + f(n) .$$

EXAMPLE : $\sum_{i=1}^{4}(2i+3) = (2(1)+3) + (2(2)+3) + (2(3)+3) + (2(4)+3)$
$= 5 + 7 + 9 + 11 = 32$.

PROBLEMS :

1.) Evaluate the following sums.

a.) $\sum_{i=1}^{5}(7-4i)$

b.) $\sum_{i=6}^{11} 2i^2$

c.) $\sum_{i=1}^{100} 8$

d.) $\sum_{i=63}^{582} 2$

Some Useful Summation Formulae

1.) a.) If c is a constant, then $\sum_{i=1}^{n} c = c + c + c + \cdots + c \; (n \; times) = nc$.

b.) $\sum_{i=1}^{n}(f(i) \pm g(i)) = \sum_{i=1}^{n} f(i) \pm \sum_{i=1}^{n} g(i)$.

c.) $\sum_{i=1}^{n} cf(i) = c\sum_{i=1}^{n} f(i)$, where c is a constant.

2.) $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Example : $1 + 2 + 3 + 4 + \cdots + 20 = \dfrac{20(20+1)}{2} = 210$

3.) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Example : $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 30^2 = \dfrac{30(30+1)(2(30)+1)}{6} = 9455$

4.) (Geometric Sum) Let r be a nonzero real number. Then
$\sum_{i=0}^{n} r^i = 1 + r + r^2 + r^3 + \cdots + r^n = \dfrac{r^{n+1}-1}{r-1}$.

Example : $1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{15} = \dfrac{3^{15+1}-1}{3-1} = 21,523,360$

PROBLEMS :

2.) Use appropriate formulae to simplify or solve each of the following problems.

a.) $\sum_{i=1}^{1000} (3i - 4)$

b.) $\sum_{i=17}^{83} i^2$

c.) $\sum_{i=1}^{50} i(i+2)$

d.) $\sum_{i=1}^{30} [(i+1)^2 - i^2]$

e.) $\sum_{i=1}^{30} [(i+1)^6 - i^6]$

f.) $\sum_{i=1}^{200}(\sqrt{i+5}-\sqrt{i+4})$

3.) Evaluate the following expressions.

a.) $1+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots+\left(\frac{1}{3}\right)^{15}$

b.) $\left(\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^6-\left(\frac{1}{2}\right)^7+\cdots+\left(\frac{1}{2}\right)^{20}$

c.) $5^2+5^4+5^6+5^8+\cdots+5^{30}$

d.) $1+2+3+4+\cdots+100$

e.) $73+74+75+76+\cdots+342$

f.) $17+21+25+29+\cdots+401$

g.) $25+36+49+64+\cdots+2500$

h.) $3+6+10+15+\cdots+1891$

4.) a.) Simplify the value of $1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^n$, where n is a positive integer.

b.) What happens to your answer in part a.) as n gets larger and larger (n goes to infinity) ?

c.) Make a conjecture about $1+r+r^2+r^3+\cdots+r^n$ as n goes to infinity. Use your conjecture to evaluate each of the following.

i.) $1+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots$

ii.) $1-\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2-\left(\frac{2}{5}\right)^3+\left(\frac{2}{5}\right)^4-\cdots$

iii.) $1+(0.7)+(0.7)^2+(0.7)^3+\cdots$

iv.) $1+(1.1)+(1.1)^2+(1.1)^3+\cdots$

v.) $1 - 1 + 1 - 1 + 1 - 1 + \cdots$

vi.) $0.2 + 0.02 + 0.002 + 0.0002 + \cdots$

vii.) $0.35 + 0.0035 + 0.000035 + 0.00000035 + \cdots$

5.) Write each repeating decimal number as a rational number.

a.) $0.111111\cdots$

b.) $0.232323\cdots$

c.) $0.456456456\cdots$

6.) A checkerboard is 8 by 8. Place 1 penny on the first square, 2 pennies on the 2nd square, 4 pennies on the 3rd square, 8 pennies on the 4th square, 16 pennies on the 5th square, and continue this pattern until all of the squares have been used.

a.) What is the total value in dollars of all of the pennies used to cover the checkerboard ?

b.) How high is the stack of pennies on the 64th square ? (Assume that a 10-penny stack is about 5/8 inches high.)

7.) Assume that a very large piece of paper is 1/64 of an inch thick. Cut the piece of paper in half and stack the halves on each other. Cut this stack in half and stack the halves on each other. Cut this stack in half and stack the halves on each other. Continue this process. How thick is the stack after 3 cuts ? after 6 cuts ? after 10 cuts ? after 16 cuts (convert to feet) ? after 23 cuts (convert to miles) ? after 34 cuts (convert to miles) ?

8.) A Super Ball is thrown straight up in the air. It goes 100 feet high and then drops to the ground and begins to bounce. Assume that on each bounce, the ball bounces 95% as high as its height on the previous bounce. What is the total (vertical) distance traveled by the Super Ball after the ball has struck the ground

i.) 1 time ?
ii.) 2 times ?
iii.) 3 times ?

iv.) 4 times ?

v.) 20 times ?

vi.) What is the total distance traveled by the Super Ball ? (For calculus students) Assuming that the acceleration due to gravity is -32 ft./sec.^{2}, how long will the ball bounce ?

9.) Use the result in part 4.)c.) to determine the value (rational number) of each infinite sum.

a.) $\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 5\left(\frac{1}{32}\right) + \cdots$

b.) $1 + 0.2 + 0.03 + 0.004 + 0.0005 + \cdots$