Lines

Definition: The slope of the line passing through points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} \]

Example: Find the slope of the line passing through the given points.

1.) \((1, -\frac{1}{3})\) and \((\frac{1}{2}, -2)\) →
\[ m = \frac{-2 - \left(-\frac{1}{3}\right)}{\frac{1}{2} - 1} = \frac{-\frac{6}{3} + \frac{1}{3}}{\frac{1}{2} - \frac{2}{2}} \]
\[ = \frac{-\frac{5}{3}}{-1} = \frac{5}{3} \cdot \frac{2}{1} = \frac{10}{3} \]

2.) \((3, -1)\) and \((-2, 7)\) (Opt. Proc.)

3.) \((-\frac{1}{2}, \frac{3}{4})\) and \((\frac{2}{3}, -1)\) (Opt. Proc.)

**Slope-Intercept Form:** Line with slope \(m\) and \(y\)-intercept \(b\) is

\[ y = mx + b \]

**Point-Slope Form:** Line with slope \(m\) and passing through the point \((x_1, y_1)\) is

\[ y - y_1 = m(x - x_1) \]
**Parallel Lines**: Slopes are equal

\[
\begin{align*}
Y &= 3X + 2 \\
Y &= 3X - 7
\end{align*}
\]

\( \{ \) parallel lines

**Perpendicular Lines**: Slopes are negative reciprocals

\[
\begin{align*}
Y &= \frac{2}{3}X + 4 \\
Y &= -\frac{3}{2}X - 5
\end{align*}
\]

\( \{ \) perpendicular lines

**Similar Triangles**: Two triangles with the same angles

\[
\frac{a}{b} = \frac{c}{d}
\]

Fact
Example: Solve for \( z \) where the triangles are similar.

\[
\frac{4}{3} = \frac{z}{2} \quad \Rightarrow \quad z = 2 \left( \frac{4}{3} \right) = \frac{8}{3}
\]

Question: How can we prove that the slopes of perpendicular lines are negative reciprocals?

Without loss of generality, assume that lines \( L_1 \) and \( L_2 \) are perpendicular.
and intersect at the origin. Assume that the slope of \( L_1 \) is \( \frac{b}{c} \) and the slope of \( L_2 \) is \( \frac{-z}{w} \).

It is easy to show that the two shaded right triangles are similar, so that

\[
\frac{w}{z} = \frac{b}{c} \quad \Rightarrow \quad w = \frac{b}{c} z
\]

then the slope of \( L_2 \) is

\[
m = \frac{-z}{w} = \frac{-\frac{b}{c}}{z} = \frac{-c}{b}, \quad \text{i.e.,}
\]

the slopes are negative reciprocals.

Example: (Opt. Prac.) Solve for \( x \) in the following diagram.

\[
\text{ANS: } x = \frac{15}{4}
\]
Functions

Definition: In an equation with variables \( x \) and \( y \), variable \( y \) is a function of \( x \) if

1.) each \( x \)-value has exactly one \( y \)-value

2.) the graph of the equation passes the vertical line test.

Example:

\[
\begin{align*}
\text{IS a function} & \quad \text{NOT a function}
\end{align*}
\]

Example: Show that the following equation represents \( y \) as a function of \( x \), or conclude that
y is not a function of $x$.

1.) $xy = \frac{x - y}{x + 3} \rightarrow$

$xy (x + 3) = x - y \rightarrow$

$x^2y + 3xy = x - y \rightarrow$

$x^2y + 3xy + y = x \rightarrow$

$y(x^2 + 3x + 1) = x \rightarrow$

$y = \frac{x}{x^2 + 3x + 1}$, so clearly each $x$-value has exactly one $y$-value, and $y$ is a function of $x$.

2.) $xy = x^2 - y^2 + q \rightarrow$

$y^2 + xy = x^2 + q$ \hspace{1em} \text{NOTE: } x = 0 \rightarrow y^2 + (0)y = (0)^2 + q$

$\rightarrow y^2 = q \rightarrow y = 3, y = -3$, so $x = 0$ has two $y$-values.
and \( y \) is NOT a function of \( x \).

**Function Notation:** In the previous example 1.)

\[
y = \frac{x}{x^2 + 3x + 1}
\]

is a function of \( x \), so write 

\[
f(x) = \frac{x}{x^2 + 3x + 1}
\]

then for example,

\[
f(1) = \frac{(1)}{(1)^2 + 3(1) + 1} = \frac{1}{5}
\]

\[
f(-2) = \frac{(-2)}{(-2)^2 + 3(-2) + 1} = \frac{-2}{-1} = 2
\]

\[
f\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) + 1} = \frac{\frac{1}{2}}{\frac{1}{4} + \frac{6}{4} + \frac{4}{4}}
\]

\[
= \frac{\frac{1}{2}}{\frac{11}{4}} = \frac{1}{2} \cdot \frac{4}{11} = \frac{2}{11}
\]
**Definition:** Function \( y = f(x) \) is one-to-one if

1.) each \( y \)-value has a unique (exactly one) \( x \)-value.

2.) the graph of \( f \) passes the horizontal line test.

**Algebraic Definition of one-to-one:**
Function \( y = f(x) \) is 1-1:
if \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

**Fact:** If function \( y = f(x) \) is 1-1, then \( f \) has an inverse function, \( f^{-1} \).

**Example:** Use the algebraic definition of 1-1 to show that the following function is 1-1. Then find its inverse function:

\[ f(x) = \frac{x-3}{2x+5} \]
\[ f(x_1) = f(x_2) \rightarrow \frac{x_1 - 3}{2x_1 + 5} = \frac{x_2 - 3}{2x_2 + 5} \]

\[ \rightarrow (x_1 - 3)(2x_2 + 5) = (x_2 - 3)(2x_1 + 5) \]

\[ \rightarrow 2x_1x_2 + 5x_1 - 6x_2 - 15 = 2x_1x_2 + 5x_2 - 6x_1 - 15 \]

\[ \rightarrow 5x_1 + 6x_1 = 5x_2 + 6x_2 \]

\[ \rightarrow 11x_1 = 11x_2 \]

\[ \rightarrow x_1 = x_2 \text{, i.e., } f \text{ is 1-1} \]

Find inverse function:
\[ y = \frac{x - 3}{2x + 5} \] (switch x and y)

\[ \rightarrow x = \frac{y - 3}{2y + 5} \] (now solve for y)

\[ \rightarrow x(2y + 5) = y - 3 \]

\[ \rightarrow 2xy + 5x = y - 3 \]

\[ \rightarrow 2xy - y = -5x - 3 \]
\[ y(2x-1) = -5x-3 \]
\[ y = \frac{-5x-3}{2x-1} = f^{-1}(x) . \]

**Functional Composition**

If \( y = f(x) \) and \( y = g(x) \) are functions, then

\[(f \circ g)(x) = f(g(x))\]

and
\[(g \circ f)(x) = g(f(x))\]

**Example:** Find and simplify \( f \circ g \) and \( g \circ f \) for

\[ f(x) = \frac{x+1}{x-1} \text{ and } g(x) = \frac{2-x}{x+3} . \]

\[(f \circ g)(x) = f(g(x))
= f \left( \frac{2-x}{x+3} \right)\]
\[
\left( \frac{2-x}{x+3} \right) + 1 = \frac{\frac{2-x}{x+3} + 1}{\frac{2-x}{x+3} - 1} \cdot \frac{x+3}{x+3}
\]

\[
= \frac{(2-x) + (x+3)}{(2-x) - (x+3)} = \frac{5}{5-2x}
\]

\[
(g \circ f)(x) = g\left( f(x) \right)
\]

\[
= g\left( \frac{x+1}{x-1} \right)
\]

\[
= 2 - \left( \frac{x+1}{x-1} \right)
\]

\[
= \frac{\left( \frac{x+1}{x-1} \right) + 3}{\frac{x+1}{x-1} + 3} \cdot \frac{x-1}{x-1}
\]

12
\[
\begin{align*}
\frac{2(x-1) - (x+1)}{(x+1) + 3(x-1)} &= \frac{2x-2-x-1}{x+1+3x-3} = \frac{x-3}{4x-2}.
\end{align*}
\]

**Domain and Range of a Function**

**Definition:** Let \( y = f(x) \) be a function.

I.) The **Domain** of \( f \) is the set of admissible \( x \)-values.

II.) The **Range** of \( f \) is the set of corresponding \( y \)-values.

**Example:** Find the Domain and Range for each function.

1.) \( f(x) = 4 + 3 \sin x \)
Domain: all $x$-values

Range: We know $-1 \leq \sin x \leq 1$ 

$\rightarrow -3 \leq 3 \sin x \leq 3$ 

$\rightarrow 4-3 \leq 4+3 \sin x \leq 4+3$ 

$\rightarrow 1 \leq 4+3 \sin x \leq 7$, so 

$Y$ 

Range is $1 \leq Y \leq 7$.

2.) $f(x) = 2 + \sqrt{3-x}$ ; we need 

$3-x \geq 0 \rightarrow x \leq 3$ so 

Domain: all $x \leq 3$

Range: $\sqrt{3-x} = 0$ when $x = 3$ and $\sqrt{3-x} \geq 0$ for $x \leq 3$. If 

$x \rightarrow -\infty$, then $\sqrt{3-x} \rightarrow +\infty$, so 

Range is all $y \geq 2$
3.) \( f(x) = \sqrt{9-x^2} \); we need
\[
9 - x^2 = (3-x)(3+x) \geq 0, \text{ so} \]
\[
\begin{array}{cccc}
3-x & 0 & 3+x \\
\hline
x=3 & - & x=-3 \\
\end{array}
\]

**Domain:** \(-3 \leq x \leq 3\)

**Range:** \( y = \sqrt{9-x^2} \geq 0 \rightarrow \)
\[
y^2 = (\sqrt{9-x^2})^2 = 9-x^2 \rightarrow \\
x^2 + y^2 = 9 \}
\]
so \( f \) is **TOP half of circle**

of radius 3:

**Range:** \( 0 \leq y \leq 3 \)