Math 16A
Kouba
Functions– Review

**DEFINITION**: In an equation composed of $x$'s and $y$'s, variable $y$ is a function of $x$ if each admissible $x$-value has exactly one $y$-value.

**NOTE**: The graph of a function passes the *vertical line test*. That is, a vertical line passed through the graph will touch the graph in at most one point.

**EXAMPLE**: Assume that $xy - 3 = x^2 + 2y$. Then $xy - 2y = x^2 + 3 \quad \longrightarrow$

$$(x - 2)y = x^2 + 3 \quad \longrightarrow$$

$$y = \frac{x^2 + 3}{x - 2} \quad \longrightarrow$$

$y$ is a function of $x$.

**EXAMPLE**: Assume that $xy^2 - 1 = x + y$. If $x = 1$, then

$$y^2 - 1 = 1 + y \quad \longrightarrow$$

$$y^2 - y - 2 = 0 \quad \longrightarrow$$

$$(y - 2)(y + 1) = 0 \quad \longrightarrow$$

$$y = 2 \text{ or } y = -1 \quad \longrightarrow$$

$x = 1$ has TWO $y$-values $\quad \longrightarrow$

$y$ is NOT a function of $x$.

**NOTATION**: If $y$ is a function of $x$, then we write $y = f(x)$. 

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**EXAMPLE**: If $y = x^2 + x$, then $y$ is a function of $x$ and we write $f(x) = x^2 + x$; then

a.) $f(-2) = (-2)^2 + (-2) = 4 - 2 = 2$.  
b.) $f(2x - 1) = (2x - 1)^2 + (2x - 1) = 4x^2 - 4x + 1 + 2x - 1 = 4x^2 - 2x$. 

**DEFINITION**: Assume that $y = f(x)$ is a function. The **domain** of function $f$ is the set of all admissible $x$-values. The **range** of function $f$ is the set of all corresponding $y$-values.

**EXAMPLE**: Consider function $f(x) = \sqrt{2x - 6}$. Then $2x - 6 \geq 0 \quad \rightarrow \quad 2x \geq 6 \quad \rightarrow \quad x \geq 3 \quad \rightarrow$

**DOMAIN**: $x \geq 3$.

Since $\sqrt{2x - 6} \geq 0$, $f(3) = 0$, and $2x - 6$ gets infinitely large as $x$ gets infinitely large, it follows that

**RANGE**: $y \geq 0$.

**DEFINITION**: A function $y = f(x)$ is **one-to-one** if each $y$-value has exactly one $x$-value. More precisely, a one-to-one function has the property that if $f(x_1) = f(x_2)$ ($y$-values are equal), then $x_1 = x_2$ ($x$-values are equal).

**NOTE**: The graph of a one-to-one function passes the **horizontal line test**. That is, a horizontal line passed through the graph will touch the graph in at most one point.

**EXAMPLE**: Consider the function (parabola) $y = x^2 - 5$. If $y = 4$, then

$$4 = x^2 - 5 \quad \rightarrow$$

$$x^2 = 9 \quad \rightarrow$$

$$x = 3 \text{ or } x = -3 \quad \rightarrow$$

$y = 4$ has **TWO** $x$-values \quad \rightarrow
function $y$ is NOT one-to-one.

**EXAMPLE**: Consider the function $f(x) = \frac{x}{x + 3}$. Prove that $f$ is one-to-one:

\[
f(x_1) = f(x_2) \quad \rightarrow \\
\frac{x_1}{x_1 + 3} = \frac{x_2}{x_2 + 3} \quad \rightarrow \\
x_1(x_2 + 3) = x_2(x_1 + 3) \quad \rightarrow \\
x_1x_2 + 3x_1 = x_1x_2 + 3x_2 \quad \rightarrow \\
3x_1 = 3x_2 \quad \rightarrow \\
x_1 = x_2 \quad \rightarrow 
\]

function $f$ IS one-to-one.

**DEFINITION**: Assume that $y = f(x)$ and $y = g(x)$ are functions. The composition of functions $f$ and $g$ is

\[(f \circ g)(x) = f(g(x)) .\]

**EXAMPLE**: Consider the functions $f(x) = \frac{x}{10 - x}$ and $g(x) = \frac{1}{x + 8}$.

Then

\[(f \circ g)(x) = f(g(x)) = f \left( \frac{1}{x + 8} \right) = \frac{1}{x + 8} \left( \frac{1}{x + 8} \right) = \frac{1}{10 - \left( \frac{1}{x + 8} \right)} .\]
\[ \frac{1}{10 - \left( \frac{1}{x + 8} \right)} \cdot \frac{x + 8}{x + 8} = \frac{1}{10(x + 8) - 1} = \frac{1}{10x + 79}. \]

**DEFINITION**: The inverse function of function \( y = f(x) \) is the function \( y = f^{-1}(x) \) for which
\[ f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x. \]

**FACT**: If \( y = f(x) \) is a one-to-one function, then \( f \) has an inverse function.

**SEE INVERSE FUNCTION HANDOUT.**

**EXAMPLE**: The function \( f(x) = \frac{x}{x + 3} \) is one-to-one. Find its inverse:

\[ y = \frac{x}{x + 3} \quad \rightarrow \quad \text{(Switch variables.)} \quad x = \frac{y}{y + 3} \]

(Solve for \( y \).) \( x(y + 3) = y \quad \rightarrow \)

\[ xy + 3x = y \quad \rightarrow \]

\[ xy - y = -3x \quad \rightarrow \]

\[ y(x - 1) = -3x \quad \rightarrow \]

\[ y = \frac{-3x}{x - 1} \quad \rightarrow \quad \text{inverse function is} \quad f^{-1}(x) = \frac{3x}{1 - x}. \]

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