Math 16A
Section 3.6

Limits to $\pm$ Infinity ($\pm\infty$)

RECALL: Indeterminate Forms are "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$0\cdot\infty$", "$\infty-\infty$", "1^{∞}", "∞^{0}", and "0^{0}".

Shortcut Fact: In the context of limits $\frac{c}{\pm\infty} = 0$ for any finite number c.

Example: Determine the following limits.

1.) $\lim_{x\to\infty} \frac{1}{x} = "\frac{1}{\infty}" = 0$

2.) $\lim_{x\to-\infty} \frac{30}{x+100} = "\frac{30}{-\infty}" = 0$

3.) $\lim_{x \to \infty} (x^3 - 2x^2 + 5x - 100)$

$= \lim_{x \to \infty} (x(x^2 - 2x + 5) - 100)$

$= \lim_{x \to \infty} (x(x(x-2) + 5) - 100)$

$= \infty$

4.) $\lim_{x \to -\infty} \dfrac{x^2 + 3x - 5}{3x^2 - 2x + 4} = \text{"}\dfrac{\infty}{\infty}\text{"}$

(divide by highest power of x in the denominator)

$= \lim_{x \to -\infty} \dfrac{x^2 + 3x - 5}{3x^2 - 2x + 4} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x \to -\infty} \dfrac{1 + \frac{3}{x} - \frac{5}{x^2}}{3 - \frac{2}{x} + \frac{4}{x^2}} = \dfrac{1 + (0) - (0)}{3 - (0) + (0)}$

$= \dfrac{1}{3}$

5.) $\lim_{x \to \infty} \dfrac{2x^2 - x^3}{x^2 + x + 1} = \text{"}\dfrac{-\infty}{\infty}\text{"}$

$= \lim_{x \to \infty} \dfrac{2x^2 - x^3}{x^2 + x + 1} \cdot \dfrac{1/x^2}{1/x^2}$

$$= \lim_{x\to\infty} \frac{2-x}{1+\frac{1}{x}+\frac{1}{x^2}} = \frac{2-(\infty)}{1+(0)+(0)} = -\infty$$

6.) $\lim_{x\to\infty} \left(x+\sqrt{x^2+100}\right) = \infty + \sqrt{\infty}$

$= \infty + \infty = \infty$

7.) $\lim_{x\to\infty} \left(x-\sqrt{x^2+9}\right) = "\infty - \infty"$

(use a conjugate)

$$= \lim_{x\to\infty} \left(x-\sqrt{x^2+9}\right) \frac{\left(x+\sqrt{x^2+9}\right)}{\left(x+\sqrt{x^2+9}\right)}$$

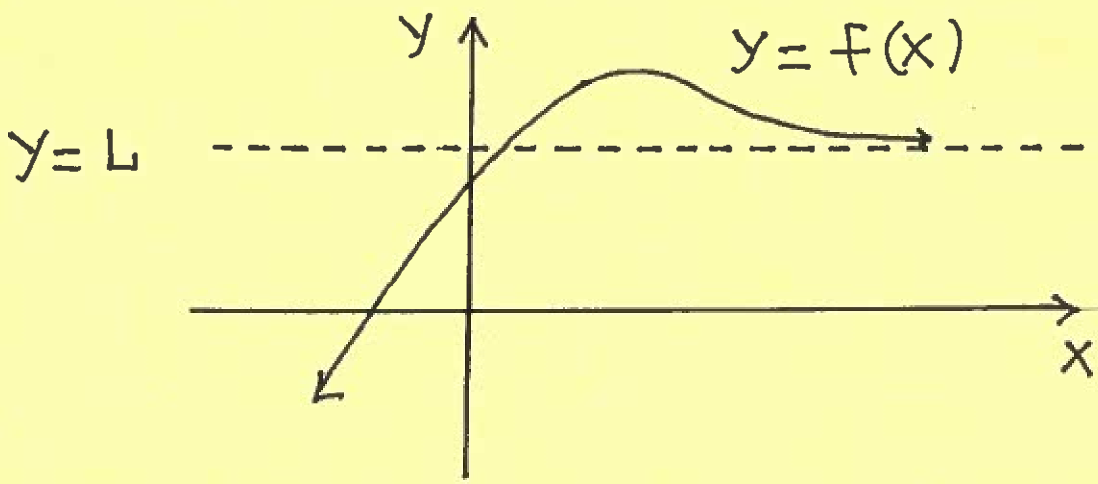
$$= \lim_{x\to\infty} \frac{x^2-(x^2+9)}{x+\sqrt{x^2+9}}$$

$$= \lim_{x\to\infty} \frac{-9}{x+\sqrt{x^2+9}} = \frac{-9}{\infty+\infty}$$

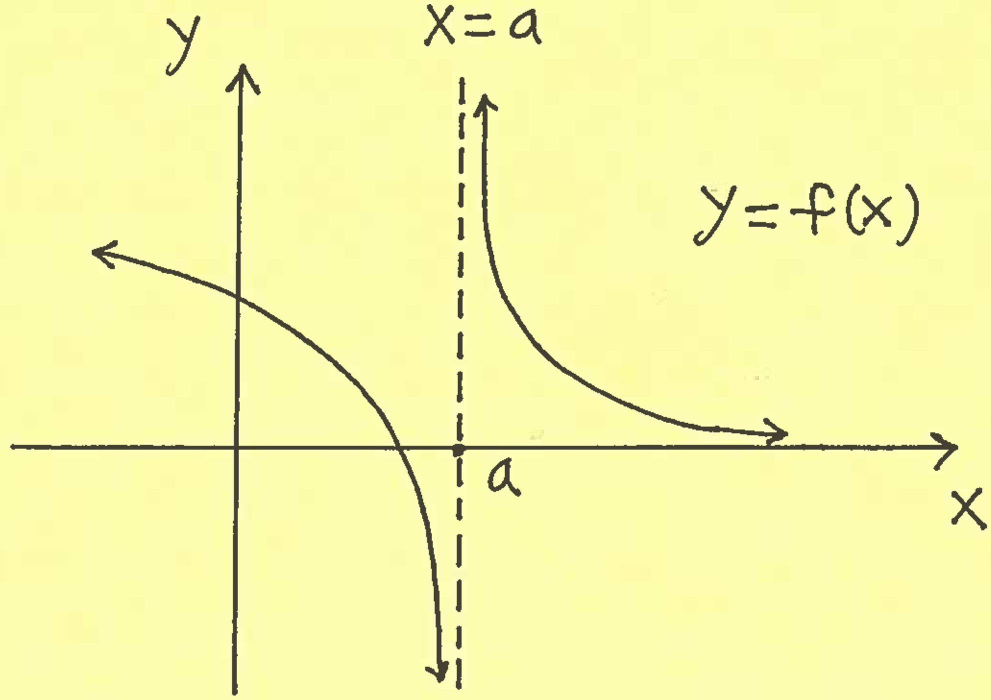
$$= \frac{-9}{\infty} = 0$$

Asymptotes – Horizontal, Vertical, and Tilted

Definition: If $\lim_{x \to \pm\infty} f(x) = L$, a FINITE #, then the line $\boxed{y = L}$ is a Horizontal Asymptote for the graph of $y = f(x)$.



Definition: If $\lim_{x \to a^{\pm}} f(x) = \pm\infty$, then the line $\boxed{x = a}$ is a Vertical Asymptote for the graph of $y = f(x)$.



<u>Definition</u>: If $y = f(x)$ is a rational function with the degree of the numerator exactly one more than the degree of the denominator, then the graph of f has a <u>Tilted Asymptote</u> (Slant Asymptote), which can be found using Polynomial Division.

Example: Use limits to find all Horizontal Asymptotes (H.A.) and Vertical Asymptotes (V.A.) for each function. Then use a graphing calculator to sketch the graph of f.

1.) $y = \dfrac{6x-12}{3x+4}$;

$$\lim_{x\to\pm\infty} \frac{6x-12}{3x+4} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\pm\infty} \frac{6-\frac{12}{x}}{3+\frac{4}{x}}$$

$$= \frac{6-0}{3+0} = 2, \text{ so H.A. is } \boxed{y=2} \ ;$$

$$3x+4=0 \rightarrow 3x=-4 \rightarrow x = {}^{-4}/_{3},$$

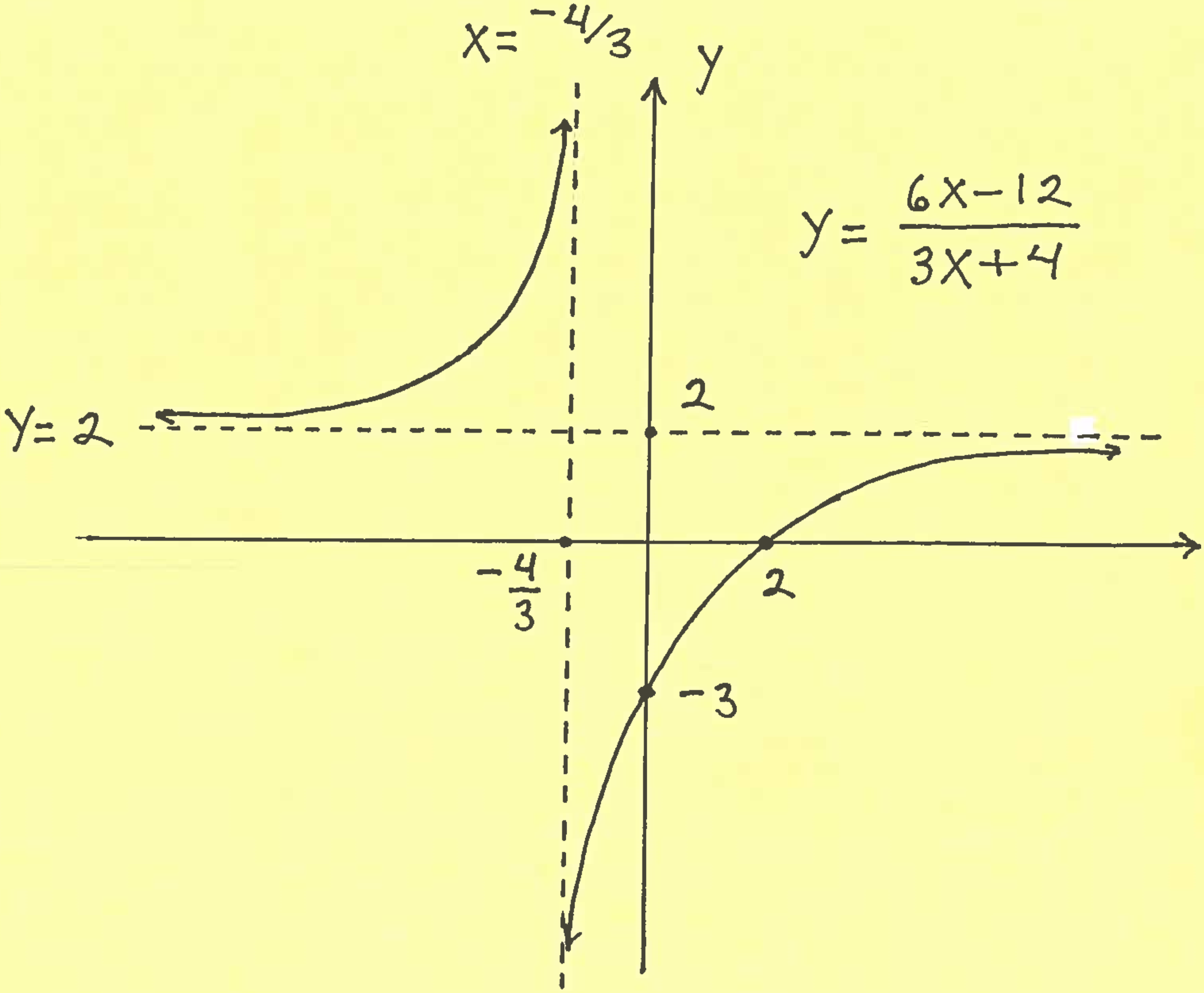
$$\text{then } \lim_{x\to -\frac{4}{3}^{+}} \frac{6x-12}{3x+4} = \frac{-20}{0^{+}} = -\infty,$$

-1.4 $\quad -1.3$

$x = \dfrac{-4}{3}$

so V.A. is $\boxed{x = \dfrac{-4}{3}}$;

$$\text{and } \lim_{x\to -\frac{4}{3}^{-}} \frac{6x-12}{3x+4} = \frac{-20}{0^{-}} = +\infty$$



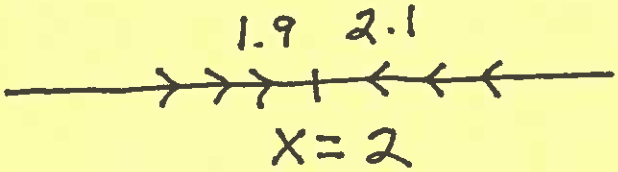
2.) $y = \dfrac{x-4}{4-x^2}$;

$$\lim_{x \to \pm\infty} \frac{x-4}{4-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x \to \pm\infty} \frac{\frac{1}{x} - \frac{4}{x^2}}{\frac{4}{x^2} - 1}$$

$$= \frac{0-0}{0-1} = 0,$$ so H.A. is $\boxed{y=0}$;

$4 - x^2 = (2-x)(2+x) = 0 \rightarrow x = 2,\ x = {}^{-}2$;

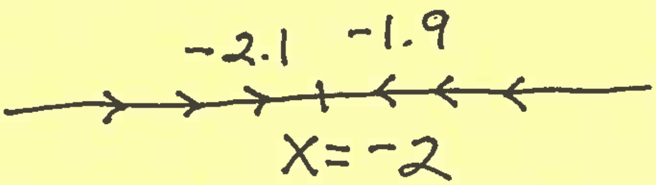
$$\lim_{x \to 2^+} \frac{x-4}{4-x^2} = \frac{-2}{0^-} = +\infty \text{, so}$$



V.A. is $\boxed{x=2}$;

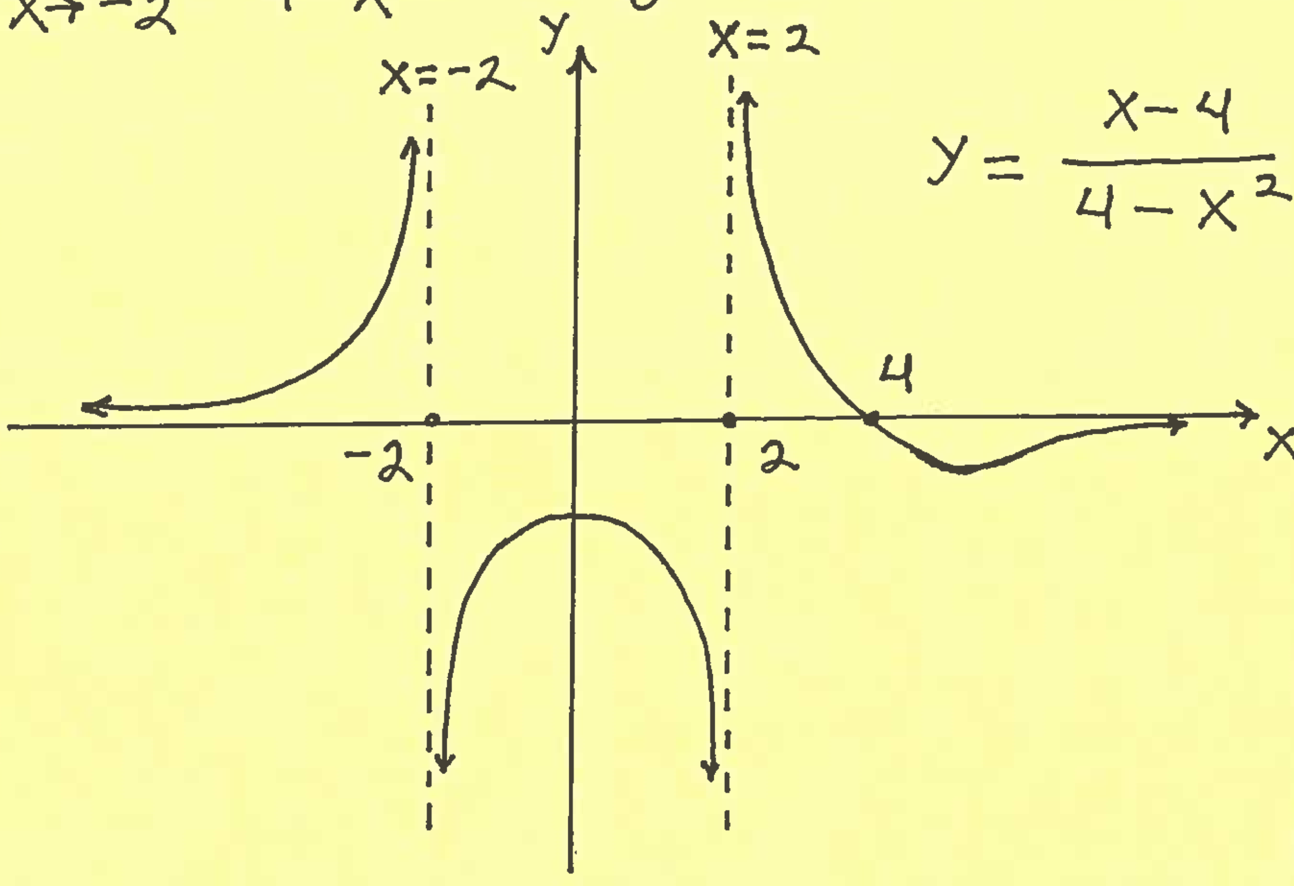
$$\lim_{x \to 2^-} \frac{x-4}{4-x^2} = \frac{-2}{0^+} = -\infty \text{ ;}$$

$$\lim_{x \to -2^+} \frac{x-4}{4-x^2} = \frac{-6}{0^+} = -\infty \text{, so}$$



V.A. is $\boxed{x=-2}$;

$$\lim_{x \to -2^-} \frac{x-4}{4-x^2} = \frac{-6}{0^-} = +\infty$$



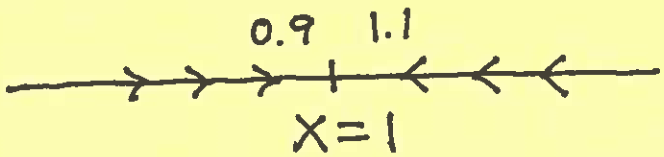
3.) $y = \dfrac{x^2}{x-1}$;

$$\lim_{x \to \pm\infty} \frac{x^2}{x-1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x \to \pm\infty} \frac{x}{1 - \frac{1}{x}}$$

$$= \frac{\pm\infty}{1-0} = \pm\infty$$ so $\underline{\underline{\text{NO}}}$ $\underline{\underline{\text{H.A.'s}}}$;

$x - 1 = 0 \longrightarrow x = 1$, so

$$\lim_{x \to 1^+} \frac{x^2}{x-1} = \frac{1}{0^+} = \infty \text{ , so}$$



V.A. is $\boxed{x=1}$;

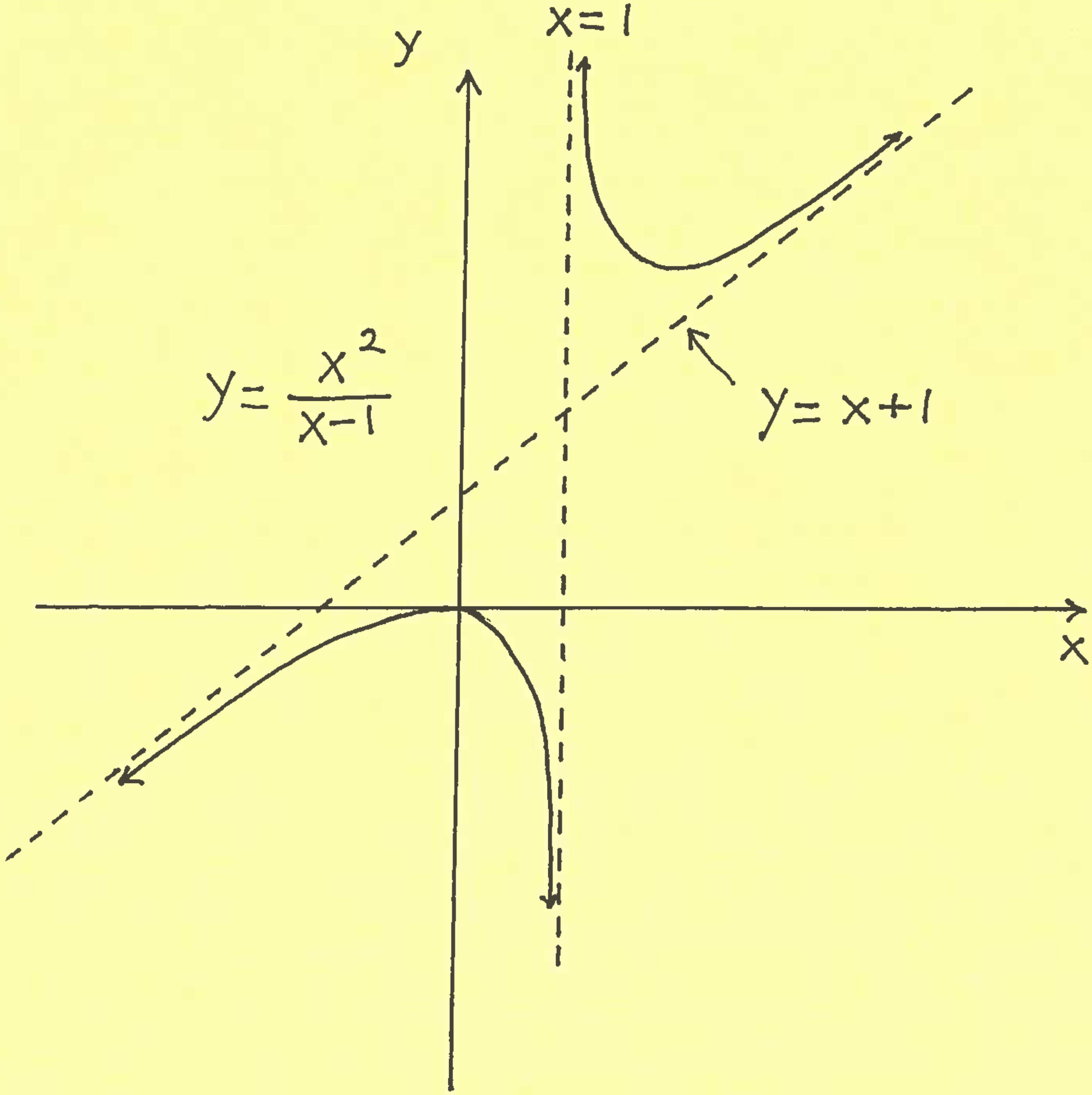
$$\lim_{x \to 1^-} \frac{x^2}{x-1} = \frac{1}{0^-} = -\infty$$

Tilted Asymptote: $y = \dfrac{x^2}{x-1}$

$$\begin{array}{r|l} & \quad x+1 \\ \hline x-1 & x^2 \\ & -(x^2 - x) \\ \hline & \qquad x \\ & \quad -(x-1) \\ \hline & \qquad\quad 1 \end{array}$$

$= \underbrace{x+1}_{\downarrow} + \dfrac{1}{x-1}$,

so $\boxed{y = x+1}$ is a T.A.



<u>Example</u>: Find all possible H.A.'s for $y = \dfrac{\sqrt{4x^2+9}}{x-3}$.

$$\lim_{x \to +\infty} \frac{\sqrt{4x^2+9}}{x-3} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{x^2(4 + 9/x^2)}}{x - 3}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{x^2}\,\sqrt{4 + 9/x^2}}{x - 3}$$

(<u>RECALL</u>: $\sqrt{z^2} = |z| = \begin{cases} z, & \text{if } z \geq 0 \\ -z, & \text{if } z < 0. \end{cases}$)

$$= \lim_{\substack{x \to +\infty \\ x > 0}} \frac{|x|}{x - 3} \cdot \sqrt{4 + 9/x^2}$$

$$= \lim_{x \to +\infty} \frac{x}{x - 3} \cdot \sqrt{4 + 9/x^2}$$

$$= \lim_{x \to +\infty} \frac{x}{x - 3} \cdot \frac{1/x}{1/x} \cdot \sqrt{4 + \frac{9}{x^2}}$$

$$= \lim_{x \to +\infty} \frac{1}{1 - \frac{3}{x}} \cdot \sqrt{4 + 9/x^2}$$

$$= \frac{1}{1 - 0} \sqrt{4 + 0} = 2, \text{ so}$$

H.A. is $\boxed{y = 2}$;

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2 + 9}}{x - 3} = \lim_{\substack{x \to -\infty \\ x < 0}} \frac{\sqrt{x^2} \cdot \sqrt{4 + 9/x^2}}{x - 3}$$

$$= \lim_{\substack{x \to -\infty \\ x<0}} \frac{|x|}{x-3} \sqrt{4 + 9/x^2}$$

$$= \lim_{x \to -\infty} \frac{-x}{x-3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \cdot \sqrt{4 + 9/x^2}$$

$$= \lim_{x \to -\infty} \frac{-1}{1 - \frac{3}{x}} \cdot \sqrt{4 + 9/x^2}$$

$$= \frac{-1}{1-0} \sqrt{4+0} = {}^{-}2, \text{ so}$$

H.A. is $\boxed{y = {}^{-}2}$.

<u>Example</u>: Use polynomial division to find the T.A. for each function.

1.) $y = \dfrac{2x^2 - 3x + 7}{x+2}$

$$\begin{array}{r} 2x - 7 \\ x+2 \,\overline{)\, 2x^2 - 3x + 7} \\ -(2x^2 + 4x) \\ \hline -7x + 7 \\ -(-7x - 14) \\ \hline 21 \end{array}$$

$$= \underbrace{2x - 7}_{\downarrow} + \frac{21}{x+2}, \text{ so}$$

$\boxed{y = 2x - 7}$ is a T.A.

2.) $y = \dfrac{x^3 - x^2 + x - 1}{x^2 + 2x + 2}$

$$\left(\begin{array}{r|l} & \quad x - 3 \\ \hline x^2 + 2x + 2 & x^3 - x^2 + x - 1 \\ & -(x^3 + 2x^2 + 2x) \\ \hline & \quad -3x^2 - x - 1 \\ & \quad -(-3x^2 - 6x - 6) \\ \hline & \qquad 5x + 5 \end{array}\right)$$

$$= \underbrace{x - 3}_{\downarrow} + \frac{5x + 5}{x^2 + 2x + 2}, \text{ so}$$

$\boxed{y = x - 3}$ is a T.A.