Math 16A
Section 1.5

Limits of Functions

Definition: (Non-technical)
The limit as x approaches #a of f(x) equals L, written

$$\lim_{x \to a} f(x) = L,$$

means as x-values get closer and closer to #a, the corresponding y-values get closer and closer to #L. (Note: x approaches #a from both sides.)

Note: Think of the limit L as an "expected" y-value as x-values get closer and closer to #a.

Example: Find the following limits.

1.) $\lim_{x \to 2} (x^2 - 3x + 5) = (2)^2 - 3(2) + 5$

$$= 4 - 6 + 5 = 3$$

2.) $$\lim_{x \to -1} \frac{2-x}{x^3+4} = \frac{2-(-1)}{(-1)^3+4} = \frac{2+1}{-1+4}$$

$$= \frac{3}{3} = 1$$

3.) $$\lim_{x \to \frac{\pi}{2}} (\sin x + \cos(2x))$$

$$= \sin\left(\frac{\pi}{2}\right) + \cos\left(2\left(\frac{\pi}{2}\right)\right)$$

$$= 1 + \cos \pi = 1 + (-1) = 0$$

(NOTE: In the context we will often encounter the following INDETERMINATE Forms: "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$0 \cdot \infty$", "$\infty - \infty$", "1^{∞}", "∞^{0}", and "0^{0}". These forms simply mean we are "stuck" and need to do additional work to get an answer.)

4.) $\lim_{x\to 2} \dfrac{x^2+4x-12}{x^2-4} = \dfrac{(2)^2+4(2)-12}{(2)^2-4}$

$= \dfrac{4+8-12}{4-4} = \text{"}\dfrac{0}{0}\text{"}$ (so stuck; let's try factoring)

$= \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x+6)}{\cancel{(x-2)}(x+2)} = \dfrac{(2)+6}{(2)+2} = 2$

5.) $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-4x+3} = \text{"}\dfrac{0}{0}\text{"}$ (so factor)

$= \lim_{x\to 1} \dfrac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x-3)} = \dfrac{3}{-2} = \dfrac{-3}{2}$

6.) $\lim_{x\to -4} \dfrac{\frac{1}{x}+\frac{1}{4}}{x+4} = \text{"}\dfrac{0}{0}\text{"}$ (add fractions)

$= \lim_{x\to -4} \dfrac{\frac{1}{x}\cdot\frac{4}{4}+\frac{1}{4}\cdot\frac{x}{x}}{\frac{x+4}{1}} = \lim_{x\to -4} \dfrac{\cancel{4+x}}{4x}\cdot\dfrac{1}{\cancel{x+4}}$

$= \lim_{x\to -4} \dfrac{1}{4x} = \dfrac{-1}{16}$

7.) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} = \text{"}\dfrac{0}{0}\text{"}$ (use conjugate)

$$= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}$$ (RECALL: $(a-b)(a+b)=a^2-b^2$)

$$= \lim_{x\to 9} \frac{(\sqrt{x})^2-(3)^2}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

8.) $\lim_{x\to -2} \frac{x^2-4}{x^3+8} = \text{"}\frac{0}{0}\text{"}$

(RECALL: I.) $a^3+b^3=(a+b)(a^2-ab+b^2)$

II.) $a^3-b^3=(a-b)(a^2+ab+b^2)$)

$$= \lim_{x\to -2} \frac{x^2-4}{x^3+2^3} = \lim_{x\to -2} \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}(x^2-2x+4)}$$

$$= \frac{(-2)-2}{(-2)^2-2(-2)+4} = \frac{-4}{12} = -\frac{1}{3}$$

9.) $\lim_{x\to 1} \frac{x-1}{x^{1/3}-1} = \text{"}\frac{0}{0}\text{"}$ (rewrite top)

$$= \lim_{x\to 1} \frac{(x^{1/3})^3-(1)^3}{x^{1/3}-1}$$ (factor)

$$= \lim_{x\to 1} \frac{\cancel{(x^{1/3}-1)}\left((x^{1/3})^2+(x^{1/3})(1)+(1)^2\right)}{\cancel{x^{1/3}-1}}$$

$$= \lim_{x\to 1} (x^{2/3}+x^{1/3}+1)$$

$$= (1)^{2/3}+(1)^{1/3}+1 = 1+1+1 = 3$$

10.) Let $f(x) = \begin{cases} 1 & , \text{ if } x<-1 \\ 2 & , \text{ if } x=-1 \\ |x| & , \text{ if } -1<x\le 1 \\ 1+\sqrt{x} & , \text{ if } x>1 \end{cases}$;

sketch the graph of f and use the graph to determine the following limits :

a.) $\lim_{x\to 0} f(x)$ b.) $\lim_{x\to -2} f(x)$

c.) $\lim_{x\to 9} f(x)$ d.) $\lim_{x\to -\frac{3}{4}} f(x)$

e.) $\lim_{x\to -1} f(x)$

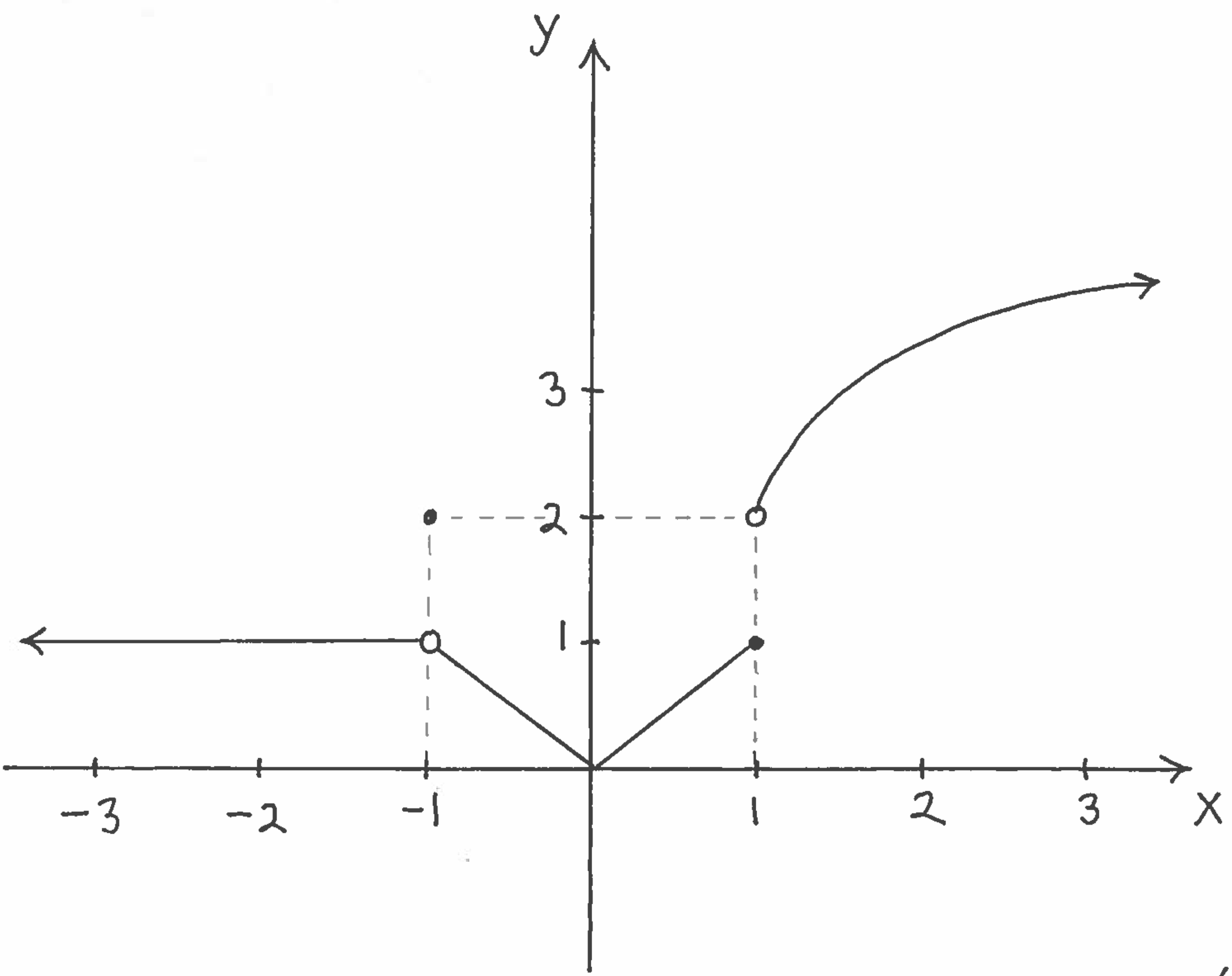
(ONE-SIDED LIMITS)

f.) $\lim_{x\to 1^+} f(x)$ (Right-hand limit) g.) $\lim_{x \to 1^-} f(x)$ (Left-hand limit)

h.) $\lim_{x \to 1} f(x)$ (Two-sided limit)

(LIMITS TO $\pm\infty$)

i.) $\lim_{x \to +\infty} f(x)$ j.) $\lim_{x \to -\infty} f(x)$



a.) $\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = |0| = 0$

b.) $\lim_{x\to -2} f(x) = \lim_{x\to -2} (1) = 1$

c.) $\lim_{x\to 9} f(x) = \lim_{x\to 9} (1+\sqrt{x})$

$= 1+\sqrt{9} = 1+3 = 4$

d.) $\lim_{x\to -\frac{3}{4}} f(x) = \lim_{x\to -\frac{3}{4}} |x|$

$= \left|-\frac{3}{4}\right| = \frac{3}{4}$

e.) $\lim_{x\to -1} f(x) = 1$ ("expected" y-value)

f.) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1+\sqrt{x})$

$(x>1)$

$= 1+\sqrt{1} = 1+1 = 2$

g.) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} |x|$

$(x<1)$

$= |1| = 1$

h.) $\lim_{x \to 1} f(x)$ Does Not Exist (D.N.E.)
because two possibilities
(2 and 1)

i.) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (1+\sqrt{x})$

$= 1 + \sqrt{\infty} = 1 + \infty = \infty$

j.) $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (1) = 1$

11.) $\lim_{x \to 3} \frac{|x-3|}{x-3} = \text{"}\frac{0}{0}\text{"}$;

$\lim_{\substack{x \to 3^+ \\ (x>3)}} \frac{|x-3|}{x-3} = \lim_{\substack{x \to 3^+ \\ (x-3>0)}} \frac{\cancel{x-3}}{\cancel{x-3}}$

$= \lim_{x \to 3^+} 1 = 1$;

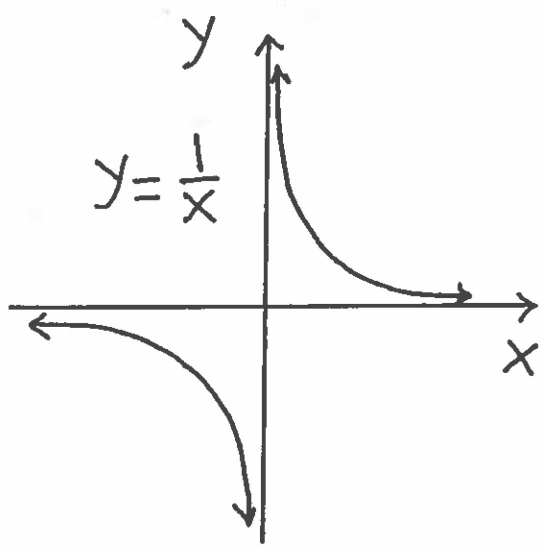
$\lim_{\substack{x \to 3^- \\ (x<3)}} \frac{|x-3|}{x-3} = \lim_{\substack{x \to 3^- \\ (x-3<0)}} \frac{-\cancel{(x-3)}}{\cancel{x-3}}$

$$= \lim_{x \to 3^-} -1 = -1 \quad ; \text{ so}$$

$$\lim_{x \to 3} \frac{|x-3|}{x-3} \text{ D.N.E. because}$$

TWO possibilities (−1 and 1)

Infinite Limits

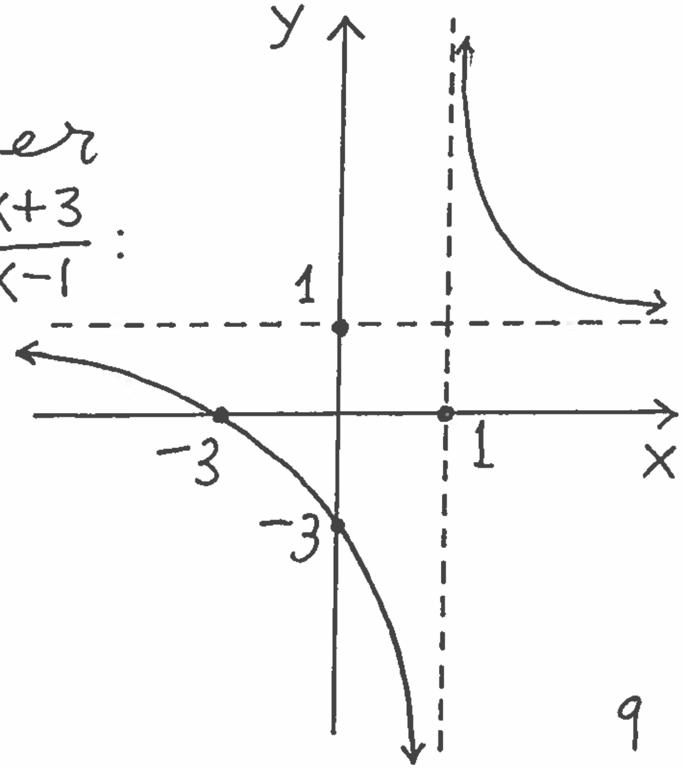


Example: Consider the graph of $y = \frac{1}{x}$:

a.) $\lim_{x \to 0^+} \frac{1}{x} = \text{"}\frac{1}{0^+}\text{"} = +\infty$

b.) $\lim_{x \to 0^-} \frac{1}{x} = \text{"}\frac{1}{0^-}\text{"} = -\infty$

Example: Consider the graph of $y = \frac{x+3}{x-1}$:



a.) $\lim_{x \to 1^+} \frac{x+3}{x-1} = \text{"}\frac{4}{0^+}\text{"}$

$= +\infty$

b.) $\lim\limits_{x \to 1^-} \dfrac{x+3}{x-1} = \text{"}\dfrac{4}{0^-}\text{"} = -\infty$

<u>SHORTCUT FACT</u>: In the context of limits $\text{"}\dfrac{c}{0}\text{"} = \pm\infty$, where $c \neq 0$.

<u>Example</u>: Determine the following limits.

1.) $\lim\limits_{x \to -2^+} \dfrac{x-3}{x+2} = \dfrac{-5}{0^+} = -\infty$

-1.9

$x = -2$

2.) $\lim\limits_{x \to 1^-} \dfrac{x^3-4}{x^2-1} = \dfrac{-3}{0^-} = \dfrac{3}{0^+} = +\infty$

0.9

$x = 1$